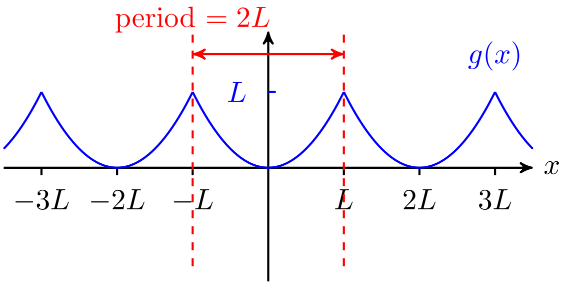
# Solutions Serie 6

**1.** Let $g$ be the $2L$-periodic extensions to $\mathbb{R}$ of $x^2$ from $[-L, L)$.

**a)** Sketch a graph of this function.

*Solution:*




**b)** Prove that the Fourier series of $g$ is

$$f(x) = \frac{L^2}{3} + \sum_{n=1}^{+\infty} (-1)^n \frac{4L^2}{\pi^2 n^2} \cos\left(\frac{n\pi}{L}x\right)$$

*Solution:*

$g$ is an even function, so it will have just $a_n$ coefficients. We have

$$a_0 = \frac{1}{2L} \int_{-L}^{L} g(x)\, dx = \frac{1}{L} \int_{0}^{L} x^2\, dx = \frac{L^2}{3}$$

The other $a_n$ coefficients can be either found explicitely by integrating twice by parts (left to you), or using the following trick.
The $b_n$ coefficients of the derivative $g'$ are related to our coefficients by

$$b_n(g') = \frac{1}{L} \int_{-L}^{L} g'(x) \sin\left(\frac{n\pi}{L}x\right) d\,x =$$

$$= \frac{1}{L} \left( \cancel{g(x) \sin\left(\frac{n\pi}{L}x\right) \Big|_{-L}^{L}} - \frac{n\pi}{L} \int_{-L}^{L} g(x) \cos\left(\frac{n\pi}{L}x\right) d\,x \right) =$$

$$= -\frac{n\pi}{L} a_n(g)$$

From Serie 5 exercice 3 b) we know that that the Fourier coefficient of the function $x$ are given by

$$b_n(x) = -\frac{2L}{\pi n}(-1)^n,$$

And using $b_n(g') = 2\, b_n(x)$ we get

$$a_n(g) = -\frac{L}{n\pi} b_n(g') = -\frac{2L}{n\pi} b_n(x) = (-1)^n \frac{4L^2}{\pi^2 n^2}$$

from which

$$f(x) = \frac{L^2}{3} + \sum_{n=1}^{+\infty} (-1)^n \frac{4L^2}{\pi^2 n^2} \cos\left(\frac{n\pi}{L} x\right)$$

**c)** Because $g$ is well-behaved and continuous everywhere, its Fourier series $G$ converge to it in every point. In particular

$$L^2 = g(L) = f(L).$$

Deduce from this equality the value of the Riemann Zeta function $\zeta(s)$ evaluated at $s = 2$

$$\zeta(2) := \sum_{n=1}^{+\infty} \frac{1}{n^2}$$

*Solution:*

From point c)

$$L^2 = f(L) = \frac{L^2}{3} + \sum_{n=1}^{+\infty} (-1)^n \frac{4L^2}{\pi^2 n^2} \cos\left(\frac{n\pi}{\cancel{L}} \cancel{L}\right) = \frac{L^2}{3} + \sum_{n=1}^{+\infty} \frac{4L^2}{\pi^2 n^2} =$$

$$= \frac{L^2}{3} + \frac{4L^2}{\pi^2}\zeta(2) \quad \Longrightarrow \quad \zeta(2) = \frac{\pi^2}{4L^2}\left(L^2 - \frac{L^2}{3}\right) = \frac{\pi^2}{6}.$$

**2.** For $a > 0$, consider the function $\cosh(ax)$ on the interval $[-\pi, \pi)$ and extend it on all $\mathbb{R}$ to a function of period $2\pi$.

**a)** Compute its complex Fourier series.

*Solution:*

The coefficients of the complex Fourier series are

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cosh(ax) e^{-inx}\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{e^{ax} + e^{-ax}}{2}\right) e^{-inx} dx =$$

$$= \frac{1}{4\pi} \int_{-\pi}^{\pi} \left(e^{(a-in)x} + e^{-(a+in)x}\right) dx = \frac{1}{4\pi} \left(\frac{e^{(a-in)x}}{a - in} - \frac{e^{-(a+in)x}}{a + in}\right)\Bigg|_{\pi}^{\pi} =$$

$$= \frac{1}{4\pi} \left(\frac{e^{a\pi} e^{-in\pi} - e^{-a\pi} e^{in\pi}}{a - in} - \frac{e^{-a\pi} e^{-in\pi} - e^{a\pi} e^{in\pi}}{a + in}\right) =$$

[To proceed remember that $\mathrm{e}^{i\pi} = -1$, therefore $\mathrm{e}^{\pm in\pi} = (-1)^n$ ]

$$= (-1)^n \frac{1}{2\pi}\left(\frac{\mathrm{e}^{a\pi} - \mathrm{e}^{-a\pi}}{2}\right)\left(\frac{1}{a-in} + \frac{1}{a+in}\right) = (-1)^n \frac{1}{2\pi}\sinh(a\pi)\frac{2a}{n^2+a^2} =$$
$$= \frac{a\sinh(a\pi)}{\pi}\frac{(-1)^n}{n^2+a^2}.$$

Therefore we obtain complex Fourier series:

$$\frac{a\sinh(a\pi)}{\pi}\sum_{n=-\infty}^{+\infty}\frac{(-1)^n}{n^2+a^2}\mathrm{e}^{inx}.$$

**b)** Use this result to find the value of the following series:

$$\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^2+a^2} = ?$$

*Solution:*

The complex Fourier series will be equal to $\cosh(ax)$ for each $x \in [-\pi, \pi]$, or equivalently:

$$\frac{\pi\cosh(ax)}{a\sinh(a\pi)} = \sum_{n=-\infty}^{+\infty}\frac{(-1)^n}{n^2+a^2}\mathrm{e}^{inx}, \qquad \forall x \in [-\pi, \pi]. \tag{1}$$

In particular for $x_0 = 0$ we obtain something very similar to what we need. Observe that the right hand side becomes

$$\sum_{n=-\infty}^{+\infty}\frac{(-1)^n}{n^2+a^2}\underbrace{\mathrm{e}^{in0}}_{=1} = \sum_{n=-\infty}^{+\infty}\frac{(-1)^n}{n^2+a^2} = \underbrace{\frac{1}{a^2}}_{\substack{n=0\\ \text{term}}} + 2\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^2+a^2} =: \frac{1}{a^2} + 2S,$$

where $S$ is the sum we wanted to compute. Therefore equation (1) in $x_0 = 0$ becomes:

$$\frac{\pi}{a\sinh(a\pi)} = \frac{1}{a^2} + 2S,$$

from which

$$S = \sum_{n=1}^{+\infty}\frac{(-1)^n}{n^2+a^2} = \frac{\pi}{2a\sinh(a\pi)} - \frac{1}{2a^2}.$$

**3.** The function $f(x) = \left|\cos\left(\frac{x}{2}\right)\right|$ is periodic of period $2\pi$.



**Please turn!**

**a)** Compute its Fourier series.

*Solution:*

The function is even, therefore it will only have $a_n$ coefficients. $L = \pi$, so:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\cos\left(\frac{x}{2}\right)\right|\,dx \overset{\text{even}}{=} \frac{1}{\pi}\int_{0}^{\pi}\left|\cos\left(\frac{x}{2}\right)\right|\,dx \overset{(\star)}{=} \frac{1}{\pi}\int_{0}^{\pi}\cos\left(\frac{x}{2}\right)\,dx =$$

$$= \frac{2}{\pi}\sin\left(\frac{x}{2}\right)\Bigg|_0^{\pi} = \frac{2}{\pi}$$

where the equality $(\star)$ is true because for $x$ in the interval $[0, \pi]$, the function $\cos\left(\frac{x}{2}\right) \geq 0$ is already nonnegative. For the other coefficients we will use the formula $2\cos(\alpha)\cos(\beta) = (\cos(\alpha+\beta) + \cos(\alpha-\beta))$:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left|\cos\left(\frac{x}{2}\right)\right|\cos(nx)\,dx = \frac{2}{\pi}\int_{0}^{\pi}\cos\left(\frac{x}{2}\right)\cos(nx)\,dx =$$

$$= \frac{1}{\pi}\int_{0}^{\pi}\left(\cos\left(\frac{(2n+1)x}{2}\right) + \cos\left(\frac{(2n-1)x}{2}\right)\right)\,dx =$$

$$= \frac{2}{\pi}\left(\frac{\sin\left(\frac{(2n+1)x}{2}\right)}{2n+1} + \frac{\sin\left(\frac{(2n-1)x}{2}\right)}{2n-1}\right)\Bigg|_0^{\pi} =$$

$$= \frac{2}{\pi}\left(\frac{\sin\left(\frac{(2n+1)\pi}{2}\right)}{2n+1} + \frac{\sin\left(\frac{(2n-1)\pi}{2}\right)}{2n-1}\right) =$$

$$= \frac{2}{\pi}\left(\frac{(-1)^n}{2n+1} + \frac{(-1)^{n+1}}{2n-1}\right) = \frac{2}{\pi}(-1)^n\left(\frac{1}{2n+1} - \frac{1}{2n-1}\right) =$$

$$= \frac{4}{\pi}\cdot\frac{(-1)^{n+1}}{4n^2-1}.$$

Therefore the Fourier series will be:

$$\frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{4n^2-1}\cos(nx)$$

**b)** Use this result to find the value of the following series:

$$\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{4n^2-1} = \,?$$

*Solution:*



**Look at the next page!**

The function is continuous everywhere, therefore

$$\left|\cos\left(\frac{x}{2}\right)\right| = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{4n^2-1}\cos(nx)\,, \quad \forall x \in \mathbb{R}. \tag{2}$$

In particular choosing the special $x_0 = 0$ we have

$$1 = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{4n^2-1} \quad \Leftrightarrow \quad \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{4n^2-1} = \frac{\pi}{4} - \frac{1}{2}.$$

**4.** Compute the Fourier integral of the function $f(x) = \mathrm{e}^{-\pi|x|}$ and use it to compute the values of the following integral:

$$\int_{\mathbb{R}} \frac{\cos(\omega x)}{\omega^2+\pi^2}\,d\omega$$

(for each $x \in \mathbb{R}$).

*Solution:*

The function $f(x) = \mathrm{e}^{-\pi|x|}$ is an even and continuous function, so its Fourier integral contains only the cosine term and it is equal to the function on each point:

$$\int_0^{+\infty} A(\omega)\cos(\omega x)\,d\omega = \mathrm{e}^{-\pi|x|}\,, \qquad \forall x \in \mathbb{R}\,. \tag{3}$$

We compute the coefficient $A(\omega)$:

$$A(\omega) = \frac{1}{\pi}\int_{\mathbb{R}} f(v)\cos(\omega v)\,dv = \frac{2}{\pi}\int_0^{+\infty} \mathrm{e}^{-\pi v}\cos(\omega v)\,dv =$$

$$\frac{2}{\pi}\left[\frac{\mathrm{e}^{-\pi v}(\omega\sin(\omega v) - \pi\cos(\omega v))}{\omega^2+\pi^2}\right]\Bigg|_{v=0}^{v=+\infty} = \frac{2}{\cancel{\pi}}\cdot\frac{\cancel{\pi}}{\omega^2+\pi^2} = \frac{2}{\omega^2+\pi^2}$$

When we insert this result in (3) we obtain that for each $x \in \mathbb{R}$:

$$2\int_0^{+\infty}\frac{\cos(\omega x)}{\omega^2+\pi^2}\,d\omega = \mathrm{e}^{-\pi|x|}\,.$$

One last observation is that the function in the integral on the left-hand side is an even function of $\omega$, therefore the left-hand side is actually equal to the integral over all $\omega \in \mathbb{R}$, which is what we need to compute:

$$\boxed{\int_{\mathbb{R}}\frac{\cos(\omega x)}{\omega^2+\pi^2}\,d\omega = \mathrm{e}^{-\pi|x|}\,, \qquad \forall x \in \mathbb{R}\,.}$$

**5.** Find the Fourier transform $\widehat{f} = \mathcal{F}(f)$ of the following functions:

**a)** $f(x) = \begin{cases} e^{2ix}, & -1 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$

*Solution:*

For $\omega \neq 2$ we have

$$\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ix\omega}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{ix(2-\omega)}\, dx = \frac{1}{\sqrt{2\pi}\, i(2-\omega)} e^{ix(2-\omega)} \Big|_{-1}^{1} =$$

$$= \frac{1}{\sqrt{2\pi}\, i(2-\omega)} \left( e^{i(2-\omega)} - e^{-i(2-\omega)} \right) = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{(2-\omega)} \cdot \frac{e^{i(2-\omega)} - e^{-i(2-\omega)}}{2i} =$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin(2-\omega)}{2-\omega}.$$

**b)** $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ -x, & -1 \le x \le 0 \\ 0, & \text{otherwise.} \end{cases}$

*Solution:*

$$\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-i\omega x}\, dx = \frac{1}{\sqrt{2\pi}} \left( -\int_{-1}^{0} x e^{-i\omega x}\, dx + \int_{0}^{1} x e^{-i\omega x}\, dx \right) =$$

$$= \frac{1}{\sqrt{2\pi}} \left( -x \frac{e^{-i\omega x}}{-i\omega} \Big|_{-1}^{0} - \frac{1}{i\omega} \int_{-1}^{0} e^{-i\omega x}\, dx + x \frac{e^{-i\omega x}}{-i\omega} \Big|_{0}^{1} + \frac{1}{i\omega} \int_{0}^{1} e^{-i\omega x}\, dx \right) =$$

$$= \frac{1}{\sqrt{2\pi}} \left( \frac{e^{i\omega}}{i\omega} + \frac{1}{(i\omega)^2} e^{-i\omega x} \Big|_{-1}^{0} - \frac{e^{-i\omega}}{i\omega} - \frac{1}{(i\omega)^2} e^{-i\omega x} \Big|_{0}^{1} \right) =$$

$$= \frac{1}{\sqrt{2\pi}} \left( \frac{e^{i\omega}}{i\omega} - \frac{e^{-i\omega}}{i\omega} + \frac{1}{(i\omega)^2}(1 - e^{i\omega}) - \frac{1}{(i\omega)^2}(e^{-i\omega} - 1) \right) =$$

$$= \frac{1}{\sqrt{2\pi}} \left( 2\frac{\sin(\omega)}{\omega} + \frac{2}{-\omega^2} + \frac{1}{\omega^2}(e^{i\omega} + e^{-i\omega}) \right) =$$

$$= \frac{1}{\sqrt{2\pi}} \left( 2\frac{\sin(\omega)}{\omega} - \frac{2}{\omega^2} + 2\frac{\cos(\omega)}{\omega^2} \right) =$$

$$= \sqrt{\frac{2}{\pi}} \left( \frac{\cos(\omega) + \omega \sin(\omega) - 1}{\omega^2} \right)$$