Embedding trees in dense graphs

Václav Rozhoň

February 14, 2019

joint work with T. Klimošová and D. Piguet
Definition (Extremal graph theory, Bollobás 1976)

Extremal graph theory, in its strictest sense, is a branch of graph theory developed and loved by Hungarians.
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Alternative definition: substructures in graphs
Theorem (Mantel 1907)

Graph $G$ has $n$ vertices. If $G$ has more than $n^2/4$ edges then it contains a triangle.

Generalisations?
Extremal graph theory

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Generalisations?
Density of edges vs. density of triangles (Razborov 2008)

(image from the book of Lovász)
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Other cliques (Turán 1941), asymptotically for all non-bipartite graphs (Erdős-Stone 1946)
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The answer for $C_4$ is of order $n^{3/2}$, lower bound via finite projective planes.
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The answer for $C_4$ is of order $n^{3/2}$, lower bound via finite projective planes.
What is the answer for trees?
Fix any tree $T$ on $k$ vertices. There are graphs with average degree $k - 2$ that do not contain $T$. 
Erdős-Sós conjecture

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**Conjecture (Erdős-Sós)**
Any graph with average degree greater than $k - 2$ contains any tree on $k$ vertices as a subgraph.
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Partial results:

- special trees (paths – Erdős, Gallai 1959)
- special graphs (without $C_4$ – Saclé, Wozniak 1997)
- $n$ and $k$ differ by constant (Görlich, Žak 2016)
Erdős-Sós conjecture

Theorem (announced by Ajtai, Komlós, Simonovits, Szemerédi)
The conjecture holds for $k \geq k_0$. 

One can get reasonably close if the size of the tree is comparable with the size of the graph. Below $\Delta$ is maximum degree and $d_{eg}$ average degree.

Theorem (R. 2019), also (Besomi, Pavez-Signé, Stein 2019+)

Let $T$ be a class of trees such that $\forall T \in T$: $\Delta(T) \in o(|T|)$.

Then any graph $G$ with $d_{eg}(G) = |T| + o(|G|)$ contains any $T \in T$. 

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Let $\mathcal{T}$ be a class of trees such that $\forall T \in \mathcal{T} : \Delta(T) \in o(|T|)$. Then any graph $G$ with $\overline{\text{deg}}(G) = |T| + o(|G|)$ contains any $T \in \mathcal{T}$.
Conjecture (Loebl, Komlós, Sós 1995)

If at least $n/2$ vertices of $G$ have degree at least $k$, then $G$ contains any tree with $k + 1$ vertices as a subgraph.
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Conjecture (Simonovits)
If at least $rn$ vertices of $G$ have degree at least $k$, then $G$ contains any tree with $k + 1$ vertices and at most $r(k + 1)$ vertices in one colour class as a subgraph.
Loebl-Komlós-Sós

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**Theorem (Klimošová, Piguet, R. 2019+)***

If at least $rn$ vertices of $G$ have degree at least $k + o(n)$, then $G$ contains any tree with $k + 1$ vertices and at most $r(k + 1)$ vertices in one colour class as a subgraph.
Conjecture (Loebl, Komlós, Sós, Simonovits)

If at least \( rn \) vertices of \( G \) have degree at least \( k \), then \( G \) contains any tree with \( k + 1 \) vertices and at most \( r(k + 1) \) vertices in one colour class as a subgraph.

Theorem (Klimošová, Piguet, R. 2019+)

If at least \( rn \) vertices of \( G \) have degree at least \( k + o(n) \), then \( G \) contains any tree with \( k + 1 \) vertices and at most \( r(k + 1) \) vertices in one colour class as a subgraph.

Turns out that one can get this \( r \) trade-off also in the proof of previous Erdős-Sós result.
General technique

- Expansion of the host graph can compensate for the lack of degree.
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General technique

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- \((\text{Pseudo})\text{random graphs have good expansion.}\)
- Szemerédi regularity lemma: dense graph = cluster graph + pseudorandomness (but we have to pay \(\varepsilon\) fraction of edges).
- This enables us to embed any small subtree \((\varepsilon' n \text{ for very small } \varepsilon')\).
- Aim is to find suitable decomposition of the tree such that small subtrees are embedded by Szemerédi and the macro structure by us.
- We reduced the problem to a certain fractional variant of itself. But now we have much simpler tree structure to work with.
Proof.

Condition on the maximum degree actually gives even simpler decomposition. After decomposition of $G$ and $T$ look at a high degree cluster of $G$ and a maximal matching in its neighbourhood. Provide (almost) greedy algorithm for embedding.
Proof of the Loebl-Komlós-Sós result

Proof.

After decomposition of $G$ and $T$ ’discharge’ into several configurations, embedding for each one being straightforward.