Solutions Serie 13

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1. Find the solution of the following Laplace equation on the disk of radius 2:

$$\begin{cases} \Delta u = 0, & (x,y) \in D_2 \\ u(x,y) = x^3. & (x,y) \in \partial D_2 \end{cases}$$

Do as follows:

a) Write the boundary condition in polar coordinates.

Solution:

The polar coordinates are linked to the standard cartesian ones by

$$\begin{cases} x = r\cos(\vartheta) \\ y = r\sin(\vartheta) \end{cases}$$

At the boundary the radius is $r = 2$ and the function becomes

$$x^3 = (2\cos(\vartheta))^3 = 8\cos^3(\vartheta).$$

b) Solve the problem in polar coordinates, using the methods/formulas from the Lecture Notes.

[*Hint:* It might be useful at some point to use the trigonometric formula

$$\cos^3(\vartheta) = \frac{3}{4}\cos(\vartheta) + \frac{1}{4}\cos(3\vartheta)\,]$$

Solution:

The Poisson integral formula always provides the solution, but it is very difficult to get an explicit solution out of it (solving the integral), so we will try to avoid it whenever possible. In fact, when the boundary condition is easy, it will be also easy to match the coefficients in the Fourier series.

Namely, from the Lecture notes we have a general solution of the Laplace equation in polar coordinates

$$u(r,\vartheta) = \sum_{n=0}^{+\infty} r^n \left(A_n \cos(n\vartheta) + B_n \sin(n\vartheta)\right).$$

We impose that on the boundary $r = 2$:

$$u(2,\vartheta) = \sum_{n=0}^{+\infty} 2^n \left(A_n \cos(n\vartheta) + B_n \sin(n\vartheta)\right) = 8\cos^3(\vartheta).$$

Which amounts in finding the Fourier series of $8\cos^3(\vartheta)$:

$$8\cos^3(\vartheta) = a_0 + \sum_{n=1}^{+\infty} a_n \cos(n\vartheta) + \sum_{n=1}^{+\infty} b_n \sin(n\vartheta)$$

The formula given in the text of the Exercise allows us to avoid computing any integral to get

$$8\cos^3(\vartheta) = 6\cos(\vartheta) + 2\cos(3\vartheta).$$

In terms of the coefficients we had to find this means

$$\begin{cases} B_n = 0 & \forall n \geq 0 \\ A_n = 0 & \forall n \geq 0,\ n \neq 1,3 \\ A_1 = \frac{a_1}{2} = 3 \\ A_3 = \frac{a_3}{2^3} = \frac{1}{4} \end{cases}$$

and the solution is

$$u(r,\vartheta) = rA_1\cos(\vartheta) + r^3 A_3 \cos(3\vartheta) = 3r\cos(\vartheta) + \frac{r^3}{4}\cos(3\vartheta).$$

c) Express the solution in the standard cartesian coordinates:

$$u(x,y) = ?$$

Solution:

If we want to express some function in terms of the standard cartesian coordinates (x,y) we are ok with terms like $r, \cos^k(\vartheta), \sin^k(\vartheta)$. So in this case we just need to write the function $\cos(3\vartheta)$ in these terms:

$$\cos(3\vartheta) = 4\cos^3(\vartheta) - 3\cos(\vartheta)$$

and

$$\begin{aligned} & 3r\cos(\vartheta) + \frac{r^3}{4}\cos(3\vartheta) = 3r\cos(\vartheta) + \frac{r^3}{4}\left(4\cos^3(\vartheta) - 3\cos(\vartheta)\right) = \\ = \ & 3(r\cos(\vartheta)) + (r\cos(\vartheta))^3 - \frac{3}{4}r^2(r\cos(\vartheta)) = 3x + x^3 - \frac{3}{4}(x^2+y^2)x = \\ = \ & 3x + x^3 - \frac{3}{4}x^3 - \frac{3}{4}xy^2 = 3x + \frac{1}{4}x^3 - \frac{3}{4}xy^2. \end{aligned}$$

2. a) Find the solution $u(r,\vartheta)$ of the following Dirichlet problem on the disk of radius R in polar coordinates:

$$\begin{cases} \Delta u = 0, & 0 \leq r \leq R, 0 \leq \vartheta \leq 2\pi \\ u(R,\vartheta) = \sin^2(\vartheta). & 0 \leq \vartheta \leq 2\pi \end{cases}$$

[*Hint:* Remember the trigonometric formula

$$\sin^2(\vartheta) = \frac{1}{2} - \frac{1}{2}\cos(2\vartheta)\ ]$$

Solution:

The solution will be

$$u(r,\vartheta) = \sum_{n=0}^{+\infty} r^n \left(A_n \cos(n\vartheta) + B_n \sin(n\vartheta)\right),$$

with coefficients found imposing

$$u(R,\vartheta) = \sum_{n=0}^{+\infty} R^n \left(A_n \cos(n\vartheta) + B_n \sin(n\vartheta)\right) = \sin^2(\vartheta).$$

Using the trigonometric formula

$$\sin^2(\vartheta) = \frac{1}{2} - \frac{1}{2}\cos(2\vartheta)$$

we obtain coefficients

$$\begin{cases} B_n = 0, & \forall n \geq 0 \\ A_n = 0, & \forall n \geq 0,\, n \neq 0, 2 \\ A_0 = \frac{1}{2}, \\ A_2 = -\frac{1}{2R^2}. \end{cases}$$

Finally

$$u(r,\vartheta) = \frac{1}{2} - \frac{r^2}{2R^2}\cos(2\vartheta).$$

b) Find the maximum of $u(r,\vartheta)$. In which point(s) is it reached?

Solution:

The constant $1/2$ plays no role in finding where is the maximum. The other term is a product of functions dependent from different variables. We then just need to maximize these functions independently:

$$\begin{cases} -\cos(2\vartheta) = 1 \\ r = R \end{cases} \quad \Leftrightarrow \quad \begin{cases} \vartheta = \frac{\pi}{2}, \frac{3}{2}\pi \\ r = R \end{cases} \quad \rightsquigarrow \quad (r,\vartheta) = \left(R, \frac{\pi}{2}\right), \left(R, \frac{3}{2}\pi\right) =: P_1, P_2$$

So the maximum is reached in the two points P_1, P_2 and it's equal to

$$u(P_1) = u(P_2) = \frac{1}{2} + \frac{R^2}{2R^2} = \frac{1}{2} + \frac{1}{2} = 1.$$

Remark: the maximum is reached only the boundary. Inside the disk the function is always strictly smaller than this value. You will see in the next lecture that this is not a coincidence.

c) Express the solution in the standard cartesian coordinates.

Solution:

We transform the term

$$\cos(2\vartheta) = \cos^2(\vartheta) - \sin^2(\vartheta) \quad \rightsquigarrow \quad r^2\cos(2\vartheta) = x^2 - y^2$$

and the solution will be

$$u(x,y) = \frac{1}{2} - \frac{1}{2R^2}(x^2 - y^2).$$

Please turn!

3. Prove, without computing explicitely the integrals, that for each $0 \leq r < 1$ and for each $0 \leq \vartheta \leq 2\pi$:

a)

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\vartheta-\varphi)+r^2}\,d\varphi = 1$$

Solution:

Let $D = \{(r,\vartheta)\,|\,0 \leq r \leq 1, 0 \leq \vartheta \leq 2\pi\}$ be the unit disk and u the constant function equal to 1

$$\begin{aligned} u &: D \to \mathbb{R} \\ (r,\vartheta) &\mapsto u(r,\vartheta) \equiv 1. \end{aligned}$$

Then clearly u is the solution of the Dirichlet problem

$$\begin{cases} \Delta u = 0, & \text{in } D \\ u(1,\vartheta) = 1, & 0 \leq \vartheta \leq 2\pi \end{cases}$$

and therefore for each $(r,\vartheta) \in [0,1) \times [0,2\pi]$ it must be represented by the Poisson integral formula:

$$u(r,\vartheta) = \frac{1}{2\pi}\int_0^{2\pi} K(r,\vartheta,1,\varphi)u(1,\varphi)\,d\varphi \qquad \rightsquigarrow$$

$$\rightsquigarrow \qquad 1 = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\vartheta-\varphi)+r^2}\,d\varphi.$$

b)

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{(1-r^2)\big(\cos^3(\varphi)\sin(\varphi)-\sin^3(\varphi)\cos(\varphi)\big)}{1-2r\cos(\vartheta-\varphi)+r^2}\,d\varphi = \frac{r^4}{4}\sin(4\vartheta).$$

Solution:

We would like to do as we did before. That is, consider the function

$$f(\vartheta) = \cos^3(\vartheta)\sin(\vartheta) - \sin^3(\vartheta)\cos(\vartheta), \qquad 0 \leq \vartheta \leq 2\pi.$$

We want to solve the Dirichlet problem on the unit disk with this boundary condition

$$\begin{cases} \Delta u = 0, & \text{in } D \\ u(1,\vartheta) = f(\vartheta), & 0 \leq \vartheta \leq 2\pi \end{cases}$$

We can proceed by matching the coefficients in the Fourier series as did in the Exercises **1.** and **2.**.
Even better, we can also recognise that the boundary condition

$$\cos^3(\vartheta)\sin(\vartheta) - \sin^3(\vartheta)\cos(\vartheta) = x^3y - xy^3$$

is already a well defined harmonic function on the whole plane

$$\Delta(x^3y - xy^3) = 6xy - 6xy \equiv 0.$$

So it must be itself the solution of the Dirichlet problem above.
Now we just need to manipulate it in polar coordinates to obtain

$$x^3y - xy^3 = xy(x^2 - y^2) = r^4 \cos(\vartheta)\sin(\vartheta)\big(\cos^2(\vartheta) - \sin^2(\vartheta)\big) =$$
$$= \frac{r^4}{2}\sin(2\vartheta)\cos(2\vartheta) = \frac{r^4}{4}\sin(4\vartheta),$$

and finally using the Poisson integral formula we have proved that

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{(1-r^2)\big(\cos^3(\varphi)\sin(\varphi) - \sin^3(\varphi)\cos(\varphi)\big)}{1 - 2r\cos(\vartheta - \varphi) + r^2}\, d\varphi = \frac{r^4}{4}\sin(4\vartheta).$$

4. Consider the following Neumann problem (Laplace equation with fixed normal derivative on the boundary):

$$\begin{cases} \nabla^2 u = 0, & \text{in } D_R \\ \partial_n u(R,\vartheta) = \vartheta(2\pi - \vartheta)(\vartheta^2 - 12), & 0 \le \vartheta \le 2\pi \ \big(\text{ parametrising } \partial D_R\big) \end{cases}$$

with D_R the disk center in the origin and radius R.

Is there a solution?

Solution:

Let $A \subset \mathbb{R}^2$ be a (regular) region of the plane and the curve $\gamma = \partial A$ its boundary. As explained in the lecture notes, if u solves the Neumann problem on A

$$\begin{cases} \nabla^2 u = 0, & \text{in } A \\ \partial_n u = g, & \text{on } \gamma \end{cases}$$

then the integral of g on the boundary must vanish because of the divergence theorem

$$\int_\gamma g\, d\gamma = \int_\gamma (\partial_n u)\, d\gamma = \int_\gamma (\nabla u \cdot n)\, d\gamma = \int_A \operatorname{div}(\nabla u)\, dA = \int_A (\nabla^2 u)\, dA = \int_A 0\, dA = 0.$$

In our case the region is a disk $A = D_R$ and the integral on the boundary is

$$\int_\gamma g\, d\gamma = \int_0^{2\pi} \vartheta(2\pi - \vartheta)(\vartheta^2 - 12)\, d\vartheta = \int_0^{2\pi} \big(-\vartheta^4 + 2\pi\vartheta^3 + 12\vartheta^2 - 24\pi\vartheta\big)\, d\vartheta =$$
$$= \left(-\frac{\vartheta^5}{5} + \frac{\pi\vartheta^4}{2} + 4\vartheta^3 - 12\pi\vartheta^2\right)\Bigg|_0^{2\pi} = -\frac{32}{5}\pi^5 + 8\pi^5 + 32\pi^3 - 48\pi^3 =$$
$$= -16\pi^3 + \left(8 - \frac{32}{5}\right)\pi^5 = -16\pi^3 + \frac{8}{5}\pi^5 = \frac{8}{5}\pi^3(\pi^2 - 10) \neq 0.$$

This means that the problem is ill-posed and can't have a solution.

Please turn!

5. Consider the Dirichlet problem for the Laplace equation,

$$R = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 1,\ 0 \le y \le 2\},$$

$$\begin{cases} \Delta u = 0, & (x, y) \in R, \\ u(0, y) = u(1, y) = 0, & 0 \le y \le 2, \\ u(x, 0) = 0, & 0 \le x \le 1, \\ u(x, 2) = f(x) & 0 \le x \le 1, \end{cases}$$

where f is a given continuous function. Show that there exists a unique solution.

[*Hint:* Assume to have two solutions u_1 and u_2 and consider the Dirichlet problem for the Laplace equation for $v = u_1 - u_2$.]

Solution:
By the lecture notes we know that there exists a solution. We can show that there is a unique solution in the following way:

Let assume that u_1 and u_2 are two solutions of the Laplace equation. And consider $v = u_1 - u_2$. Then by linearity of the Laplace equation we have,

$$\Delta v = \Delta u_1 - \Delta u_2 = 0.$$

And for the boundary conditions we have

$$\begin{aligned} v(0, y) &= u_1(0, y) - u_2(0, y) = 0, && 0 \le y \le 2, \\ v(1, y) &= u_1(1, y) - u_2(1, y) = 0, && 0 \le y \le 2, \\ v(x, 0) &= u_1(x, 0) - u_2(x, 0) = 0, && 0 \le x \le 1, \\ v(x, 2) &= u_1(x, 2) - u_2(x, 2) = f(x) - f(x) = 0 && 0 \le x \le 1. \end{aligned}$$

Therefore, we get for v,

$$\begin{cases} \Delta v = 0, & (x, y) \in R, \\ v_(0, y) = v(1, y) = 0, & 0 \le y \le 2, \\ v(x, 0) = 0, & 0 \le x \le 1, \\ v(x, 2) = 0 & 0 \le x \le 1. \end{cases}$$

Hence, $v = 0$ and $u_1 = u_2$.