D-MAVT
D-MATL

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Analysis III

# Solutions Serie 12

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**1.** Find the general solution $u = u(x, y)$ of the following Laplace equation on a rectangle, with nonzero boundary conditions on two edges of the rectangle:

$$\begin{cases} \Delta u = 0, & 0 \leq x \leq a, 0 \leq y \leq b \\ u(x,0) = u(0,y) = 0, & 0 \leq x \leq a, 0 \leq y \leq b \\ u(x,b) = f(x), & 0 \leq x \leq a \\ u(a,y) = g(y), & 0 \leq y \leq b \end{cases}$$

where $f(x)$ and $g(y)$ are two arbitrary functions with the only (obvious) condition that they are compatible with the other boundary conditions, which means:

$$f(0) = f(a) = g(b) = g(0) = 0.$$

Choose your preferred method to solve it, in particular you are allowed to use any formula previously learned from the Lecture notes or from any other exercise.

[*Hint:* You can combine the results from §4.5.1. of the Lecture notes and Exercise **4.** of Serie 11.]

*Solution:*

We remember that we already know how to solve the problem in the cases in which either $g = 0$ (§4.5.1 of the Lecture notes) or $f = 0$ (Serie 11 - Exercise **4.**). Let us denote by $u_1, u_2$ the solutions to those two problems, that is:

$$\begin{cases} \Delta u_1 = 0, & 0 \leq x \leq a, 0 \leq y \leq b \\ u_1(x,0) = u_1(0,y) = 0, & 0 \leq x \leq a, 0 \leq y \leq b \\ u_1(x,b) = f(x), & 0 \leq x \leq a \\ u_1(a,y) = 0, & 0 \leq y \leq b \end{cases}$$

$$\begin{cases} \Delta u_2 = 0, & 0 \leq x \leq a, 0 \leq y \leq b \\ u_2(x,0) = u_2(0,y) = 0, & 0 \leq x \leq a, 0 \leq y \leq b \\ u_2(x,b) = 0, & 0 \leq x \leq a \\ u_2(a,y) = g(y), & 0 \leq y \leq b \end{cases}$$

But then the function $u = u_1 + u_2$ is a solution of the original problem with both $f$ and $g$. Now we just have to remember the formulas for $u_1$ and $u_2$ found, respectively, in §4.5.1. of the Lecture notes and in Exercise **4.** of Serie 11:

$$u_1(x,y) = \sum_{n=1}^{+\infty} A_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right), \qquad A_n = \frac{2}{a\sinh\left(\frac{n\pi}{a}b\right)}\int_0^a f(s)\sin\left(\frac{n\pi}{a}s\right)\,ds\,,$$

$$u_2(x,y) = \sum_{n=1}^{+\infty} A_n^* \sin\left(\frac{n\pi}{b}y\right)\sinh\left(\frac{n\pi}{b}x\right), \qquad A_n^* = \frac{2}{b\sinh\left(\frac{n\pi}{b}a\right)}\int_0^b g(s)\sin\left(\frac{n\pi}{b}s\right)\,ds\,,$$

$$\Longrightarrow \qquad u(x,y) = \sum_{n=1}^{+\infty}\left[A_n \sin\left(\frac{n\pi}{a}x\right)\sinh\left(\frac{n\pi}{a}y\right) + A_n^*\sin\left(\frac{n\pi}{b}y\right)\sinh\left(\frac{n\pi}{b}x\right)\right],$$

with coefficients $A_n$ and $A_n^*$ as above.

**2.** Find the solution of the following wave equation (with **inhomogeneous boundary conditions**) on the interval $[0,\pi]$:

$$u = u(x,t) \text{ such that } \begin{cases} u_{tt} = c^2 u_{xx}, & t \geq 0,\ x\in[0,\pi] \\ u(0,t) = 3, & t \geq 0 \\ u(\pi,t) = 5, & t\geq 0 \\ u(x,0) = x^2 + \frac{1}{\pi}(2-\pi^2)x + 3, & x\in[0,\pi] \\ u_t(x,0) = 0. & x \in [0,\pi] \end{cases} \tag{1}$$

You must proceed as follows.

**a)** Find the unique function $w = w(x)$ with $w'' = 0$, $w(0) = 3$, and $w(\pi) = 5$.

*Solution:*

The only functions with second derivative zero are the linear functions

$$w(x) = \alpha x + \beta, \quad \alpha, \beta \in \mathbb{R}.$$

Imposing the boundary conditions we find the right coefficients

$$\begin{cases} 3 = w(0) = \alpha\cdot 0 + \beta \\ 5 = w(\pi) = \alpha\cdot\pi + \beta \end{cases} \quad\Leftrightarrow\quad \begin{cases} \alpha = \frac{2}{\pi} \\ \beta = 3 \end{cases} \quad\Leftrightarrow\quad \boxed{w(x) = \frac{2}{\pi}x + 3.}$$

**b)** Define $v(x,t) := u(x,t) - w(x)$. Formulate the corresponding problem for $v$, equivalent to (1).

*Solution:*

The ODE doesn't change because $w$ is independent of time and has second derivative zero. The boundary conditions become homogeneous (that's why we chose this $w$)

$$v(0,t) = u(0,t) - w(0) = 3 - 3 = 0 \quad \textit{and} \quad v(\pi,t) = u(\pi,t) - w(\pi) = 5 - 5 = 0.$$

The initial position of the wave changes in

$$v(x,0) = u(x,0) - w(x) = x^2 + \frac{1}{\pi}(2-\pi^2)x + \cancel{3} - \frac{2}{\pi}x - \cancel{3} = x^2 + \cancel{\frac{2}{\pi}x} - \pi x - \cancel{\frac{2}{\pi}x} = x^2 - \pi x,$$

while the initial speed doesn't change (because, again, $w$ is independent of time). Finally

$$\boxed{\begin{cases} v_{tt} = c^2 v_{xx}, & t \geq 0,\ x \in [0,\pi] \\ v(0,t) = v(\pi,t) = 0, & t \geq 0 \\ v(x,0) = x^2 - \pi x, & x \in [0,\pi] \\ v_t(x,0) = 0. & x \in [0,\pi] \end{cases}}$$

**c)** (i) Find, using the formula from the script, the solution $v(x,t)$ of the problem you have just formulated.

[*Hint:* You can use the following formulas:

$$\int x \sin(nx)\, dx = \frac{\sin(nx) - nx\cos(nx)}{n^2} \quad (+ \text{ constant})$$

$$\int x^2 \sin(nx)\, dx = \frac{(2 - n^2x^2)\cos(nx) + 2nx\sin(nx)}{n^3} \quad (+ \text{ constant})]$$

*Solution:*

This is a standard homogeneous wave equation with homogeneous boundary conditions. The formula from the script is

$$v(x,t) = \sum_{n=1}^{+\infty} \left(B_n \cos(\lambda_n t) + B_n^* \sin(\lambda_n t)\right) \sin\left(\frac{n\pi}{L}x\right), \qquad \lambda_n = \frac{cn\pi}{L}$$

$$\overset{(L=\pi)}{=} \sum_{n=1}^{+\infty} (B_n \cos(cnt) + B_n^* \sin(cnt) \sin(nx).$$

The coefficients $B_n^* = 0$, because the initial speed is zero, while the coefficients $B_n$ are the Fourier series coefficients of the odd, $2\pi$-periodic extension of the initial datum $x^2 - \pi x$, that is:

$$B_n = \frac{2}{\pi} \int_0^\pi (x^2 - \pi x)\sin(nx)\, dx = \frac{2}{\pi} \int_0^\pi x^2 \sin(nx)\, dx - 2\int_0^\pi x\sin(nx)\, dx = \dots$$

[we can continue the computation using the hint]

$$\dots = \frac{2}{\pi}\left( \frac{(2-n^2x^2)\cos(nx) + 2nx\sin(nx)}{n^3}\Bigg|_0^\pi \right) - 2\left( \frac{\sin(nx) - nx\cos(nx)}{n^2}\Bigg|_0^\pi \right) =$$

$$= \frac{2}{\pi}\left( \frac{(2-n^2\pi^2)\cos(n\pi) - 2}{n^3} \right) - 2\left( -\frac{n\pi\cos(n\pi)}{n^2} \right) =$$

$$= \frac{4\cos(n\pi) - \cancel{2n^2\pi^2\cos(n\pi)} - 4 + \cancel{2n^2\pi^2\cos(n\pi)}}{\pi n^3} =$$

$$= \frac{4}{\pi n^3}(\cos(n\pi) - 1) = \frac{4}{\pi n^3}((-1)^n - 1) = -\frac{4}{\pi n^3} \cdot \begin{cases} 0 & n = 2j \\ 2 & n = 2j+1 \end{cases}$$

Finally we get the following equivalent expressions

$$\boxed{\begin{aligned} v(x,t) &= \frac{4}{\pi}\sum_{n=1}^{+\infty}\frac{((-1)^n-1)}{n^3}\cos(cnt)\sin(nx) = \\ &= -\frac{8}{\pi}\sum_{j=0}^{+\infty}\frac{1}{(2j+1)^3}\cos\big(c(2j+1)t\big)\sin\big((2j+1)x\big). \end{aligned}}$$

(ii) Write down explicitly the solution $u(x,t)$ of the original problem (1).
*Solution:*
We get the following equivalent expressions

$$\boxed{\begin{aligned} u(x,t) = v(x,t) + w(x) &= \frac{4}{\pi}\left(\sum_{n=1}^{+\infty}\frac{((-1)^n-1)}{n^3}\cos(cnt)\sin(nx)\right) + \frac{2}{\pi}x + 3 \\ &= -\frac{8}{\pi}\left(\sum_{j=0}^{+\infty}\frac{1}{(2j+1)^3}\cos\big(c(2j+1)t\big)\sin\big((2j+1)x\big)\right) + \frac{2}{\pi}x + 3. \end{aligned}}$$

**3.** Find the solution of the heat equation on an infinite bar

$$\begin{cases} u_t = c^2 u_{xx}, & x \in \mathbb{R},\ t \geq 0 \\ u(x,0) = f(x) = \begin{cases} \sinh(x), & |x| \leq 1 \\ 0, & \text{otherwise} \end{cases} & x \in \mathbb{R} \end{cases}$$

in Fourier integral form - formula (4.24) of the Lecture notes.

*Solution:*

As shown by formula (4.24) at pag. 65 of the Lecture notes in the case of an infinite bar, the general solution can be written in Fourier integral form as:

$$u(x,t) = \int_0^{+\infty} \big(A(p)\cos(px) + B(p)\sin(px)\big)\mathrm{e}^{-c^2p^2t}\,dp$$

where

$$A(p) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(v)\cos(pv)\,dv \quad \text{und} \quad B(p) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(v)\sin(pv)\,dv.$$

As $f$ is an odd function, we have that $A(p) = 0$. For $B(p)$ we obtain:

$$
\begin{aligned}
B(p) &= \frac{2}{\pi}\int_0^1 \sinh(v)\sin(pv)\,dv = \\
&= \frac{2}{\pi}\cosh(v)\sin(pv)\Big|_0^1 - \frac{2p}{\pi}\int_0^1 \cosh(v)\cos(pv)\,dv = \\
&= \frac{2}{\pi}\cosh(1)\sin(p) - \frac{2p}{\pi}\sinh(v)\cos(pv)\Big|_0^1 - \frac{2p^2}{\pi}\int_0^1 \sinh(v)\sin(pv)\,dv = \\
&= \frac{2}{\pi}\cosh(1)\sin(p) - \frac{2p}{\pi}\sinh(1)\cos(p) - p^2 B(p) = \\
&\iff B(p) = \frac{1}{1+p^2}\left(\frac{2}{\pi}\cosh(1)\sin(p) - \frac{2p}{\pi}\sinh(1)\cos(p)\right)
\end{aligned}
$$

Consequently the solution is given by

$$
\boxed{u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{1}{1+p^2}\left(\cosh(1)\sin(p) - p\sinh(1)\cos(p)\right)\sin(px)e^{-c^2p^2t}\,dp.}
$$

**4.** Solve the following heat equation on an infinite bar:

$$
\begin{cases}
u_t = \frac{1}{2}u_{xx}, & x \in \mathbb{R},\ t \geq 0 \\
u(x,0) = x\mathrm{e}^{-\frac{1}{2}x^2}, & x \in \mathbb{R}
\end{cases}
$$

via the Fourier transform with respect to $x$.
*Hint:* Use that, for $a > 0$,

$$
\mathcal{F}\left(x\mathrm{e}^{-ax^2}\right)(\omega) = \frac{-i\omega}{(2a)^{3/2}}\mathrm{e}^{-\frac{\omega^2}{4a}}.
$$

*Solution:*

Let us denote by $\widehat{u}(w,t) := \mathcal{F}\big(u(\cdot,t)\big)(w)$ the Fourier transform with respect to the space variable. Then, as shown at pag. 67 of the Lecture notes, the PDE transforms into

$$
\begin{cases}
\widehat{u}_t = -\frac{1}{2}\omega^2\widehat{u} \\
\widehat{u}(\omega,0) = -i\omega\mathrm{e}^{-\frac{1}{2}\omega^2}
\end{cases}
$$

This is nothing but an ODE for $\widehat{u}$ in the variable $t$. The general solution is simply given by

$$
\widehat{u}(\omega,t) = \widehat{u}(\omega,0)\mathrm{e}^{-\frac{1}{2}\omega^2 t} = -i\omega\mathrm{e}^{-\frac{1}{2}\omega^2}\mathrm{e}^{-\frac{1}{2}\omega^2 t} = -i\omega\mathrm{e}^{-\frac{1}{2}(1+t)\omega^2}.
$$



**Please turn!**

To obtain the solution $u$ we need to apply the Fourier inverse transform on the above equation. The left hand side is then equal to $u(x,t)$, whereas the right hand side can be taken care by rewriting the hint as

$$\mathscr{F}^{-1}(-\mathrm{i}\omega \mathrm{e}^{-b\omega^2}) = \frac{x}{(2b)^{3/2}} \mathrm{e}^{-\frac{x^2}{4b}},$$

and setting $b = \frac{1}{2}(1+t)$ in the above formula. The solution is therefore given by

$$\boxed{u(x,t) = \frac{x}{(1+t)^{3/2}} \mathrm{e}^{-\frac{1}{2(1+t)}x^2}.}$$