Solutions Serie 4

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1. This exercise relates the periodicity of a function with its derivative(s) and its properties of boundedness.
Let $f : \mathbb{R} \to \mathbb{R}$ be any function.

a) Prove that if f is periodic and continuous, then it is bounded.

Solution:

The point is that the values assumed by a periodic function of period P are already assumed by its restriction to any bounded interval of length P, and here we can use the boundedness theorem.

$$\max |f| = \max_{x\in\mathbb{R}} |f(x)| = \max_{x\in[0,P]} |f(x)| =: M < +\infty$$

b) Prove that if f is differentiable, and periodic of period P, then also f' is periodic with the same period.

Solution:

For any real number x we have, using the periodicity of f:

$$f'(x+P) = \lim_{t\to 0} \frac{f(x+P+t) - f(x+P)}{t} = \lim_{t\to 0} \frac{f(x+t) - f(x)}{t} = f'(x)$$

c) From **a)** and **b)** deduce that if f is periodic and smooth, then it is bounded and all its derivative are bounded as well.

Solution:

f is bounded by **a)** and its derivative f' is periodic by **b)** and continuous, thus bounded (by **a)** again). This applies for all derivatives.

d) Use **a)** and **b)** to give a very simple proof that $\sin(x^2)$ is not periodic.

Solution:

This function is differentiable, and its derivative $2x\cos(x^2)$ is not bounded, therefore it is not periodic.

2. Suppose that f and g are periodic functions of fundamental periods P and Q, respectively. What can you say about their sum $f+g$? More precisely, give minimal sufficient conditions on P and Q the sum $f+g$ is periodic, and if so, which is the period.

Solution:

Let us try to understand for which $W > 0$ the sum $f + g$ could be periodic. In other words we want the following equality for all $x \in \mathbb{R}$:

$$f(x+W) + g(x+W) \stackrel{!}{=} f(x) + g(x)\,.$$

Without assuming anything about the relationship between f and g (because we want *minimal* sufficient conditions), we need to impose that equality holds term by term, that is:

$$\begin{cases} f(x+W) \stackrel{!}{=} f(x) & \forall x \in \mathbb{R} \\ g(x+W) \stackrel{!}{=} g(x) & \forall x \in \mathbb{R} \end{cases}$$

from which it follows that W must be a positive multiple of both P and Q, that is there must exist positive natural numbers $n, m > 0$ for which $W = nP = mQ$. In other words, if we define the "least common multiple" function $LCM(\alpha, \beta)$ of two positive real numbers α, β as:

$$LCM(\alpha, \beta) = \min\{\gamma \,|\, \exists m, n \in \mathbb{N} \text{ such that } \gamma = m\alpha = n\beta\}$$

then we have the following result:

"The minimal sufficient conditions on the periods P and Q so that the sum $f+g$ is a periodic function is that $\Gamma := LCM(P, Q) < \infty$, in which case the sum is periodic with period Γ. Without further assumptions on the functions f and g, this is also the fundamental period."

3. Determine which of the following functions is periodic and which is not. For the periodic ones, determine their fundamental period. For the non-periodic ones, explain/prove why they are not periodic.

[*Hint: Using Exercises **1.** and **2.** helps.*]

a) $\cos(\frac{2\pi x}{L})$, where $L > 0$ is a constant.

*Solution:*The function is periodic of fundamental period L.

b) $\sin(2x) + x^3$

Solution:

It is not periodic. Indeed the function is not bounded because of the x^3.

c) $\cos(4x) + 2\cos(2x)$

Solution: It is periodic of fundamental period π.
If $f(x)$ is periodic of period P_1 and $g(x)$ is periodic of period P_2, then their sum $f(x) + g(x)$ is periodic of period the least common multiple

$$P = \mathrm{LCM}(P_1, P_2)$$

of the two periods[1]. In this case $\cos(4x)$ is periodic of fundamental period $2\pi/4 = \pi/2$ while $2\cos(2x)$ is periodic of fundamental period $2\pi/2 = \pi$, therefore their sum is periodic of period

$$P = \mathrm{LCM}\left(\frac{\pi}{2}, \pi\right) = \mathrm{LCM}(1, 2) \cdot \frac{\pi}{2} = 2\frac{\pi}{2} = \pi.$$

It is easy to see that no smaller number is a period.

d) $\cos(15x) + 3\sin(6x)$

Solution:

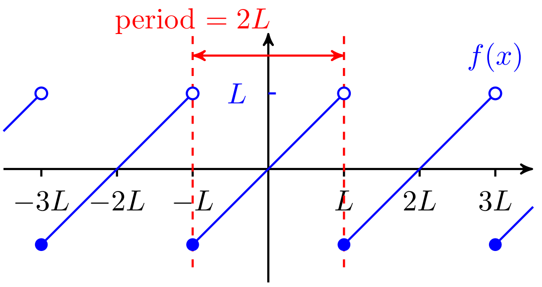
It is periodic of fundamental period $2\pi/3$.
In this case $\cos(15x)$ is periodic of fundamental period $2\pi/15$ while $3\sin(6x)$ is periodic of fundamental period $\pi/3$, therefore their sum is periodic of period

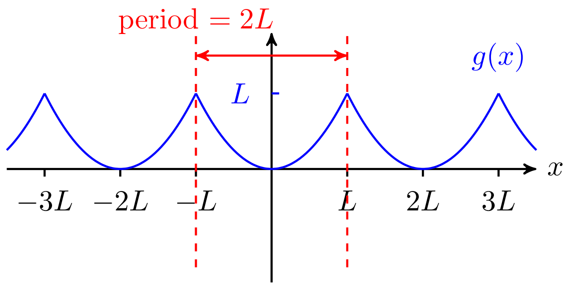
$$P = \mathrm{LCM}\left(\frac{2\pi}{15}, \frac{\pi}{3}\right) = \mathrm{LCM}(2, 5) \cdot \frac{\pi}{15} = \frac{10}{15}\pi = \frac{2}{3}\pi\,.$$

It is easy to see that no smaller number is a period.

4. Let now f and g be, respectively, the $2L$-periodic extensions to $\mathbb{R}$ of x and x^2 from $[-L, L)$. Sketch a graph of these functions.

Solution:





a) Are f and g well behaved in the sense specified above?

Solution:

Yes. f is continuous everywhere except in the odd integer multiples of L (so, anyway, a discrete set of points), and has left and right derivatives everywhere. g is continuous everywhere and has left and right derivatives everywhere.

[1] By the *least common muliple* of two real numbers we mean the smallest number P such that there are positive *integer* numbers k_1, k_2 such that $P = k_1 P_1 = k_2 P_2$. In the case that there is no such number, we define it to be $+\infty$ and the consequence is that the function is not periodic.

b) What are the points of discontinuity of f and g?

Solution:

As said before, g is continuous everywhere while f has discontinuities in the points $x_k = kL$, with k odd integer, that is $x = L, -L, 3L, -3L, \dots$.

c) What are the mean values of the left and right limit of f in its points of discontinuity?

$$\frac{1}{2}\left(f^+(x_0) + f^-(x_0)\right) = \ ?$$

Solution:

In each of these points the right limit is always $-L$ while the left limit is L, therefore the mean value is

$$\frac{1}{2}\left(f^+(x_k) + f^-(x_k)\right) = \frac{1}{2}(-L + L) = 0$$

d) The Fourier serie of f is

$$F(x) = \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{2L}{\pi n} \sin\left(\frac{n\pi}{L} x\right)$$

Does the Fourier series F of f converge to these values in these points?

Solution:

We have that the Fourier series

$$F(x) = \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{2L}{\pi n} \sin\left(\frac{n\pi}{L} x\right)$$

converges to zero in the points $x_k = kL$ with k odd integer. Indeed for these points

$$\sin\left(\frac{n\pi}{L} x_k\right) = \sin\left(\frac{n\pi}{L} kL\right) = \sin(n\pi k) = 0 \qquad \Longrightarrow \qquad F(x_k) = 0.$$

5. Compute the Fourier serie of $\cos^3(x)$ in $[-\pi, \pi]$ to show the following trigonometric identity

$$\cos^3(x) = \frac{3}{4}\cos(x) + \frac{1}{4}\cos(3x).$$

[*Hint:* *You can use that the coefficients* $b_n = 0$ *for all* n*. You will see the justification in chapter 3.2. And you can use the trigonometric formula* $\cos(x)\cos(nx) = \frac{1}{2}\left(\cos((n+1)x) + \cos((n-1)x)\right)$.]

Solution:

The $\cos^3(x)$ is even. Therefore its Fourier serie has only cosine terms (see chapter 3.2 in the lecture notes). For a_0 we have,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos^3(x)\, dx = 0.$$

For a_n we use the following trigonometric formulas

$$\cos(x)\cos(nx) = \frac{1}{2}\left(\cos((n+1)x) + \cos((n-1)x)\right).$$

and

$$\cos^2(x) = \frac{1}{2}\left(\cos(2x) + \cos(0)\right) = \frac{1}{2}\left(\cos(2x) + 1\right)$$

Therefore, for $n \geq 1$, we have

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^3(x)\cos(nx)\, dx \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2(x)\cos(x)\cos(nx)\, dx \\
&= \frac{1}{4\pi} \int_{-\pi}^{\pi} \left(\cos(2x) + 1\right)\left(\cos((n+1)x) + \cos((n-1)x)\right)\, dx \\
&= \frac{1}{4\pi} \int_{-\pi}^{\pi} \cos(2x)\cos((n+1)x)\, dx + \frac{1}{4\pi} \int_{-\pi}^{\pi} \cos(2x)\cos((n-1)x)\, dx \\
&+ \frac{1}{4\pi} \int_{-\pi}^{\pi} \left(\cos((n+1)x) + \cos((n-1)x)\right)\, dx.
\end{aligned}$$

Using the orthogonality of the trigonometric system (see lecture notes page 24) we deduce,

$$\int_{-\pi}^{\pi} \cos(2x)\cos((n+1)x)\, dx = \begin{cases} \pi, & n = 1 \\ 0, & \text{sonst} \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(2x)\cos((n-1)x)\, dx = \begin{cases} \pi, & n = 3 \\ 0, & \text{sonst} \end{cases}$$

and the last integral vanish except for $n = 1$. Therefore,

$$a_1 = \frac{1}{4\pi} \int_{-\pi}^{\pi} 1\, dx + \frac{1}{4\pi} \int_{-\pi}^{\pi} \cos^2(2x)\, dx = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$a_3 = \frac{1}{4\pi} \int_{-\pi}^{\pi} \cos^2(2x)\, dx = \frac{1}{4}.$$

Please turn!

Finally, the Fourier serie of $\cos^3(x)$ is

$$\cos^3(x) = \frac{3}{4}\cos(x) + \frac{1}{4}\cos(3x).$$