D-MAVT
D-MATL

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Analysis III

Solutions Serie 2

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1. Find the inverse Laplace transform $f = \mathscr{L}^{-1}(F)$ of the following functions:

a) $F(s) = \dfrac{3}{s^5}$

Solution:

$$\mathscr{L}^{-1}\left(\frac{3}{s^5}\right) = 3\mathscr{L}^{-1}\left(\frac{1}{4!}\cdot\frac{4!}{s^5}\right) = 3\frac{1}{4!}\cdot\mathscr{L}^{-1}\left(\frac{4!}{s^5}\right) = \frac{1}{8}t^4.$$

b) $F(s) = \dfrac{1}{s^2+25}$

Solution:

$$\mathscr{L}^{-1}\left(\frac{1}{s^2+25}\right) = \mathscr{L}^{-1}\left(\frac{1}{5}\cdot\frac{5}{s^2+25}\right) = \frac{1}{5}\cdot\mathscr{L}^{-1}\left(\frac{5}{s^2+5^2}\right) = \frac{1}{5}\sin(5t).$$

c) $F(s) = \dfrac{7}{(s+3)(s-2)}$

Solution:

By partial fraction decomposition

$$\frac{7}{(s+3)(s-2)} = 7\left(\frac{1}{5(s-2)} - \frac{1}{5(s+3)}\right) = \frac{7}{5}\left(\frac{1}{s-2} - \frac{1}{s+3}\right) \implies f(t) = \frac{7}{5}(e^{2t} - e^{-3t}).$$

d) $F(s) = \dfrac{s+1}{(s+2)(s^2+s+1)}$

Solution:

The strategy is first applying partial fraction decomposition and then trying to get some expression similar to the Laplace transforms of sine and cosine.

$$F(s) = \frac{s+1}{(s+2)(s^2+s+1)} = \frac{1}{3}\left(\frac{s+2}{s^2+s+1} - \frac{1}{s+2}\right) = \frac{1}{3}\left(\frac{s+2}{s^2+s+1}\right) - \frac{1}{3}\left(\frac{1}{s+2}\right)$$

The second term is the Laplace transform of $-1/3e^{-2t}$ so we just have to modify further the first term.

$$\frac{s+2}{s^2+s+1} = \frac{s+2}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{s+2}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} =$$

$$= \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} + \sqrt{3}\cdot\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} \quad \Longrightarrow$$

$$\Longrightarrow \quad \mathscr{L}^{-1}\left(\frac{s+2}{s^2+s+1}\right) = e^{-1/2t}\left(\cos\left(\frac{\sqrt{3}}{2}t\right)+\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}t\right)\right).$$

Finally the whole inverse Laplace transform will be

$$f = \mathscr{L}^{-1}\left(\frac{s+1}{(s+2)(s^2+s+1)}\right) = \frac{1}{3}e^{-1/2t}\left(\cos\left(\frac{\sqrt{3}}{2}t\right)+\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}t\right)\right) - \frac{1}{3}e^{-2t}.$$

e) $F(s) = \dfrac{s}{(s-1)^2(s^2+2s+5)}$

Solution:

We still use first partial fraction decomposition to get terms which are known Laplace transforms (exponential, polynomials, sine and cosine).

$$\frac{s}{(s-1)^2(s^2+2s+5)} = \frac{1}{16}\cdot\frac{1}{s-1}+\frac{1}{8}\cdot\frac{1}{(s-1)^2}-\frac{1}{16}\cdot\frac{s}{s^2+2s+5}-\frac{5}{16}\cdot\frac{1}{s^2+2s+5} =$$

$$= \frac{1}{16}\cdot\frac{1}{s-1}+\frac{1}{8}\cdot\frac{1}{(s-1)^2}-\frac{1}{16}\cdot\frac{s+1-1}{(s+1)^2+2^2}-\frac{5}{16}\cdot\frac{1}{(s+1)^2+2^2} =$$

$$= \frac{1}{16}\cdot\frac{1}{s-1}+\frac{1}{8}\cdot\frac{1}{(s-1)^2}-\frac{1}{16}\cdot\frac{s+1}{(s+1)^2+2^2}-\frac{4}{16}\cdot\frac{1}{(s+1)^2+2^2} \quad \Longrightarrow$$

$$\Longrightarrow \quad f = \mathscr{L}^{-1}\left(\frac{s}{(s-1)^2(s^2+2s+5)}\right) = \frac{e^t}{16}+\frac{te^t}{8}-\frac{1}{16}e^{-t}\left(\cos(2t)+2\sin(2t)\right).$$

2. Find the Laplace transform of the following functions.

a) $f(t) = -8t^2\mathrm{e}^{-4t}$

Solution:

$$\mathcal{L}\left(-8t^2\mathrm{e}^{-4t}\right)(s) = -8\mathcal{L}(t^2)\,(s+4) = -8\cdot\frac{2!}{(s+4)^3} = -\frac{16}{(s+4)^3}.$$

b) $f(t) = \cosh^2(t/2)$

Solution:

$$\cosh^2(t/2) = \left(\frac{e^{t/2}+e^{-t/2}}{2}\right)^2 = \frac{1}{4}\left(e^t + e^{-t} + 2\right) \implies$$
$$\implies \mathcal{L}\left(\cosh^2(t/2)\right)(s) = \frac{1}{4}\left(\frac{1}{s-1} + \frac{1}{s+1} + \frac{2}{s}\right) = \frac{1}{4}\frac{4s^2-2}{s(s^2-1)} = \frac{2s^2-1}{2s(s^2-1)}.$$

c) $f(t) = t\sinh(3t)$

Solution:

$$\mathcal{L}(\sinh(3t))(s) = \mathcal{L}\left(\frac{\mathrm{e}^{3t}-\mathrm{e}^{-3t}}{2}\right) = \frac{1}{2}\left(\frac{1}{s-3} - \frac{1}{s+3}\right) = \frac{3}{s^2-9}$$
$$\implies \mathcal{L}(t\sinh(t))(s) = -\frac{d}{ds}\mathcal{L}(\sinh(t))(s) = -\frac{d}{ds}\left(\frac{3}{s^2-9}\right) = -\frac{-6s}{(s^2-9)^2} = \frac{6s}{(s^2-9)^2}.$$

d) $f(t) = t^2\cos(\pi t)$

Solution:

$$\mathcal{L}\left(t^2\cos(\pi t)\right)(s) = \frac{d^2}{ds^2}\mathcal{L}\left(\cos(\pi t)\right)(s) = \frac{d^2}{ds^2}\left(\frac{s}{s^2+\pi^2}\right) = \frac{d}{ds}\left(\frac{-s^2+\pi^2}{(s^2+\pi)^2}\right) =$$
$$= \frac{2s(s^2-3\pi^2)}{(s^2+\pi^2)^3}.$$

e) $f(t) = u(t-3)(t-3)^2$

Solution:

$$\mathcal{L}\left(u(t-3)(t-3)^2\right)(s) = e^{-3s}\mathcal{L}(t^2)(s) = e^{-3s}\frac{2!}{s^3} = \frac{2e^{-3s}}{s^3}.$$

f) $f(t) = u(t-5\pi)\cos(t)$

Solution:

$$\cos(t) = -\cos(t-5\pi) \implies \mathcal{L}\left(u(t-5\pi)\cos(t)\right)(s) = -\mathcal{L}\left(u(t-5\pi)\cos(t-5\pi)\right)(s) =$$
$$= -e^{-5\pi s}\mathcal{L}(\cos(t))(s) = -\frac{e^{-5\pi s}s}{s^2+1}.$$

Please turn!

g) $f(t) = -u(t-3)t^2 + u(t-5)\cos(t)$

Solution:

Some explanations are required here, as the function we need to transform is more complicated. In fact neither of the two terms has been shifted by the proper amount. Let us start with the first one, which needs to be shifted by 3, and therefore we make this shift appear:

$$-t^2 = -(t-3+3)^2 = -\left((t-3)^2 + 6(t-3) + 9\right).$$

Using this expression we know how to transform the first term:

$$\mathcal{L}(-u(t-3)t^2) = -\mathcal{L}\left(u(t-3)((t-3)^2 + 6(t-3) + 9)\right) = -\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right)\mathrm{e}^{-3s}.$$

The second term needs to be shifted by 5:

$$\cos(t) = \cos(t-5+5) = \cos(t-5)\cos(5) - \sin(t-5)\sin(5).$$

Therefore we also know how to transform this term:

$$\mathcal{L}(u(t-5)\cos(t)) = \left(\frac{s\cos(5)}{s^2+1} - \frac{\sin(5)}{s^2+1}\right)\mathrm{e}^{-5s} = \left(\frac{s\cos(5) - \sin(5)}{s^2+1}\right)\mathrm{e}^{-5s}.$$

Finally the whole thing is:

$$\mathcal{L}\left(-u(t-3)t^2 + u(t-5)\cos(t)\right)(s) = -\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right)\mathrm{e}^{-3s} + \left(\frac{s\cos(5) - \sin(5)}{s^2+1}\right)\mathrm{e}^{-5s}.$$

3. Find the Laplace transforms of the following functions

a) $f(t) = \begin{cases} \sin(t), & 0 \le t \le 4\pi \\ 0. & \text{otherwise} \end{cases}$

Solution:

We could either compute the Laplace transform directly, or we can observe that in terms of Heaviside functions we can write[1]:

$$f(t) = \sin(t) \cdot (u(t) - u(t-4\pi))$$

For the first term we can use the t-shifting property in its original form and for the second term we can use the t-shifting property in its modified form:

$$\mathcal{L}(g(t)u(t-a)) = \mathrm{e}^{-as}\mathcal{L}(g(t+a))$$

[1] To be precise, for how we defined the Heaviside function (see Serie 2), the equality $f(t) = \cos(t)(u(t) - u(t-2\pi))$ holds for all t except $t = 2\pi$, in which the right-hand side is zero and the left-hand side is 1. However when we compute the Laplace transform we integrate this function, therefore this difference will make no contribution and the result will be the same. A more precise way to say this: the Laplace transforms of two functions which differ from each other only on a set of measure zero are the same.

in this case $g(t) = \sin(t)$ and $a = 4\pi$ for which $\sin(t + 4\pi) = \sin(t)$, so actually

$$\mathcal{L}(\sin(t)u(t-4\pi)) = \mathrm{e}^{-4\pi s}\mathcal{L}(\sin(t+4\pi)) = \mathrm{e}^{-4\pi s}\mathcal{L}(\sin(t)) = \mathrm{e}^{-4\pi s}\frac{1}{s^2+1}$$

in conclusion

$$\mathcal{L}(f) = (1 - \mathrm{e}^{-4\pi s})\frac{1}{s^2+1}.$$

b) $f(t) = \begin{cases} 5t^2, & t \geq 100 \\ 0. & \text{otherwise} \end{cases}$

Solution:

In this case $f(t) = 5t^2 \cdot u(t - 100)$, so that using the t-shifting property:

$$\mathcal{L}(f) = \mathrm{e}^{-100s}\mathcal{L}(5(t+100)^2) = 5\mathrm{e}^{-100s}\mathcal{L}(t^2+200t+100^2) = 5\mathrm{e}^{-100s}\left(\frac{2}{s^3} + \frac{200}{s^2} + \frac{100^2}{s}\right).$$

4. First application to ODEs

Solve the following initial value problems with constant coefficients.

a) $\begin{cases} 2y'' + 2y' - 4y = 0 \\ y(0) = 1 \\ y'(0) = 5 \end{cases}$

Solution:

We transform both sides of the differential equation and we get, calling $Y(s)$ the Laplace transform of $y(t)$:

$$\begin{aligned}
&\mathcal{L}(2y'' + 2y' - 4y)(s) = 2\mathcal{L}(y'')(s) + 2\mathcal{L}(y')(s) - 4\mathcal{L}(y)(s) \\
&= 2(s^2Y(s) - sy(0) - y'(0)) + 2(sY(y)(s) - y(0)) - 4Y(s) \\
&= (2s^2 + 2s - 4)Y(s) - 2(s+1)y(0) - 2y'(0) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow \quad Y(s) &= \frac{2y'(0) + 2(s+1)y(0)}{2s^2 + 2s - 4} = \frac{12 + 2s}{(2s+4)(s-1)} \\
&= \frac{7}{3(s-1)} - \frac{4}{3(s+2)} = \frac{7}{3}\mathcal{L}(e^t)(s) - \frac{4}{3}\mathcal{L}(e^{-2t})(s).
\end{aligned}$$

Thus the solution is

$$y(t) = \frac{7}{3}e^t - \frac{4}{3}e^{-2t}.$$

b) $\begin{cases} 2y'' + 3y' - 2y = te^{-2t} \\ y(0) = 0 \\ y'(0) = -2 \end{cases}$

Solution:

As before we take the Laplace transform of both terms in the differential equation, and we plug in also the initial conditions to get:

$$2\left(s^2Y(s) - sy(0) - y'(0)\right) + 3\left(sY(s) - y(0)\right) - 2Y(s) = \frac{1}{(s+2)^2} \Leftrightarrow$$
$$\Leftrightarrow (2s^2 + 3s - 2)Y(s) + 4 = \frac{1}{(s+2)^2} \Leftrightarrow (2s-1)(s+2)Y(s) + 4 = \frac{1}{(s+2)^2}$$

Now solve for $Y(s)$

$$Y(s) = \frac{1}{(2s-1)(s+2)^3} - \frac{4}{(2s-1)(s+2)} = \frac{1 - 4(s+2)^2}{(2s-1)(s+2)^3} = \frac{-4s^2 - 16s - 15}{(2s-1)(s+2)^3}$$

and we use partial fraction decomposition to get, after a few computations,

$$Y(s) = \frac{A}{(2s-1)} + \frac{B}{(s+2)} + \frac{C}{(s+2)^2} + \frac{D}{(s+2)^3}$$

with

$$A = -\frac{192}{125}, \quad B = \frac{96}{125}, \quad C = -\frac{2}{25}, \quad D = -\frac{1}{5}.$$

It's convenient to get the common denominator of 125 on all these coefficients to get

$$Y(s) = \frac{1}{125}\left(-\frac{192}{2(s-1/2)} + \frac{96}{s+2} - \frac{10}{(s+2)^2} - \frac{25}{(s+2)^3}\right)$$

which, taking the inverse transform, gives

$$y(t) = \frac{1}{125}\left(-96e^{t/2} + 96e^{-2t} - 10te^{-2t} - \frac{25}{2}t^2e^{-2t}\right).$$

c) $\begin{cases} 3y'' = -8 + u(t-4)(t-4) \\ y(0) = 3 \\ y'(0) = 1 \end{cases}$

Solution:

The transformed equation is

$$3(s^2Y(s) - sy(0) - y'(0)) = \frac{-8}{s} + \frac{e^{-4s}}{s^2} \Leftrightarrow$$
$$\Leftrightarrow 3s^2Y(s) = 9s + 3 - \frac{8}{s} + \frac{e^{-4s}}{s^2} \Leftrightarrow$$
$$\Leftrightarrow Y(s) = \frac{3}{s} + \frac{1}{s^2} - \frac{8}{3s^3} + \frac{e^{-4s}}{3s^4}$$

Look at the next page!

which, taking the inverse transform, gives:

$$y(t) = 3 + t - \frac{4}{3}t^2 + \frac{1}{18}u(t-4)(t-4)^3.$$

5. ODEs with nonconstant coefficients

Solve the following initial value problem with nonconstant coefficients.

$$\begin{cases} ty'' - ty' + y = 2 \\ y(0) = 2 \\ y'(0) = -4 \end{cases}$$

Solution:

Transforming a function multiplied by t corresponds to taking the (minus) derivative of the Laplace transform of the function. In this case we are interested in

$$\mathcal{L}\left(ty'\right) = -\frac{d}{ds}\mathcal{L}\left(y'\right) = -\frac{d}{ds}\left(sY(s) - y(0)\right) = -sY'(s) - Y(s)$$
$$\mathcal{L}\left(ty''\right) = -\frac{d}{ds}\mathcal{L}\left(y''\right) = -\frac{d}{ds}\left(s^2Y(s) - sy(0) - y'(0)\right) = -s^2Y'(s) - 2sY(s) + y(0)$$

Taking the Laplace transform of both sides of the differential equation and plugging in the initial conditions yields

$$\begin{aligned} -s^2Y'(s) - 2sY(s) + y(0) - \left(-sY'(s) - Y(s)\right) + Y(s) &= \frac{2}{s} \\ \Leftrightarrow \qquad s(1-s)Y'(s) + 2(1-s)Y(s) + 2 &= \frac{2}{s} \\ \Leftrightarrow \qquad s\cancel{(1-s)}Y'(s) + 2\cancel{(1-s)}Y(s) &= \frac{2\cancel{(1-s)}}{s} \\ \Leftrightarrow \qquad Y'(s) + \frac{2}{s}Y(s) &= \frac{2}{s^2} \end{aligned}$$

This is a first order differential equation of the form $Y'(s) + f(s)Y(s) = g(s)$, for which the homogeneous solution is (look at the theory reminder for Ex.5 of Serie 0)

$$Y(s) = ce^{-\int f(s)ds} = ce^{-\int \frac{2}{s}ds} = ce^{-2\ln(s)} = c\left(e^{\ln(s)}\right)^{-2} = \frac{c}{s^2}, \qquad c \in \mathbb{R}.$$

A particular solution can be found by variation of constant, or assuming it to be of the form αs^k. This assumption is justified by the fact that all terms involved are of this kind, and we will in fact find that $2/s$ is a particular solution. So

$$Y(s) = \frac{2}{s} + \frac{c}{s^2}$$

for some $c \in \mathbb{R}$. We will find this constant when imposing again the initial conditions, but first we take the inverse transform to get

$$y(t) = 2 + ct$$

which can be consistent with $y'(0) = -4$ if and only if $c = -4$. So, finally

$$y(t) = 2 - 4t.$$

6. **a)** Using the Laplace transform of the derivative, and the Laplace transform of $1/\sqrt{t}$ found in Exercise **1.c)** of Serie 1, prove that

$$\mathcal{L}\left(\sqrt{t}\right) = \frac{\sqrt{\pi}}{2s\sqrt{s}}.$$

Solution:

We have that

$$\left(\sqrt{t}\right)' = \frac{1}{2\sqrt{t}}$$

and consequently the Laplace transforms of these functions are related by

$$\mathcal{L}\left(\frac{1}{\sqrt{t}}\right) = 2\mathcal{L}\left(\left(\sqrt{t}\right)'\right) = 2\left(s\mathcal{L}\left(\sqrt{t}\right) - \sqrt{t}\,_{|t=0}\right) = 2s\mathcal{L}\left(\sqrt{t}\right)$$

from which, combining with the result found in Exercise **1.c)** of Serie 1, we get

$$\mathcal{L}(\sqrt{t}) = \frac{1}{2s} \cdot \mathcal{L}\left(\frac{1}{\sqrt{t}}\right) = \frac{1}{2s} \cdot \frac{\sqrt{\pi}}{\sqrt{s}} = \frac{\sqrt{\pi}}{2s\sqrt{s}}$$

b) More generally, prove that for every integer $n \geq 0$:

$$\mathcal{L}\left(t^n\sqrt{t}\right) = \frac{(2n+2)!\sqrt{\pi}}{(4s)^{n+1}(n+1)!\sqrt{s}}.$$

Solution:

Call $g_n(t) = t^n\sqrt{t}$. We use induction on $n \geq 0$, where the starting case $n = 0$ is exactly what we proved in the previous point. So assume $n \geq 1$ and that the claim holds for $n-1$.

We use the relation

$$g_n'(t) = \left(n + \frac{1}{2}\right) g_{n-1}(t) = \frac{2n+1}{2} \cdot g_{n-1}(t) \tag{1}$$

which allows us to compute the Laplace transform of the n-th function, $G_n(s)$, from the Laplace transform of the $(n-1)$-th function, of which we know the expression. In fact we have, for the rule on derivatives,

$$\mathcal{L}(g_n')(s) = s\mathcal{L}(g_n)(s) - \cancel{g_n(0)} = s\mathcal{L}(g_n)(s) = sG_n(s)$$

and combining with equation (1) we get

$$G_n(s) = \frac{1}{s} \cdot \mathcal{L}(g_n')(s) \overset{(1)}{=} \frac{2n+1}{2s} G_{n-1}(s) \overset{\text{induction}}{=} \frac{2n+1}{2s} \frac{(2n)!\sqrt{\pi}}{(4s)^n n! \sqrt{s}} =$$
$$= \frac{(2n+2)!\sqrt{\pi}}{4s \cdot (4s)^n (n+1)! \sqrt{s}} = \frac{(2n+2)!\sqrt{\pi}}{(4s)^{n+1} (n+1)! \sqrt{s}}.$$