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Analysis III

Solutions Serie 7

1. Discrete Fourier transform (DFT)

Let $N = 4$ and f be a function whose the following values,

$$f(0) = 2, \quad f\left(\frac{2\pi}{N}\right) = 0, \quad f\left(2\frac{2\pi}{N}\right) = 6, \quad f\left(3\frac{2\pi}{N}\right) = 3.$$

Find the discrete Fourier transform (DFT) of the function f with the numerical values given above. And write down the finite trigonometric representation of the function f with the coefficients that you found.

Steps:

1) Find the value of w_4.
2) Compute the entries of the matrix $\mathbf{M}_4^{-1}$ using the formula

$$\mathbf{M}^{-1} = \frac{1}{N}[w^{-jk}].$$

3) Use the formula

$$\mathbf{C} = \mathbf{M}^{-1}\mathbf{F},$$

where $\mathbf{F} = \begin{bmatrix} 2 & 0 & 6 & 3 \end{bmatrix}^\top$ to find $\mathbf{C} = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 \end{bmatrix}^\top$.

4) Use Euler's formula to pass from the finite complex Fourier series

$$f(t) = c_0 + c_1 e^{it} + c_2 e^{2it} + c_3 e^{3it}$$

to the finite trigonometric representation.

Solution:

The measurements (sample values) are given by $\mathbf{F} = \begin{bmatrix} 2 & 0 & 6 & 3 \end{bmatrix}^\top$.

We have $N = 4$, then

$$w = w_N = e^{2\pi i/N} = e^{\pi i/2} = i.$$

Our goal is to find the vector $\mathbf{C} = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 \end{bmatrix}^\top$ using the formula

$$\mathbf{C} = \mathbf{M}^{-1}\mathbf{F},$$

where $\mathbf{M}^{-1} = \frac{1}{N}[w^{-jk}]$. Since

$$w^{-jk} = i^{-jk},$$

the matrix $\mathbf{M}^{-1}$ is given by

$$\frac{1}{N}\begin{bmatrix} w^0 & w^0 & w^0 & w^0 \\ w^0 & w^{-1} & w^{-2} & w^{-3} \\ w^0 & w^{-2} & w^{-4} & w^{-6} \\ w^0 & w^{-3} & w^{-6} & w^{-9} \end{bmatrix} = \frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}.$$

Therefore by computation we get,

$$\mathbf{C} = \mathbf{M}^{-1}\mathbf{F} = \frac{1}{N}\begin{bmatrix} w^0 & w^0 & w^0 & w^0 \\ w^0 & w^{-1} & w^{-2} & w^{-3} \\ w^0 & w^{-2} & w^{-4} & w^{-6} \\ w^0 & w^{-3} & w^{-6} & w^{-9} \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 6 \\ 3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 6 \\ 3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 11 \\ -4+3i \\ 5 \\ -4-3i \end{bmatrix}.$$

Hence, the finite complex Fourier series is given by

$$f(t) = \frac{11}{4} + \frac{-4+3i}{4}e^{it} + \frac{5}{4}e^{2it} + \frac{-4-3i}{4}e^{3it}.$$

Using Euler's formula $e^{ix} = \cos(x) + i\sin(x)$, we get the finite trigonometric representation

$$\begin{aligned} f(t) = &\frac{1}{4}\left(11 - 4\cos(t) - 3\sin(t) + 5\cos(2t) - 4\cos(3t) + 3\sin(3t)\right) \\ &+ \frac{i}{4}\left(-4\sin(t) + 3\cos(t) + 5\sin(2t) - 4\sin(3t) - 3\cos(3t)\right). \end{aligned}$$

2. Write out the matrix $\mathbf{M}_8$ in terms of w_8, expressing each entry as the lowest possible positive power of w_8. You do not need to write w_8 explicitly.

Do the same for the inverse matrix $\mathbf{M}_8^{-1}$.

Solution:

The matrix $\mathbf{M}_8$ is given by

$$\mathbf{M} = \begin{bmatrix} w^0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 \\ w^0 & w^1 & w^2 & w^3 & w^4 & w^5 & w^6 & w^7 \\ w^0 & w^2 & w^4 & w^6 & w^8 & w^{10} & w^{12} & w^{14} \\ w^0 & w^3 & w^6 & w^9 & w^{12} & w^{15} & w^{18} & w^{21} \\ w^0 & w^4 & w^8 & w^{12} & w^{16} & w^{20} & w^{24} & w^{28} \\ w^0 & w^5 & w^{10} & w^{15} & w^{20} & w^{25} & w^{30} & w^{35} \\ w^0 & w^6 & w^{12} & w^{18} & w^{24} & w^{30} & w^{36} & w^{42} \\ w^0 & w^7 & w^{14} & w^{21} & w^{28} & w^{35} & w^{42} & w^{49} \end{bmatrix}.$$

Recall

$$w = w_8 = e^{2\pi i/8},$$

 Look at the next page!

and $w^8 = (e^{2\pi i/8})^8 = e^{2\pi i} = 1$. Therefore, we have $w^{8+n} = w^n$. For example $w^{49} = w^{8*6+1} = (w^8)^6 w = w$. We can reduce $\mathbf{M}$ as follow

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & w^1 & w^2 & w^3 & w^4 & w^5 & w^6 & w^7 \\ 1 & w^2 & w^4 & w^6 & 1 & w^2 & w^4 & w^6 \\ 1 & w^3 & w^6 & w^1 & w^4 & w^7 & w^2 & w^5 \\ 1 & w^4 & 1 & w^4 & 1 & w^4 & 1 & w^4 \\ 1 & w^5 & w^2 & w^7 & w^4 & w^1 & w^6 & w^3 \\ 1 & w^6 & w^4 & w^2 & 1 & w^6 & w^4 & w^2 \\ 1 & w^7 & w^6 & w^5 & w^4 & w^3 & w^2 & w^1 \end{bmatrix} \tag{1}$$

The next part was not asked for exercise.

Moreover we have

$$\begin{aligned} w^2 &= (e^{2\pi i/8})^2 = e^{\pi i/2} = i, \\ w^3 &= w^2 w = iw, \\ w^4 &= w^2 w^2 = i^2 = -1, \\ w^5 &= w^4 w = (-1)w = -w, \\ w^6 &= w^4 w^2 = (-1)i = -i, \\ w^7 &= w^6 w = -iw. \end{aligned}$$

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & w & i & iw & -1 & -w & -i & -iw \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & iw & -i & w & -1 & -iw & i & w \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -w & i & -iw & -1 & w & -i & iw \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & -iw & -i & -w & -1 & iw & i & w \end{bmatrix}$$

We can further develop this matrix with the following

$$w = e^{\frac{2\pi i}{8}} = \cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

Hence, we finally have

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1+i}{\sqrt{2}} & i & \frac{-1+i}{\sqrt{2}} & -1 & \frac{-1-i}{\sqrt{2}} & -i & \frac{1-i}{\sqrt{2}} \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & \frac{-1+i}{\sqrt{2}} & -i & \frac{1+i}{\sqrt{2}} & -1 & \frac{1-i}{\sqrt{2}} & i & \frac{-1-i}{\sqrt{2}} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \frac{-1-i}{\sqrt{2}} & i & \frac{1-i}{\sqrt{2}} & -1 & \frac{1+i}{\sqrt{2}} & -i & \frac{-1+i}{\sqrt{2}} \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & \frac{1-i}{\sqrt{2}} & -i & \frac{-1-i}{\sqrt{2}} & -1 & \frac{-1+i}{\sqrt{2}} & i & \frac{1+i}{\sqrt{2}} \end{bmatrix}.$$

Please turn!

For $\mathbf{M}_8^{-1}$ we have

$$\mathbf{M}_8^{-1} = \frac{1}{8}\begin{bmatrix} w^0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 & w^0 \\ w^0 & w^{-1} & w^{-2} & w^{-3} & w^{-4} & w^{-5} & w^{-6} & w^{-7} \\ w^0 & w^{-2} & w^{-4} & w^{-6} & w^{-8} & w^{-10} & w^{-12} & w^{-14} \\ w^0 & w^{-3} & w^{-6} & w^{-9} & w^{-12} & w^{-15} & w^{1-8} & w^{-21} \\ w^0 & w^{-4} & w^{-8} & w^{-12} & w^{-16} & w^{-20} & w^{-24} & w^{-28} \\ w^0 & w^{-5} & w^{-10} & w^{-15} & w^{-20} & w^{-25} & w^{-30} & w^{-35} \\ w^0 & w^{-6} & w^{-12} & w^{-18} & w^{-24} & w^{-30} & w^{-36} & w^{-42} \\ w^0 & w^{-7} & w^{-14} & w^{-21} & w^{-28} & w^{-35} & w^{-42} & w^{-49} \end{bmatrix}.$$

Note that

$$w^{-1} = w^7 = -iw,$$

therefore

$$\begin{aligned} w^{-2} &= (e^{-2\pi i/8})^2 = e^{-\pi i/2} = -i, \\ w^{-3} &= w^{-2}w^{-1} = (-i)(-iw) = -w, \\ w^{-4} &= w^{-2}w^{-}2 = (-i)(-i) = -1, \\ w^{-5} &= w^{-4}w^{-1} = (-1)(-iw) = iw, \\ w^{-6} &= w^{-4}w^{-2} = -1(-i) = i, \\ w^{-7} &= w^{-6}w^{-1} = i(-iw) = w. \end{aligned}$$

Hence, we get

$$\mathbf{M}^{-1} = \frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -iw & -i & -w & -1 & iw & i & -w \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & -w & i & -iw & -1 & w & -i & iw \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & iw & -i & w & -1 & -iw & i & -w \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & w & i & iw & -1 & -w & -i & -iw \end{bmatrix}$$

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3. Fast Fourier Transform (FFT)

Compute the Fast Fourier Transform (FFT) of the same function given in exercise 1. Check that you get the same result.

Steps:

1) Find the value of w_M, where $M = \frac{N}{2}$.
2) Compute the even and odd coefficients $\mathbf{C}^{(o)}$ and $\mathbf{C}^{(e)}$ using the formula

$$\mathbf{C}^{(o)} = \begin{bmatrix} c_0^{(o)} \\ c_1^{(o)} \end{bmatrix} = \mathbf{M}_2^{-1}\mathbf{f}^{(o)}, \qquad \text{and} \qquad \mathbf{C}^{(e)} = \begin{bmatrix} c_0^{(e)} \\ c_1^{(e)} \end{bmatrix} = \mathbf{M}_2^{-1}\mathbf{f}^{(e)}$$

Look at the next page!

3) Find the value of w_N.
4) Compute the coefficient c_k using the formulas for $k < M$

$$c_k = \frac{1}{2}\left(c_k^{(\mathrm{o})} + w_N^{-k} c_k^{(\mathrm{e})}\right).$$

And for the coefficient c_k with $k \geq M$,

$$c_{k+M} = \frac{1}{2}\left(c_k^{(\mathrm{o})} - w_N^{-k} c_k^{(\mathrm{e})}\right).$$

Solution:

We have $N = 4$ and $M = N/2 = 2$, hence

$$w = w_M = e^{2\pi i/2} = e^{\pi i} = -1.$$

Let us denote $\mathbf{F} = \begin{bmatrix} f_0 & f_1 & f_2 & f_3 \end{bmatrix}^\top$. Consequently,

$$\mathbf{C}^{(\mathrm{o})} = \begin{bmatrix} c_0^{(\mathrm{o})} \\ c_1^{(\mathrm{o})} \end{bmatrix} = \mathbf{M}_2^{-1}\mathbf{f}^{(\mathrm{o})} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} f_0 \\ f_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} f_0 + f_2 \\ f_0 - f_2 \end{bmatrix}$$

$$\mathbf{C}^{(\mathrm{e})} = \begin{bmatrix} c_0^{(\mathrm{e})} \\ c_1^{(\mathrm{e})} \end{bmatrix} = \mathbf{M}_2^{-1}\mathbf{f}^{(\mathrm{e})} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} f_1 \\ f_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} f_1 + f_3 \\ f_1 - f_3 \end{bmatrix}.$$

For $N = 4$, w_N is given by

$$w_N = e^{2\pi i/4} = e^{\pi i/2} = i.$$

From this, we obtain from the formula

$$c_0 = \frac{1}{2}\left(c_0^{(\mathrm{o})} + w_N^0 c_0^{(\mathrm{e})}\right) = \frac{1}{2}\left(\frac{1}{2}(f_0 + f_2) + \frac{1}{2}(f_1 + f_3)\right) = \frac{1}{4}(f_0 + f_1 + f_2 + f_3)$$

$$c_1 = \frac{1}{2}\left(c_1^{(\mathrm{o})} + w_N^{-1} c_1^{(\mathrm{e})}\right) = \frac{1}{2}\left(\frac{1}{2}(f_0 - f_2) - \frac{1}{2}i(f_1 - f_3)\right) = \frac{1}{4}(f_0 - if_1 - f_2 + if_3)$$

And for the coefficient c_k with $k \geq M$, we get

$$c_2 = \frac{1}{2}\left(c_0^{(\mathrm{o})} - w_N^0 c_0^{(\mathrm{e})}\right) = \frac{1}{2}\left(\frac{1}{2}(f_0 + f_2) - \frac{1}{2}(f_1 + f_3)\right) = \frac{1}{4}(f_0 - f_1 + f_2 - f_3)$$

$$c_3 = \frac{1}{2}\left(c_1^{(\mathrm{o})} - w_N^{-1} c_1^{(\mathrm{e})}\right) = \frac{1}{2}\left(\frac{1}{2}(f_0 - f_2) + \frac{1}{2}i(f_1 - f_3)\right) = \frac{1}{4}(f_0 + if_1 - f_2 - if_3).$$

Using the numerical values given by $\mathbf{F} = \begin{bmatrix} f_0 & f_1 & f_2 & f_3 \end{bmatrix}^\top = \begin{bmatrix} 2 & 0 & 6 & 3 \end{bmatrix}^\top$, we have

$$c_0 = \frac{11}{4}, \quad c_1 = \frac{-4+3i}{4}, \quad c_2 = \frac{5}{4}, \quad c_3 = \frac{-4-3i}{4}.$$

Please turn!

$$f(t) = \frac{11}{4} + \frac{-4+3i}{4}e^{it} + \frac{5}{4}e^{2it} + \frac{-4-3i}{4}e^{3it}.$$

Using Euler's formula $e^{ix} = \cos(x) + i\sin(x)$, we get the finite trigonometric representation

$$\begin{aligned} f(t) = &\frac{1}{4}\left(11 - 4\cos(t) - 3\sin(t) + 5\cos(2t) - 4\cos(3t) + 3\sin(3t)\right) \\ &+ \frac{i}{4}\left(-4\sin(t) + 3\cos(t) + 5\sin(2t) - 4\sin(3t) - 3\cos(3t)\right). \end{aligned}$$

4. Let $\mathbf{C} = \begin{bmatrix}1 & 1 & 0 & 1\end{bmatrix}^\top$.

- Find $\mathbf{F}$ using $\mathbf{F} = \mathbf{M}_4\mathbf{C}$.
- Find $\mathbf{F}$ using the fast Fourier transform.

Solution:

- (DFT)
  We have $N = 4$, then
  $$w = w_N = e^{2\pi i/N} = e^{\pi i/2} = i.$$
  Our goal is to find the vector $\mathbf{F} = \begin{bmatrix}f_0 & f_1 & f_2 & f_3\end{bmatrix}^\top$ using the formula
  $$\mathbf{F} = \mathbf{MC},$$
  The matrix $\mathbf{M}_4$ is given by
  $$\mathbf{M}_4 = \begin{bmatrix} w^0 & w^0 & w^0 & w^0 \\ w^0 & w^1 & w^2 & w^3 \\ w^0 & w^2 & w^4 & w^6 \\ w^0 & w^3 & w^6 & w^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}.$$
  Therefore by computation we get,
  $$\mathbf{F} = \mathbf{MC} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} \cdot \begin{bmatrix}1\\1\\0\\1\end{bmatrix} = \begin{bmatrix}3\\1\\-1\\1\end{bmatrix}.$$
- (FFT)
  We have $N = 4$ and $M = N/2 = 2$, hence
  $$w = w_M = e^{2\pi i/2} = e^{\pi i} = -1.$$
  We have $\mathbf{C} = \begin{bmatrix}1 & 1 & 0 & 1\end{bmatrix}^\top$ Consequently,
  $$\mathbf{F}^{(o)} = \begin{bmatrix} f_0^{(o)} \\ f_1^{(o)} \end{bmatrix} = \mathbf{M}_2\mathbf{C}^{(o)} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_0 + c_2 \\ c_0 - c_2 \end{bmatrix}$$

$$\mathbf{F}^{(e)} = \begin{bmatrix} f_0^{(e)} \\ f_1^{(e)} \end{bmatrix} = \mathbf{M}_2\mathbf{C}^{(e)} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_1 + c_3 \\ c_1 - c_3 \end{bmatrix}.$$

For $N = 4$, w_N is given by

$$w_N = e^{2\pi i/4} = e^{\pi i/2} = i.$$

From this, we obtain from the formula (3.29) in the Lecture Notes

$$f_0 = f_0^{(o)} + w_N^0 f_0^{(e)} = (c_0 + c_1 + c_2 + c_3),$$

$$f_1 = f_1^{(o)} + w_N f_1^{(e)} = (c_0 - c_2) + i(c_1 - c_3) = c_0 + ic_1 - c_2 - ic_3.$$

And for the coefficient c_k with $k \geq M$, we get

$$f_2 = f_0^{(o)} - w_N^0 f_0^{(e)} = (c_0 + c_2) - (c_1 + c_3) = c_0 - c_1 - c_2 - c_3,$$

$$f_3 = f_1^{(o)} - w_N f_1^{(e)} = (c_0 - c_2) + i(c_1 - c_3) = c_0 + ic_1 - c_2 - ic_3.$$

Using the numerical values given by $\mathbf{C} = \begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}^\top$, we have

$$f_0 = 3, \quad f_1 = 1, \quad f_2 = -1, \quad f_3 = 1.$$

We have the same result as before.

5. Fast Fourier Transform (FFT)

Let $N = 4$ and f be a function whose the following values,

$$f(0) = 0, \quad f\left(\frac{2\pi}{N}\right) = 1, \quad f\left(2\frac{2\pi}{N}\right) = 2, \quad f\left(3\frac{2\pi}{N}\right) = 3.$$

Compute the Fast Fourier Transform (FFT) of the function f with the numerical values given above. And write down the finite trigonometric representation of the function f with the coefficients that you found.

Solution:

We have $N = 4$ and $M = N/2 = 2$, hence

$$w = w_M = e^{2\pi i/2} = e^{\pi i} = -1.$$

Let us denote $\mathbf{F} = \begin{bmatrix} f_0 & f_1 & f_2 & f_3 \end{bmatrix}^\top$. Consequently,

$$\mathbf{C}^{(o)} = \begin{bmatrix} c_0^{(o)} \\ c_1^{(o)} \end{bmatrix} = \mathbf{M}_2^{-1}\mathbf{f}^{(o)} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} f_0 \\ f_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} f_0 + f_2 \\ f_0 - f_2 \end{bmatrix}$$

$$\mathbf{C}^{(\mathrm{e})} = \begin{bmatrix} c_0^{(\mathrm{e})} \\ c_1^{(\mathrm{e})} \end{bmatrix} = \mathbf{M}_2^{-1}\mathbf{f}^{(\mathrm{e})} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} f_1 \\ f_3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} f_1 + f_3 \\ f_1 - f_3 \end{bmatrix}.$$

For $N = 4$, w_N is given by

$$w_N = e^{2\pi i/4} = e^{\pi i/2} = i.$$

From this, we obtain from the formula in exercise 3,

$$c_0 = \frac{1}{2}\left(c_0^{(\mathrm{o})} + w_N^0 c_0^{(\mathrm{e})}\right) = \frac{1}{2}\left(\frac{1}{2}(f_0 + f_2) + \frac{1}{2}(f_1 + f_3)\right) = \frac{1}{4}(f_0 + f_1 + f_2 + f_3)$$

$$c_1 = \frac{1}{2}\left(c_1^{(\mathrm{o})} + w_N^{-1} c_1^{(\mathrm{e})}\right) = \frac{1}{2}\left(\frac{1}{2}(f_0 - f_2) - \frac{1}{2}i(f_1 - f_3)\right) = \frac{1}{4}(f_0 - if_1 - f_2 + if_3)$$

And for the coefficient c_k with $k \geq M$, we get

$$c_2 = \frac{1}{2}\left(c_0^{(\mathrm{o})} - w_N^0 c_0^{(\mathrm{e})}\right) = \frac{1}{2}\left(\frac{1}{2}(f_0 + f_2) - \frac{1}{2}(f_1 + f_3)\right) = \frac{1}{4}(f_0 - f_1 + f_2 - f_3)$$

$$c_3 = \frac{1}{2}\left(c_1^{(\mathrm{o})} - w_N^{-1} c_1^{(\mathrm{e})}\right) = \frac{1}{2}\left(\frac{1}{2}(f_0 - f_2) + \frac{1}{2}i(f_1 - f_3)\right) = \frac{1}{4}(f_0 + if_1 - f_2 - if_3).$$

Using the numerical values given by $\mathbf{F} = \begin{bmatrix} f_0 & f_1 & f_2 & f_3 \end{bmatrix}^\top = \begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix}^\top$, we have

$$c_0 = \frac{6}{4} = \frac{3}{2}, \quad c_1 = \frac{-2 + 2i}{4} = \frac{-1 + i}{2}, \quad c_2 = \frac{-2}{4} = -\frac{1}{2}, \quad c_3 = \frac{-2 - 2i}{4} = \frac{-1 - i}{2}.$$

Hence, the finite complex Fourier series is given by

$$f(t) = \frac{3}{2} + \frac{-1 + i}{2}e^{it} - \frac{1}{2}e^{2it} + \frac{-1 - i}{2}e^{3it}.$$

Using Euler's formula $e^{ix} = \cos(x) + i\sin(x)$, we get the finite trigonometric representation

$$\begin{aligned} f(t) = &\left(\frac{3}{2} - \frac{1}{2}\cos(t) - \frac{1}{2}\sin(t) - \frac{1}{2}\cos(2t) - \frac{1}{2}\cos(3t) + \frac{1}{2}\sin(3t)\right) \\ &+ i\left(-\frac{1}{2}\sin(t) + \frac{1}{2}\cos(t) - \frac{1}{2}\sin(2t) - \frac{1}{2}\sin(3t) - \frac{1}{2}\cos(3t)\right). \end{aligned}$$