D-MAVT
D-MATL

Prof. A. Iozzi
ETH Zürich
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Analysis III

Solutions Serie 3

1. Find the inverse Laplace transform of the following functions.

a) $F(s) = \dfrac{e^{-2s}}{s^2+4}$

Solution:

The function
$$G(s) = \frac{1}{s^2+4}$$
is the Laplace transform of
$$g(t) = \frac{1}{2}\sin(2t)$$
and it's multiplied by some exponential, thus we can apply the t-shifting theorem and find
$$\frac{e^{-2s}}{s^2+4} = e^{-2s}G(s) = \mathcal{L}(u(t-2)g(t-2))(s)$$
from which
$$f(t) = \frac{1}{2}u(t-2)\sin(2(t-2)).$$

b) $F(s) = \dfrac{e^{-s}}{(s+1)^3}$

Solution:

As before we want to recognize the term multiplied by the exponential as the Laplace transform of some function. We have
$$G(s) = \frac{1}{s^3}, \quad g(t) = \frac{t^2}{2!}$$
and so by the s-shifting theorem
$$\frac{1}{(s+1)^3} = G(s+1) = \mathcal{L}(e^{-t}g(t))(s).$$
So, with $h(t) = e^{-t}g(t)$, using the t-shifting theorem
$$\frac{e^{-s}}{(s+1)^3} = e^{-s}G(s+1) = e^{-s}\mathcal{L}(e^{-t}g(t))(s) = e^{-s}\mathcal{L}(h(t))(s) = \mathcal{L}(u(t-1)h(t-1))(s)$$
and the final inverse transform is
$$f(t) = u(t-1)h(t-1) = u(t-1)\frac{e^{-(t-1)}(t-1)^2}{2}.$$

c) $F(s) = \dfrac{1}{s(s^2+1)}$

Solution 1:

We can use the properties of the convolution, because we have a product of two functions we can recognize as Laplace transform of some other functions:

$$\mathcal{L}^{-1}\left(\frac{1}{s(s^2+1)}\right) = \mathcal{L}^{-1}\left(\mathcal{L}(1)\mathcal{L}(\sin(t))\right) = 1 * \sin(t) = \int_0^t \sin(\tau)d\tau = 1 - \cos(t).$$

Solution 2:

We can also use the integration property

$$\mathcal{L}\left(\int_0^t h(x)dx\right) = \frac{1}{s}\mathcal{L}\left(h(t)\right) \qquad \Longrightarrow \qquad \mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}(h(t))\right) = \int_0^t h(x)dx$$

with $h(x) = \sin(x)$ and we get the same result:

$$\mathcal{L}^{-1}\left(\frac{1}{s(s^2+1)}\right) = \mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}(\sin(t))\right) = \int_0^t \sin(x)dx = 1 - \cos(t).$$

d) $F(s) = \dfrac{1}{(s^2+1)^2}$

Solution:

We have, with $h(t) = \sin(t)$,

$$F(s) = \frac{1}{(s^2+1)^2} = \mathcal{L}(h)\mathcal{L}(h)$$

and therefore

$$f(t) = \mathcal{L}^{-1}(F)(t) = (h * h)(t)$$

which can be computed explicitly. For this is useful to remind the following trigonometric identity

$$\sin(\alpha)\sin(\beta) = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right) \tag{\#}$$

and

$$(h*h)(t) = \int_0^t \sin(\tau)\sin(t-\tau)d\tau \overset{(\#)}{=} \frac{1}{2}\int_0^t \cos(2\tau - t) - \cos(t)d\tau =$$

$$= \frac{1}{2}\left.\frac{\sin(2\tau-t)}{2}\right|_0^t - \frac{1}{2}t\cos(t) = \frac{1}{2}\left(\frac{\sin(t)}{2} - \frac{\sin(-t)}{2}\right) - \frac{1}{2}t\cos(t) =$$

$$= \frac{1}{2}\sin(t) - \frac{1}{2}t\cos(t) = \frac{1}{2}\left(\sin(t) - t\cos(t)\right).$$

Look at the next page!

e) $F(s) = \dfrac{s}{(s^2-16)^2}$

Solution 1:

We use the differentiation rule $\mathcal{L}'(f) = -\mathcal{L}(tf(t))$, so we need to recognise our function as the derivative of a Laplace transform. In fact:

$$\frac{s}{(s^2-16)^2} = -\frac{1}{2}\left(\frac{1}{s^2-16}\right)'$$

And this function is the Laplace transform of

$$-\frac{1}{2}\left(\frac{1}{s^2-16}\right) = -\frac{1}{8}\left(\frac{4}{s^2-16}\right) = -\frac{1}{8}\left(\mathcal{L}(\sinh(4t))\right).$$

Hence

$$\mathcal{L}^{-1}\left(\frac{s}{(s^2-16)^2}\right) = -\frac{1}{8}\mathcal{L}^{-1}\left((\mathcal{L}(\sinh(4t))'\right) = \frac{1}{8}t\sinh(4t).$$

Solution 2:

The second solution is to use the integration rule:

$$\int_s^{+\infty} \mathcal{L}(f)(s')\,ds' = \mathcal{L}\left(\frac{f(t)}{t}\right)(s)$$

where our function is $\mathcal{L}(f)$. We plug the function into the integral on the left-hand side of the integral equation. This gives

$$\int_s^{+\infty} \mathcal{L}(f)(s')\,ds' = \int_s^{+\infty} \frac{s'}{((s')^2-16)^2}ds' = -\frac{1}{2}\left(\frac{1}{(s')^2-16}\right)\Bigg|_s^{+\infty} = \frac{1}{2}\left(\frac{1}{s^2-16}\right).$$

According to the integration rule this is equal to the Laplace transform of

$$\frac{1}{2}\left(\frac{1}{s^2-16}\right) = \mathcal{L}\left(\frac{f(t)}{t}\right).$$

But we recognise this as the Laplace transform of:

$$\frac{1}{2}\left(\frac{1}{s^2-16}\right) = \frac{1}{8}\left(\frac{4}{s^2-16}\right) = \frac{1}{8}\mathcal{L}(\sinh(4t)).$$

Finally we get

$$\frac{f(t)}{t} = \frac{1}{8}\sinh(4t) \quad \Leftrightarrow \quad f(t) = \frac{1}{8}t\sinh(4t).$$

which agrees with the result we found before.

Please turn!

2. Compute the following convolutions.

a) $e^{at} * e^{bt} \quad (a, b \in \mathbb{R})$

Solution 1 (direct computation):

Let's first consider the case $a \neq b$, in which

$$e^{at} * e^{bt} = \int_0^t e^{a\tau} e^{b(t-\tau)}\, d\tau = e^{bt} \int_0^t e^{(a-b)\tau}\, d\tau = \frac{e^{bt}}{a-b} e^{(a-b)\tau} \Big|_0^t$$
$$= \frac{e^{bt}}{a-b}(e^{(a-b)t} - 1) = \frac{e^{at} - e^{bt}}{a-b}.$$

Instead for $a = b$ we have

$$e^{at} * e^{at} = \int_0^t e^{a\tau} e^{a(t-\tau)} d\tau = e^{at} \int_0^t d\tau = t\, e^{at}.$$

Solution 2 (Laplace transform):

We can use property (2) of Laplace transform (see theory reminder of serie 3). Therefore, we get,

$$e^{at} * e^{bt} = \mathcal{L}^{-1}\left(\mathcal{L}(e^{at})\mathcal{L}(e^{bt})\right) = \mathcal{L}^{-1}\left(\frac{1}{s-a} \cdot \frac{1}{s-b}\right) = \mathcal{L}^{-1}\left(\frac{1}{(s-a)(s-b)}\right)$$

Now in the case $a \neq b$ we use partial fraction decomposition and

$$\frac{1}{(s-a)(s-b)} = \frac{1}{a-b}\left(\frac{1}{s-a} - \frac{1}{s-b}\right) \quad \Longrightarrow \quad e^{at} * e^{bt} = \frac{e^{at} - e^{bt}}{a-b}$$

while in the case $a = b$

$$\frac{1}{(s-a)^2} = \mathcal{L}\left(te^{at}\right) \quad \Longrightarrow \quad e^{at} * e^{at} = t\, e^{at}.$$

b) $\sin(t) * \cos(t)$

Solution:

With the following trigonometric identity

$$\sin(\alpha)\cos(\beta) = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right) \qquad (*)$$

we have

$$\sin(t) * \cos(t) = \int_0^t \sin(\tau)\cos(t-\tau)\, d\tau \overset{(*)}{=} \frac{1}{2}\int_0^t \sin(t)\, d\tau + \frac{1}{2}\int_0^t \sin(2\tau - t)\, d\tau =$$
$$= \frac{1}{2}t\sin(t) - \frac{\cos(2\tau - t)}{4}\Big|_{\tau=0}^{\tau=t} = \frac{1}{2}t\sin(t) - \frac{1}{4}\left(\cancel{\cos(t) - \cos(-t)}\right) = \frac{1}{2}t\sin(t).$$

Look at the next page!

c) $t^m * t^n, \qquad m, n \in \mathbb{N}$

Solution 1 (direct computation - not recommended):

$$\int_0^t \tau^m (t-\tau)^n \, d\tau = \int_0^t \tau^m \left(\sum_{j=0}^{n} \binom{n}{j} t^{n-j} (-\tau)^j \right) d\tau =$$
$$= \sum_{j=0}^{n} \binom{n}{j} (-1)^j t^{n-j} \int_0^t \tau^{m+j} \, d\tau = \sum_{j=0}^{n} \binom{n}{j} (-1)^j t^{n-j} \cdot \frac{t^{m+j+1}}{m+j+1} =$$
$$= t^{m+n+1} \sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j}{m+j+1} = \frac{m!n!}{(m+n+1)!} t^{m+n+1}.$$

The last combinatorial identity follows from the identity:

$$\binom{m+n}{m}^{-1} = (m+n+1) \sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j}{m+j+1},$$

which is very hard to prove without knowing there is such kind of expression for the inverse of the binomial coefficient. That is way this method is not recommended, as one would probably just stop at second to last step, without finishing the computation.

Solution 2 (Laplace transform):

$$\mathcal{L}(t^m * t^n) = \mathcal{L}(t^m)\mathcal{L}(t^n) = \frac{m!n!}{s^{m+n+2}} = \mathcal{L}\left(\frac{m!n!}{(m+n+1)!} t^{m+n+1} \right)$$
$$\implies \qquad t^m * t^n = \frac{m!n!}{(m+n+1)!} t^{m+n+1}$$

3. Find the solution $f(t)$ of the following initial value problem:

$$\begin{cases} f''(t) - a^2 f(t) = a, \quad t > 0, \\ f(0) = 2, \quad f'(0) = a, \end{cases}$$

where $a > 0$ is a positive constant.

Solution:

We apply the Laplace transform to the ODE in the initial value problem. We denote by $F = \mathcal{L}(f)$ the Laplace transform of the function f, and we denote the variable in the new domain by s as usual (so $F = F(s)$).

The first term to transform is the second derivative f'', for which we use the formula:

$$\mathcal{L}(f'') = s^2 F - s f(0) - f'(0) = s^2 F - 2s - a.$$

The second term gives

$$\mathcal{L}(-a^2 f) = -a^2 F.$$

Using the formula 1) in the Laplace transform table we find the right hand side

$$\mathcal{L}(a) = \frac{a}{s}.$$

In conclusion the ODE becomes the following algebraic equation:

$$s^2 F - 2s - a - a^2 F = \frac{a}{s} \qquad \Longrightarrow \qquad \boxed{F = \frac{a}{s(s^2-a^2)} + \frac{2s}{s^2-a^2} + \frac{a}{s^2-a^2}.}$$

The last step is to take the inverse Laplace transform of F. For the first term we use the hint and the convolution property (Property 7 on page 17 in the Lecture Notes):

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{a}{s(s^2-a^2)}\right) &= \mathcal{L}^{-1}\Big(\mathcal{L}(1)\mathcal{L}(\sinh(at))\Big) \\
&= \int_0^t 1 \cdot \sinh(at')\ dt' \\
&= \frac{\cosh(at)}{a} - \frac{\cosh(0)}{a} \\
&= \frac{\cosh(at)}{a} - \frac{1}{a}.
\end{aligned}$$

For the second and third term, we use the usual formulas for the Laplace transform. Therefore we have,

$$\begin{aligned}
f(t) = \mathcal{L}^{-1}(F) &= \mathcal{L}^{-1}\left(\frac{a}{s(s^2-a^2)} + \frac{2s}{s^2-a^2} + \frac{a}{s^2-a^2}\right) \\
&= \mathcal{L}^{-1}\left(\frac{a}{s(s^2-a^2)}\right) + 2\mathcal{L}^{-1}\left(\frac{s}{s^2-a^2}\right) + \mathcal{L}^{-1}\left(\frac{a}{s^2-a^2}\right) \\
&= \frac{\cosh(at)}{a} - \frac{1}{a} + 2\cosh(at) + \sinh(at) \\
&= (2 + \frac{1}{a})\cosh(at) + \sinh(at) - \frac{1}{a}.
\end{aligned}$$

Hence,

$$\boxed{f(t) = (2 + \frac{1}{a})\cosh(at) + \sinh(at) - \frac{1}{a}.}$$

Look at the next page!

4. a) Use the Laplace transform to find the solution of the following initial value problem:

$$\begin{cases} y' = g(t) \\ y(0) = c \end{cases}$$

Solution:

Computing the Laplace transform of both sides of the differential equation we get

$$sY(s) - y(0) = G(s) \quad \Leftrightarrow \quad Y(s) = \frac{c}{s} + \frac{G(s)}{s}$$

To transform back the second summand of the right-hand side we use again the convolution property:

$$\mathcal{L}^{-1}\left(\frac{G(s)}{s}\right) = \mathcal{L}^{-1}\left(\frac{1}{s}\right) * \mathcal{L}^{-1}\left(G(s)\right) = 1 * g(t) = \int_0^t g(\tau) d\tau$$

from which we get

$$y(t) = c + \int_0^t g(\tau) d\tau$$

b) Find the solution $y : [0, \infty) \to \mathbb{R}$ of the following integral equation:

$$y(t) + \int_0^t y(\tau) \cosh(t - \tau)\, d\tau = t + \mathrm{e}^t.$$

Solution:

Let us start by rewriting

$$\int_0^t y(\tau) \cosh(t - \tau) d\tau = \big(y * \cosh\big)(t).$$

Now we apply the Laplace transform on the ODE. The left-hand side is:

$$\mathcal{L}(y) + \mathcal{L}(y * \cosh(t)) = Y(s) + Y(s)\mathcal{L}(\cosh(t)) = Y(s) + \frac{s}{s^2 - 1} Y(s) = \frac{s^2 + s - 1}{s^2 - 1} Y(s)$$

The right-hand side is:

$$\mathcal{L}(t + e^t) = \frac{1}{s^2} + \frac{1}{s - 1} = \frac{s^2 + s - 1}{s^2(s - 1)}.$$

Setting these expressions equal and solving for $Y(s)$ gives:

$$Y(s) = \frac{s + 1}{s^2} = \frac{1}{s} + \frac{1}{s^2},$$

and the solution is the obtained by applying the inverse Laplace transform:

$$y(t) = 1 + t.$$

Please turn!

5. Find the solution $f = f(t)$ of the following initial value problem:

$$\begin{cases} f''(t) + \omega^2 f(t) = \omega\, \delta(t-a), & t > 0 \\ f(0) = 1, \quad f'(0) = \omega, \end{cases}$$

where $\omega, a > 0$ are positive constants.

Solution:

We apply the Laplace transform to the ODE in the initial value problem. We denote by $F = \mathcal{L}(f)$ the Laplace transform of the function f, and we denote the variable in the new domain by s as usual (so $F = F(s)$).

The first term to transform is the second derivative f'', for which we use the formula:

$$\mathcal{L}(f'') = s^2 F - s f(0) - f'(0) = s^2 F - s - \omega\,.$$

Then we have $\mathcal{L}(\omega^2 f) = \omega^2 F$ (by linearity) and finally the term in the right-hand side becomes:

$$\mathcal{L}(\omega\, \delta(t-a)) = \omega \mathcal{L}(\delta(t-a)) = \omega \mathrm{e}^{-as}\,.$$

In conclusion the ODE becomes the following algebraic equation:

$$s^2 F - s - \omega + \omega^2 F = \omega \mathrm{e}^{-as} \qquad \Longrightarrow \qquad F = \frac{s}{s^2+\omega^2} + \frac{\omega}{s^2+\omega^2} + \mathrm{e}^{-as} \cdot \frac{\omega}{s^2+\omega^2}$$

We recognise the first and second term as Laplace transforms of cosine and sine, respectively, while for the third term we can use the t-shifting property, and obtain, applying the inverse Laplace transform:

$$\boxed{f = f(t) = \mathcal{L}^{-1}(F) = \cos(\omega t) + \sin(\omega t) + u(t-a) \sin\left(\omega(t-a)\right)\,.}$$