Solutions Serie 5

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1. Determine whether the following functions are even, odd, or neither. Justify your answer.

a) $f(x) = x^2 + 2$

Solution:

$$f(-x) = (-x)^2 + 2 = x^2 + 2 = f(x).$$

$\implies f$ is even.

b) $f(x) = x + 1$

Solution:

$$f(-x) = -x + 1 \neq \pm(x+1) = \pm f(x)$$

$\implies f$ is neither even nor odd.

c) $f(x) = \sinh(x^3 + x)$

Solution:

$$f(-x) = \sinh((-x)^3 - x) = \sinh(-x^3 - x) = \sinh(-(x^3 + x)) = -\sinh(x^3 + x) = -f(x)$$

$\implies f$ is odd.

d) $f(x) = \sin(\pi x) + \sin(x^2)$

Solution:

$$f(-x) = \sin(-\pi x) + \sin((-x)^2) = -\sin(\pi x) + \sin(x^2)$$

$\implies f$ is neither even nor odd.

e) $f(x) = \mathfrak{Re}(e^{i\sin(x)})$

Solution:

$$f(x) = \mathfrak{Re}(\cos(\sin(x)) + i\sin(\sin(x))) = \cos(\sin(x))$$
$$f(-x) = \cos(\sin(-x)) = \cos(-\sin(x)) = \cos(\sin(x))$$

$\implies f$ is even.

2. Consider the function

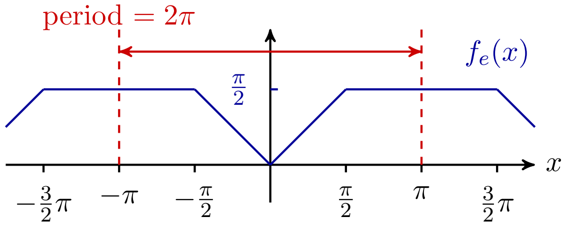
$$f(x) = \begin{cases} x, & 0 \le x \le \frac{\pi}{2} \\ \frac{\pi}{2}, & \frac{\pi}{2} \le x \le \pi \end{cases}$$

a) Extend f to an even function on the interval $[-\pi, \pi]$ and then finally to an even, 2π-periodic function on $\mathbb{R}$ and call this function f_e.
Sketch the graph of f_e and find its Fourier series.

Solution:

The even extension f_e is given, in the interval $[-\pi, \pi]$, by

$$f_e(x) = \begin{cases} \frac{\pi}{2}, & -\pi \le x \le -\frac{\pi}{2} \\ -x, & -\frac{\pi}{2} \le x \le 0 \\ x, & 0 \le x \le \frac{\pi}{2} \\ \frac{\pi}{2}, & \frac{\pi}{2} \le x \le \pi. \end{cases}$$



Being even, the b_n coefficients will vanish, while

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_e(x)\,dx = \frac{1}{\pi}\int_0^{\pi} f_e(x)\,dx = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} x\,dx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} \tfrac{\pi}{2}\,dx = \frac{1}{2\pi}x^2\Big|_0^{\frac{\pi}{2}} + \frac{x}{2}\Big|_{\frac{\pi}{2}}^{\pi} = \frac{3\pi}{8},$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_e(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} f_e(x)\cos(nx)\,dx =$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\cos(nx)\,dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} \frac{\pi}{2}\cos(nx)\,dx =$$

$$= \frac{2}{\pi}\left(\frac{x}{n}\sin(nx)\Big|_0^{\frac{\pi}{2}} - \frac{1}{n}\int_0^{\frac{\pi}{2}} \sin(nx)\,dx\right) + \frac{1}{n}\sin(nx)\Big|_{\frac{\pi}{2}}^{\pi} =$$

$$= \frac{1}{n}\sin\left(\tfrac{n\pi}{2}\right) + \frac{2}{n^2\pi}\cos(nx)\Big|_0^{\frac{\pi}{2}} + \frac{1}{n}\sin(nx)\Big|_{\frac{\pi}{2}}^{\pi} =$$

$$= \frac{2}{n^2\pi}\left(\cos\left(\tfrac{n\pi}{2}\right) - 1\right) = \begin{cases} -\frac{2}{n^2\pi}, & n = 2j+1 \\ \frac{2}{n^2\pi}((-1)^j - 1), & n = 2j. \end{cases}$$

The Fourier series is thus

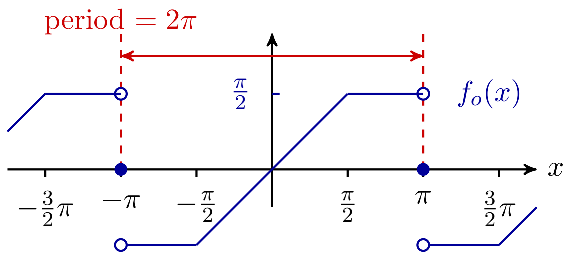
$$\frac{3\pi}{8} + \frac{2}{\pi}\sum_{j=1}^{+\infty} \frac{1}{(2j)^2}((-1)^j - 1)\cos(2jx) - \frac{2}{\pi}\sum_{j=0}^{+\infty} \frac{1}{(2j+1)^2}\cos((2j+1)x).$$

b) Do the same for the odd, 2π-periodic extension[1] of f (call this f_o).

Solution:

The odd extension f_o is given, in the interval $(-\pi, \pi]$, by

$$f_o(x) = \begin{cases} 0, & x = -\pi \\ -\frac{\pi}{2}, & -\pi < x \leq -\frac{\pi}{2} \\ x, & -\frac{\pi}{2} \leq x \leq 0 \\ x, & 0 \leq x \leq \frac{\pi}{2} \\ \frac{\pi}{2}, & \frac{\pi}{2} \leq x < \pi \\ 0, & x = \pi \end{cases}$$



Therefore here the a_n coefficients will be all zero, while

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_o(x)\sin(nx)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f_o(x)\sin(nx)\,dx =$$

$$= \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} x\sin(nx)\,dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} \frac{\pi}{2}\sin(nx)\,dx$$

$$= \frac{2}{\pi}\left(-\frac{x}{n}\cos(nx)\Big|_{0}^{\frac{\pi}{2}} + \frac{1}{n}\int_{0}^{\frac{\pi}{2}}\cos(nx)\,dx\right) - \frac{1}{n}\cos(nx)\Big|_{\frac{\pi}{2}}^{\pi}$$

$$= -\frac{1}{n}\cos\left(\tfrac{n\pi}{2}\right) + \frac{2}{n^2\pi}\sin(nx)\Big|_{0}^{\frac{\pi}{2}} - \frac{1}{n}\cos(n\pi) + \frac{1}{n}\cos\left(\tfrac{n\pi}{2}\right)$$

$$= \frac{2}{n^2\pi}\sin\left(\tfrac{n\pi}{2}\right) - \frac{1}{n}\cos(n\pi) = \begin{cases} -\frac{1}{n}, & n = 2j \\ \frac{2}{n^2\pi}(-1)^j + \frac{1}{n}, & n = 2j+1 \end{cases}$$

and the Fourier series is

$$-\sum_{j=1}^{+\infty}\frac{1}{2j}\sin(2jx) + \sum_{j=0}^{+\infty}\left(\frac{2}{(2j+1)^2\pi}(-1)^j + \frac{1}{2j+1}\right)\sin((2j+1)x).$$

3. a) Sketch the graph of the $2L$-periodic extension of

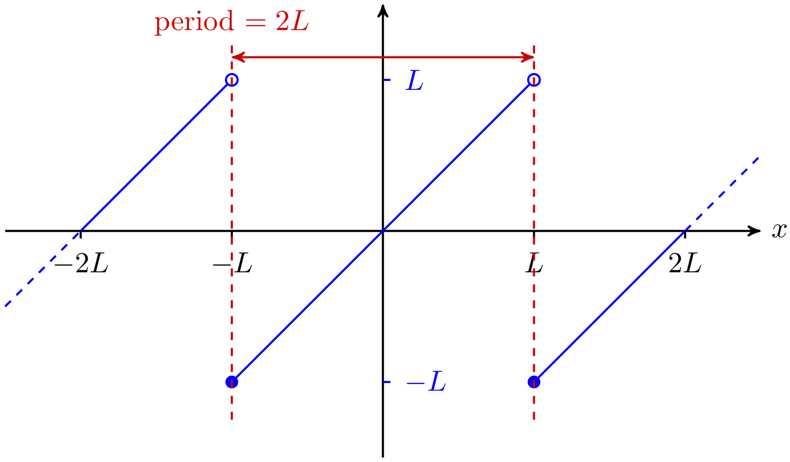
$$f(x) = x, \quad x \in [-L, L)$$

in the interval $[-2L, 2L]$. In which points this extension is not continuous?

[1] We added the condition $f(\pi) = 0$, to avoid problems when we want to extend f to an odd function.

Please turn!

Solution:



This extension is not continuous in the points $x = L + 2kL$, with $k \in \mathbb{Z}$. In these points the limit from the right is $-L$, while the limit from the left is L.

b) Compute its Fourier series.

Solution:

The extended function is odd[2] and therefore all a_n coefficients are going to vanish. Integration by parts (and a change of variable $y = \pi x/L$) yields

$$b_n = \frac{1}{L}\int_{-L}^{L} x \sin\left(\frac{n\pi}{L}x\right)dx = \frac{1}{L}\cdot\frac{L^2}{\pi^2}\int_{-\pi}^{\pi} y\sin(ny)dy =$$

$$= \frac{L}{\pi^2}\cdot\left.\frac{-ny\cos(ny)+\sin(ny)}{n^2}\right|_{-\pi}^{\pi} =$$

$$= -\frac{L}{\pi^2 n^2}\cdot\left(n\pi(-1)^n - n(-\pi)(-1)^n\right) = -\frac{2L}{\pi n}(-1)^n.$$

Therefore the Fourier series is

$$\sum_{n=1}^{+\infty} b_n \sin\left(\frac{n\pi}{L}x\right) = \sum_{n=1}^{+\infty}(-1)^{n+1}\frac{2L}{\pi n}\sin\left(\frac{n\pi}{L}x\right).$$

c) Evaluate the Fourier series at an appropriate point x_0 and use the convergence result to calculate the following numerical series

$$\sum_{k=0}^{+\infty}\frac{(-1)^k}{2k+1} = ?$$

[2]To be precise, it is odd almost everywhere, in the sense that $f(-x) = -f(x)$ for all x apart from a discrete set. In fact when we extend it to a $2L$ periodic function we get for example $f(L) = -L = f(-L)$. Anyway, because we are integrating the function, these points don't contribute.

Look at the next page!

Solution:

We know that the Fourier series converges to the function in every point in which the function is continuous. In this case, this means that every $x \in (-L, L)$ can be written as:

$$x = \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{2L}{\pi n} \sin\left(\frac{n\pi}{L} x\right). \tag{1}$$

We want to use this equality in a particular x_0 to find the value of

$$\sum_{k=0}^{+\infty} \frac{(-1)^k}{2k+1}.$$

We notice that we need only the terms $n = 2k+1$ in our Fourier series, while the others must cancel. Therefore a good attempt is $x_0 = L/2$. In fact in this way we obtain

$$\sin\left(\frac{n\pi}{L} x_0\right) = \sin\left(\frac{n\pi}{\not{L}} \cdot \frac{\not{L}}{2}\right) = \sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0, & n = 2k \\ (-1)^k. & n = 2k+1 \end{cases}$$

Plugging this in the equation (1) we get

$$\frac{L}{2} = \sum_{k=0}^{+\infty} \underbrace{(-1)^{2k+2}}_{=1} \frac{2L}{\pi(2k+1)} (-1)^k = \frac{2L}{\pi} \sum_{k=0}^{+\infty} \frac{(-1)^k}{2k+1}.$$

But then the series we were looking for is:

$$\sum_{k=0}^{+\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}.$$

4. Find the complex Fourier series of the same function $f(x)$ considered in Exercise **3**. Verify that the coefficients c_n of this series

$$\sum_{n=-\infty}^{+\infty} c_n e^{i\frac{n\pi}{L}x}$$

are related as written in the script to the real coefficients a_n, b_n found in the previous exercise.

Solution:

The complex Fourier coefficients for f are, for $n \neq 0$,

$$\begin{aligned}
c_n &= \frac{1}{2L} \int\limits_{-L}^{L} x e^{-i\frac{n\pi}{L}x}\, dx = \frac{L}{2\pi^2} \int\limits_{-\pi}^{\pi} y e^{-iny}\, dy = \\
&= \frac{L}{2\pi^2} \left(-\frac{y}{in} e^{-iny} \Big|_{-\pi}^{\pi} + \frac{1}{in} \int\limits_{-\pi}^{\pi} e^{-iny}\, dy \right) = \\
&= \frac{L}{2\pi^2} \left(-\frac{\pi}{in} e^{-in\pi} - \frac{\pi}{in} e^{in\pi} + \frac{1}{n^2} e^{-iny} \Big|_{-\pi}^{\pi} \right) = \\
&= \frac{L}{2\pi^2} \left(-\frac{\pi}{in} e^{-in\pi} - \frac{\pi}{in} e^{in\pi} + \frac{1}{n^2} e^{-in\pi} - \frac{1}{n^2} e^{in\pi} \right) = \\
&= \frac{(-1)^n L}{2\pi^2} \left(-\frac{\pi}{in} - \frac{\pi}{in} + \frac{1}{n^2} - \frac{1}{n^2} \right) \\
&= -\frac{(-1)^n L}{in\pi} = i\, \frac{(-1)^n L}{n\pi}
\end{aligned}$$

and for $n = 0$ is

$$c_0 = \frac{1}{2L} \int_{-L}^{L} x\, dx = \frac{x^2}{4L} \Big|_{-L}^{L} = 0.$$

Therefore the complex Fourier series of f is

$$\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} i\, \frac{(-1)^n L}{n\pi} e^{i\frac{n\pi}{L}x}.$$

The formula relating the real coefficients to the complex coefficients is

$$\begin{cases} a_0 = c_0 \\ a_n = c_n + c_{-n} \quad (n \geq 1) \\ b_n = i\,(c_n - c_{-n}) \end{cases}$$

and substituting we get indeed

$$\begin{cases} a_0 = c_0 = 0 \\ a_n = c_n + c_{-n} = i\, \frac{(-1)^n L}{n\pi} - i\, \frac{(-1)^n L}{n\pi} = 0 \\ b_n = i(c_n - c_{-n}) = i \left(i\, \frac{(-1)^n L}{n\pi} + i\, \frac{(-1)^n L}{n\pi} \right) = (-1)^{n+1} \frac{2L}{n\pi} \end{cases}$$

which is what we expected.