D-MAVT
D-MATL

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Analysis III

Solutions Serie 9

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1. For $k \in \mathbb{R}$, find the Fourier series solution $u = u(x,t)$ of the 1-dimensional wave equation on the interval $[0,1]$ with the following boundary and initial conditions:

$$\begin{cases} u_{tt} = u_{xx}, & \\ u(0,t) = 0 = u(1,t), & t \geq 0 \\ u(x,0) = kx(1-x^2), & 0 \leq x \leq 1 \\ u_t(x,0) = 0, & 0 \leq x \leq 1 \end{cases}$$

Use the method of separation of variables from scratch and describe each step of it.

Solution:

With variables separated $u(x,t) = F(x)G(t)$ the differential equation becomes:

$$F(x)\ddot{G}(t) = F''(x)G(t),$$

which is convenient to rewrite as

$$\frac{F''(x)}{F(x)} = \frac{\ddot{G}(t)}{G(t)}$$

because it becomes clear that we are comparing a function of x with a function of t, and the only way that this equality might be true is that both these functions are equal and constant:

$$\frac{F''(x)}{F(x)} = \frac{\ddot{G}(t)}{G(t)} = k, \qquad k \in \mathbb{R}.$$

The boundary conditions are

$$u(0,t) = F(0)G(t) = 0 \quad \text{and} \quad u(1,t) = F(1)G(t) = 0 \quad \forall t \in [0, +\infty)$$

which in order to be true, excluding the trivial solution $G(t) \equiv 0$, become:

$$F(0) = F(1) = 0.$$

In other words the initial PDE with boundary conditions becomes the system of coupled equations

$$\begin{cases} F''(x) = kF(x), \\ F(0) = F(1) = 0, \end{cases} \qquad \text{and} \quad \ddot{G}(t) = kG(t).$$

We first solve the system for $F(x)$, distinguishing the cases of k positive, zero, or negative. For $k > 0$ the general solution of the ODE is

$$F(x) = C_1 \mathrm{e}^{\sqrt{k}x} + C_2 \mathrm{e}^{-\sqrt{k}x},$$

which is, however, <u>not</u> compatible with the boundary conditions, in the sense that the only solution of this form satisfying the boundary conditions is the trivial solution: $C_1 = C_2 = 0$. In fact

$$0 = F(0) = C_1 + C_2 \quad \Leftrightarrow \quad C_2 = -C_1 \quad \implies F(x) = C_1 \left(\mathrm{e}^{\sqrt{k}x} - \mathrm{e}^{-\sqrt{k}x}\right)$$

but then imposing the other condition:

$$0 = F(1) = C_1 \left(\mathrm{e}^{\sqrt{k}} - \mathrm{e}^{-\sqrt{k}}\right) \quad \Leftrightarrow \quad \begin{array}{l} \text{either } C_1 = 0 \\ \text{or } \mathrm{e}^{2\sqrt{k}} = 1 \end{array}$$

which implies $C_1 = 0$ (and consequently $C_2 = -C_1 = 0$) because $2\sqrt{k} \neq 0$ and therefore its exponential is not 1.

For $k = 0$ the general solution is $F(x) = C_1 x + C_2$ which is also not compatible with boundary conditions unless $C_1 = C_2 = 0$. In fact

$$0 = F(0) = C_2 \quad \implies F(x) = C_1 x$$

and then

$$0 = F(1) = C_1.$$

It remains the case $k < 0$, in which its convenient to write it in the form $k = -p^2$ for positive real number p, and general solutions of $F'' = -p^2 F$ are:

$$F(x) = A\cos(px) + B\sin(px).$$

$F(0) = 0$ if and only if $A = 0$. $F(1) = 0$ if and only if $B\sin(p) = 0$, so if we want nontrivial solutions $B \neq 0$, we need to have

$$p = n\pi$$

for some integer $n \geq 1$. Conclusion: we have a nontrivial solution for each $n \geq 1$, $k = k_n = -n^2\pi^2$:

$$F_n(x) = B_n \sin(n\pi x)$$

The corresponding equation for $G(t)$ is

$$\ddot{G} = -n^2\pi^2 G$$

which has general solution

$$G_n(t) = C_n \cos(n\pi t) + D_n \sin(n\pi t).$$

The conclusion is that for every $n \geq 1$ we have a solution

$$u_n(x,t) = F_n(x)G_n(t) = \big(B_n \cos(n\pi t) + B_n^* \sin(n\pi t)\big)\sin(n\pi x), \qquad B_n, B_n^* \in \mathbb{R}.$$

and by the superposition principle general solution:

$$u(x,t) = \sum_{n=1}^{+\infty} u_n(x,t) = \sum_{n=1}^{+\infty} \big(B_n \cos(n\pi t) + B_n^* \sin(n\pi t)\big) \sin(n\pi x).$$

The coefficients are found by imposing the initial conditions. Firstly the initial position must be

$$u(x,0) = \sum_{n=1}^{+\infty} B_n \sin(n\pi x) = kx(1-x^2).$$

For this equality to hold the coefficients B_n must be the coefficients of the Fourier series of the odd, $2L(=2$ in this case)-periodic extension of the function $kx(1-x^2)$ from the interval $[0,1]$. That is:

$$\begin{aligned}
B_n &= 2\int_0^1 kx(1-x^2)\sin(n\pi x)\,dx = 2k\int_0^1 (x-x^3)\sin(n\pi x)\,dx = \\
&= 2k\left(-(x-x^3)\frac{\cos(n\pi x)}{n\pi}\bigg|_0^1 + \frac{1}{n\pi}\int_0^1 (1-3x^2)\cos(n\pi x)\,dx\right) = \\
&= \frac{2k}{n\pi}\left((1-3x^2)\frac{\sin(n\pi x)}{n\pi}\bigg|_0^1 + \frac{1}{n\pi}\int_0^1 6x\sin(n\pi x)\,dx\right) = \\
&\quad = \frac{12k}{n^2\pi^2}\left(-x\frac{\cos(n\pi x)}{n\pi}\bigg|_0^1 + \frac{1}{n\pi}\int_0^1 \cos(n\pi x)\,dx\right) = \\
&\quad = -\frac{12k}{n^3\pi^3}(-1)^n - \frac{12k}{n^4\pi^4}\sin(n\pi x)\bigg|_0^1 = \\
&\quad = \frac{12k}{n^3\pi^3}(-1)^{n+1}.
\end{aligned}$$

The initial speed instead gives trivially

$$u_t(x,0) = \sum_{n=1}^{+\infty} n\pi B_n^* \sin(n\pi x) = 0 \quad \Leftrightarrow \quad B_n^* = 0.$$

Finally, the solution is

$$\boxed{u(x,t) = \frac{12k}{\pi^3}\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^3}\cos(n\pi t)\sin(n\pi x).}$$

2. Find the solution $u = u(x,t)$ of the 1-dimensional wave equation on the interval $[0, L]$ with the following boundary and initial conditions:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & \\ u(0,t) = 0 = u(L,t), & t \geq 0 \\ u(x,0) = 0, & 0 \leq x \leq L \\ u_t(x,0) = \sin\left(\frac{\pi}{L}x\right), & 0 \leq x \leq L \end{cases}$$

Find the solution via Fourier series, you don't need to detail the steps. So use directly the formula given in §4.3 of the Lecture Notes.

Solution:

The formula for the solution via Fourier series in this case in which the initial function $f = 0$ looks like:

$$u(x,t) = \sum_{n=1}^{+\infty} B_n^* \sin\left(\frac{cn\pi}{L}t\right) \sin\left(\frac{n\pi}{L}x\right).$$

To find the coefficients B_n^* we impose the initial condition:

$$u_t(x,0) = \sum_{n=1}^{+\infty} B_n^* \frac{cn\pi}{L} \sin\left(\frac{n\pi}{L}x\right) = \sin\left(\frac{\pi}{L}x\right) \qquad \Longrightarrow \qquad \begin{cases} B_1^* = \frac{L}{c\pi}, \\ B_{n\geq 2}^* = 0. \end{cases}$$

$$\Longrightarrow \qquad \boxed{u(x,t) = \frac{L}{c\pi} \sin\left(\frac{c\pi}{L}t\right) \sin\left(\frac{\pi}{L}x\right)}$$

3. Find all possible solutions of the following PDEs of the form $u(x,t) = F(x)G(t)$ (separation of variables):

a) $xu_x + u_t = 0$

Solution:

As customary, we will denote by F' the derivative of F in the variable x and by $\dot{G}$ the derivative of G in the variable t. Plugging in the equation a function of the form $u(x,t) = F(x)G(t)$ we get

$$xF'(x)G(t) + F(x)\dot{G}(t) = 0 \qquad \Leftrightarrow \qquad x\frac{F'(x)}{F(x)} = -\frac{\dot{G}(t)}{G(t)}$$

On the left-hand side we have a function of x while on the other side we have a function of t. The equality is possible only if these expressions are constant, so there must be a $k \in \mathbb{R}$ such that

$$x\frac{F'(x)}{F(x)} = -\frac{\dot{G}(t)}{G(t)} = k \qquad \Leftrightarrow \qquad \begin{cases} F'(x) - \frac{k}{x}F(x) = 0 \\ \dot{G}(t) + kG(t) = 0 \end{cases}$$

This is a system of two homogeneous, first order, ODEs, one with non-constant and the other with constant coefficients. The solutions are

$$\begin{cases} F(x) = c_1 e^{\int \frac{k}{x}} = c_1 e^{k \int \frac{1}{x}} = c_1 \left(e^{\ln(x)}\right)^k = c_1 x^k \\ G(t) = c_2 e^{-kt} \end{cases}$$

so that, calling $c := c_1 c_2$ the product of the constants, we have solutions of the form

$$u(x,t) = F(x)G(t) = cx^k e^{-kt}, \qquad k \in \mathbb{R}.$$

b) $u_x + u_t + xu = 0$

Solution:

Here we get

$$u_x + u_t + xu = F'G + F\dot{G} + xFG,$$

which we are going to impose equal to zero. We want to keep track of the variables involved in order to have clear which function depends on which variable, to then separate the equations properly:

$$\begin{aligned} & F'(x)G(t) + F(x)\dot{G}(t) + xF(x)G(t) = 0 \quad \Leftrightarrow \\ \Leftrightarrow \quad & (F'(x) + xF(x))G(t) = -F(x)\dot{G}(t) \quad \Leftrightarrow \quad \frac{F'(x)}{F(x)} + x = -\frac{\dot{G}(t)}{G(t)} \end{aligned}$$

which, again, is an equality between some function of x and some other function of t, so there must be some constant $k \in \mathbb{R}$ for which

$$\begin{cases} \frac{F'}{F} + x = k \\ -\frac{\dot{G}}{G} = k \end{cases} \quad \Leftrightarrow \quad \begin{cases} F' + (x-k)F = 0 \\ \dot{G} + kG = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} F(x) = c_1 e^{-\int (x-k)} = c_1 e^{-\frac{x^2}{2} + kx} \\ G(t) = c_2 e^{-kt} \end{cases}$$

$$\rightsquigarrow \qquad u(x,t) = F(x)G(t) = ce^{-\frac{x^2}{2}+kx} e^{-kt} = ce^{-\frac{x^2}{2}+k(x-t)}.$$

c) $t^3 u_x + \cos(x)u - 2u_{xt} = 0$

Solution:

The equation with separated variables becomes

$$t^3 F'(x)G(t) + \cos(x)F(x)G(t) - 2F'(x)\dot{G}(t) = 0.$$

We want to get some equation in which there are just two terms, each of which is a product of a function of x and t. So we need to gather the first term with the third term:

$$F'(x)\left(t^3 G(t) - 2\dot{G}(t)\right) = -\cos(x)F(x)G(t) \quad \Leftrightarrow \quad \frac{\cos(x)F(x)}{F'(x)} = \frac{2\dot{G}(t) - t^3 G(t)}{G(t)}$$

Again we need to impose both terms to be constantly equal to some $k \in \mathbb{R}$. In the case $k \neq 0$ we can divide by it and getting the following system of ODEs:

$$\begin{cases} F' - \frac{\cos(x)}{k} F = 0 \\ \dot{G} - \frac{(t^3+k)}{2} G = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} F(x) = c_1 e^{\int \frac{\cos(x)}{k}} = c_1 e^{\frac{1}{k}\int \cos(x)} = c_1 e^{\frac{\sin(x)}{k}} \\ G(t) = c_2 e^{\int \frac{(t^3+k)}{2}} = c_2 e^{\frac{t^4}{8} + \frac{kt}{2}} = c_2 e^{\frac{t}{8}(t^3+4k)} \end{cases}$$

$$\rightsquigarrow \quad \text{family of solutions:} \qquad u(x,t) = c e^{\frac{\sin(x)}{k}} e^{\frac{t}{8}(t^3+4k)}, \qquad k \in \mathbb{R} \setminus \{0\}.$$

As one can easily observe instead the case $k = 0$ forces $F = 0$ and so also $u = 0$, which is the trivial solution and it's anyway already been considered above if the constant $c = 0$.

Remark (unimportant for the purpose of the exercises): what we did in each exercise is finding admissible values $k \in \mathbb{R}$ for which there exists some solution of the form[1]

$$u_k(x,t) = c(k) F_k(x) G_k(t)$$

Because each of these PDEs was homogeneous, by the so-called superposition principle, then also the sum of any of these solutions is a solution.
More generally any expression of the form

$$u(x,t) = \int_{-\infty}^{+\infty} c(k) F_k(x) G_k(t)\, dk$$

will be a solution, *providing* that there are some convergence conditions (i.e. the integral converges, and it does it in such a way that this expression will be differentiable, etc. etc.).
This is in what fully consists the method of separation of variables: find the values $k \in \mathbb{R}$ for which exists a specific solution with separated variables (and this is usually easy just because we have separated the variables); then 'sum' in some sense over all admissible values of k (take a series, if it's a discrete set, take the integral, if it's continuous) these solutions to get a more general solution.
This is, more or less, how to get any possible solution of the simplest PDEs.

4. Wave Equation with inhomogeneous boundary conditions

Find the solution of the following wave equation (**with inhomogeneous boundary conditions**) on the interval $[0, \pi]$:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & t \geq 0,\ x \in [0,\pi] \\ u(0,t) = 3\pi^2, & t \geq 0 \\ u(\pi,t) = 7\pi, & t \geq 0 \\ u(x,0) = 2\sin(5x) + \sin(4x) + (7 - 3\pi)x + 3\pi^2, & x \in [0,\pi] \\ u_t(x,0) = 0. & x \in [0,\pi] \end{cases} \tag{1}$$

You must proceed as follows.

[1] in what follows the subscript is there to indicate a dependence on k, it's not a derivative!

a) Find the unique function $w = w(x)$ with $w''(x) = 0$, $w(0) = 3\pi^2$, and $w(\pi) = 7\pi$.

Solution:

The only functions with second derivative zero are the linear functions

$$w(x) = \alpha x + \beta, \quad \alpha, \beta \in \mathbb{R}.$$

Imposing the boundary conditions we find the right coefficients

$$\begin{cases} 3\pi^2 = w(0) = \alpha \cdot 0 + \beta \\ 7\pi = w(\pi) = \alpha \cdot \pi + \beta \end{cases} \Leftrightarrow \begin{cases} \alpha = \frac{7\pi - 3\pi^2}{\pi} \\ \beta = 3\pi^3 \end{cases} \Leftrightarrow \boxed{w(x) = (7 - 3\pi)x + 3\pi^2.}$$

b) Define $v(x,t) := u(x,t) - w(x)$. Formulate the corresponding problem for v, equivalent to (1).

Solution:

The PDE doesn't change because w is independent of time and has second derivative in x zero. The boundary conditions become homogeneous (that's why we chose this w)

$$\begin{aligned} v(0,t) &= u(0,t) - w(0) = 3\pi^2 - 3\pi^2 = 0 \\ v(\pi,t) &= u(\pi,t) - w(\pi) = 7\pi - (7 - 3\pi)\pi - 3\pi^2 = 0. \end{aligned}$$

The initial position of the wave changes in

$$\begin{aligned} v(x,0) = u(x,0) - w(x) &= 2\sin(5x) + \sin(4x) + (7 - 3\pi)x + 3\pi^2 - (7 - 3\pi)x - 3\pi^2 \\ &= 2\sin(5x) + \sin(4x), \end{aligned}$$

while the initial speed doesn't change (because, again, w is independent of time). Finally

$$\boxed{\begin{cases} v_{tt} = c^2 v_{xx}, & t \geq 0,\ x \in [0,\pi] \\ v(0,t) = v(\pi,t) = 0, & t \geq 0 \\ v(x,0) = 2\sin(5x) + \sin(4x), & x \in [0,\pi] \\ v_t(x,0) = 0. & x \in [0,\pi] \end{cases}}$$

c) (i) Find, using the formula from the script, the solution $v(x,t)$ of the problem you have just formulated.

Solution:

This is a standard homogeneous wave equation with homogeneous boundary conditions. The formula from the script is

$$\begin{aligned} v(x,t) &= \sum_{n=1}^{+\infty} \big(B_n \cos(\lambda_n t) + B_n^* \sin(\lambda_n t)\big) \sin\left(\frac{n\pi}{L} x\right), \quad \lambda_n = \frac{cn\pi}{L} \\ &\overset{(L=\pi)}{=} \sum_{n=1}^{+\infty} \Big(B_n \cos(cnt) + B_n^* \sin(cnt)\Big) \sin(nx). \end{aligned}$$

The coefficients $B_n^* = 0$, because the initial speed is zero, while the coefficients B_n are the Fourier coefficients of the odd, 2π-periodic extension of the initial position datum $v(x,0) = 2\sin(5x) + \sin(4x)$, that is:

$$\sum_{n=1}^{+\infty} B_n \sin(nx) = 2\sin(5x) + \sin(4x).$$

By identifying the term we have, $B_4 = 1$, $B_5 = 2$ and $B_n = 0$ otherwise. Finally we get the following solution

$$\boxed{v(x,t) = \cos(4ct)\sin(4x) + 2\cos(5ct)\sin(5x).}$$

(ii) Write down explicitly the solution $u(x,t)$ of the original problem (1).

Solution:

The solution $u(x,t)$ of the inhomogeneous problem is

$$\boxed{u(x,t) = \cos(4ct)\sin(4x) + 2\cos(5ct)\sin(5x) + (7-3\pi)x + 3\pi^2.}$$