The Alon-Boppana Bound

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In spectral graph theory, the second largest eigenvalue of the graph Laplacian plays a significant role in many results due to its close connection with many geometric properties of a graph such as connectedness. There are thus many well-known bounds on this value such as Cheeger's inequality. The Alon-Boppana bound, proven by Alon and Boppana (see [Nil91]), is another important bound used in the definition of Ramanjuan graphs.

1 Preliminaries

Definition 1.1 (Graph, Extremities)**.** A *graph* $G = (V, E, \mathrm{ep})$ is a triple consisting of a set of vertices V, a set of edges E and a function

$$\begin{aligned} \mathrm{ep} :& E \to V \\ & \alpha \mapsto \{\{v_1, v_2\} \in V : v_1 \text{ and } v_2 \text{ are connected by } \alpha\} \end{aligned}$$

The set $\mathrm{ep}(\alpha)$ are called the *extremities* of the edge α

Definition 1.2 (Adjacency matrix)**.** The *adjacency matrix* $M_G \in \mathbb{R}^{n \times n}$ is

$$M_G(i,j) = \begin{cases} 1 & \exists \alpha \in E \text{ with } \{i,j\} \in \mathrm{ep}(\alpha) \\ 0 & \text{otherwise} \end{cases}$$

where the rows and columns are indexed by the vertices.

Definition 1.3 (Degree matrix)**.** The *degree matrix* D_G of a graph is a diagonal matrix with the degree of each vertex along the diagonal:

$$D_G(i,j) = \begin{cases} \deg(i) & i = j \\ 0 & \text{otherwise} \end{cases}$$

Definition 1.4 (Graph Laplacian)**.** The Laplacian of a graph G is

$$\mathcal{L} = D_G - M_G$$

Definition 1.5 (Ball)**.** Let $G = (V, E, \mathrm{ep})$ be a graph and $v \in V$ be arbitrary, then for any $n \in \mathbb{N}$ the ball of radius n around v is

$$B_v(n) = \{w \in V : d(v,w) \leq n\}$$

Remark 1. Let $\alpha \in E$ be arbitrary then define the ball of radius n around the *edge* α as

$$B_\alpha(n) = \bigcup_{v \in \mathrm{ep}(\alpha)} B_v(n)$$

In other words the ball of radius n around an edge α is the union of the balls of radius n around the vertices at the *extremities ep*(α of α

Remark 2. Trivially $B_v(i) \subset B_v(j) \ \ \forall i \leq j$

Lemma 1.1. *Let $G = (V, E, ep)$ be a graph and let $d = \max_{v \in V}\{\deg(v)\}$ denote the maximum degree of G, then $|B_v(n+1)| \leq (d-1)|B_v(n)|$*

Proof. Any element in $B_v(n+1)$ is either already in $B_v(n)$ or is connected to an element of $B_v(n)$. Since d is the maximum degree of the graph, a vertex $v \in B_v(n)$ has at most d edges. Now at least one of these edges goes to another vertex $w \in B_v(n)$, leaving a maximum of $d-1$ edges connecting w to $B_v(n+1)$, yielding the bound. ■

2 Alon-Boppana Bound

Theorem 2.1 (Alon-Boppana)**.** *Let $G = (V, E, ep)$ be a graph with edges $\alpha_1, \alpha_2 \in E$ such that $d(\alpha_1, \alpha_2) \geq 2k+2$.*
Further let $d = \max_{v \in V}\{\deg(v)\}$ denote the maximum degree of G. Let $\mathcal{L}$ denote the graph Laplacian of G and λ the second smallest eigenvalue of $\mathcal{L}$
Then

$$\lambda \leq d - 2\sqrt{d-1} + \frac{2\sqrt{d-1} - 1}{k} \tag{2.1}$$

Remark 3. By the distance between two edges we mean the minimum of the distances between the extremities of the edges or formally

$$d(\alpha_1, \alpha_2) = \min\{d(v, w) : v \in \mathrm{ep}(\alpha_1), w \in \mathrm{ep}(\alpha_2)\}$$

In particular this means $B_{\alpha_1}(k+1) \cap B_{\alpha_2}(k+1) = \emptyset$ since the two edges are $2k+2$ apart.

Remark 4. Note that the bound on the lower bound on the distance between two edges can equivalently be expressed as a lower bound on the diameter of the graph.

Proof. Define

$$W_i = B_i(\alpha_1) \setminus \bigcup_{k=0}^{i-1} B_k(\alpha_1) \ \ \forall 1 \leq i \leq k \qquad \text{vertices exactly } i \text{ away from } \alpha_1 \tag{2.2}$$

$$U_i = B_i(\alpha_2) \setminus \bigcup_{k=0}^{i-1} B_k(\alpha_2) \ \ \forall 1 \leq i \leq k \qquad \text{analog} \tag{2.3}$$

Now let $a \in \mathbb{R}_{>0}, b \in \mathbb{R}_{<0}$ and define

$$\psi : V \to \mathbb{R}$$
$$v \mapsto \begin{cases} a(d-1)^{\frac{-i}{2}} & v \in W_i \ , 1 \leq i \leq k \\ b(d-1)^{\frac{-i}{2}} & v \in U_i \ , 1 \leq i \leq k \\ 0 & \text{otherwise} \end{cases}$$

We can choose a, b such that $\langle 1, \psi \rangle = \sum_{v \in V} \psi(v) = 0$ allowing us to apply the Courant-Fischer theorem:

$$\lambda = \min_{\substack{\varphi \in \mathbb{R}^{|V|} \\ \varphi \neq 0}} \frac{\langle \mathcal{L}\varphi, \varphi \rangle}{\langle \varphi, \varphi \rangle}$$

And so in particular:

$$\lambda \leq \frac{\langle \mathcal{L}\psi, \psi \rangle}{\langle \psi, \psi \rangle} \tag{2.4}$$

We will show that

$$\frac{\langle \mathcal{L}\psi, \psi \rangle}{\langle \psi, \psi \rangle} \leq d - 2\sqrt{d-1} + \frac{2\sqrt{d-1} - 1}{k}$$

Now

$$\langle \psi, \psi \rangle = \sum_{v \in V} \psi(v)^2$$

But the only terms that contribute to the sum are for vertices that lie in W_i or in U_i since we defined $\psi(v) = 0$ otherwise.

$$\begin{aligned} \sum_{v \in V} \psi(v)^2 &= \sum_{\substack{v \in W_i \\ 1 \leq i \leq k}} \psi(v)^2 + \sum_{\substack{v \in U_i \\ 1 \leq i \leq k}} \psi(v)^2 \\ &= \sum_{i=1}^{k} \frac{a^2 |W_i|}{(d-1)^i} + \sum_{i=1}^{k} \frac{b^2 |U_i|}{(d-1)^i} \end{aligned} \tag{2.5}$$

Next by basic properties of the graph Laplacian $\mathcal{L}$

$$\langle \mathcal{L}\psi, \psi \rangle = \sum_{(v,w) \in E} (\psi(v) - \psi(w))^2 \tag{2.6}$$

We now want to bound (2.6). We first note that there are no edges connecting W_i and U_i. This is due to the fact that the two edges are at least $2k+2$ apart and only vertices of at most k away from each edge are included in W_i and U_i respectively. Also trivially per definition there are no edges between V_i and V_j unless $j = i + 1$

We then observe that for $z_i, v_i \in W_i$ we have $\psi(z_i) - \psi(v_i) = a(d-1)^{\frac{i}{2}} - a(d-1)^{\frac{i}{2}} = 0$ and analogous for $z_i, v_i \in U_i$.

Hence the only edges that contribute to the sum in (2.6) are edges between V_i and V_{i+1} or between W_i and W_{i+1}. Let $e \in E$ be an edge between V_i and V_{i+1}, then

$$(\psi(v_0) - \psi(v_1))^2 = a^2((d-1)^{-\frac{i}{2}} - (d-1)^{-\frac{i+1}{2}})^2$$

By Lemma 1.1 there are at most $(d-1)|V_i|$ from V_i to V_{i+1}.
We have to be careful here because the edges leaving V_k (and not going back to V_{k-1}) go to vertices where ψ is zero and hence $(\psi(v_0)-\psi(v_1))^2 = \psi(v_0)^2 = a^2(d-1)^{-k}$ where $v_0 \in V_k$ and $v_1 \in V \setminus B_k(\alpha)$
The entire analysis here is symmetric and applies for the W_i's as well yielding:

$$\langle \mathcal{L}\psi, \psi \rangle \leq A + B$$

where

$$A = \sum_{i=1}^{k-1} a^2(d-1)|V_i|((d-1)^{-\frac{i}{2}} - (d-1)^{-\frac{i+1}{2}})^2 + a^2(d-1)|V_k|(d-1)^{-k}$$
$$B = \sum_{i=1}^{k-1} b^2(d-1)|W_i|((d-1)^{-\frac{i}{2}} - (d-1)^{-\frac{i+1}{2}})^2 + b^2(d-1)|W_k|(d-1)^{-k} \tag{2.7}$$

We will work with A, with analagous holding for B.

$$\begin{aligned} A &= \sum_{i=1}^{k-1} a^2(d-1)|V_i|(\frac{1}{(d-1)^i} + \frac{1}{(d-1)^{i+1}} - \frac{2}{(d-1)^{\frac{2i+1}{2}}}) + a^2(d-1)|V_k|(d-1)^{-k} \\ &= \sum_{i=1}^{k-1} \frac{a^2|V_i|}{(d-1)^i}(d-2\sqrt{d-1}) + a^2(d-1)|V_k|(d-1)^{-k} \qquad \text{factoring out } (d-1)^{-i} \\ &= (d-2\sqrt{d-1}) \sum_{i=1}^{k-1} \frac{a^2|V_i|}{(d-1)^i} + \frac{a^2(d-2\sqrt{d-1}+2\sqrt{d-1}-1)|V_k|}{(d-1)^k} \\ &= (d-2\sqrt{d-1}) \sum_{i=1}^{k} \frac{a^2|V_i|}{(d-1)^i} + \frac{a^2(2\sqrt{d-1}-1)|V_k|}{(d-1)^k} \end{aligned} \tag{2.8}$$

(2.8) now looks very similar to the first term of (2.5) which is what we want to bound the ratio in Courant-Fischer.
We now use Lemma 1.1 to bound the second sum, in particular

$$\frac{V_{i+1}}{(d-1)^{i+1}} \leq \frac{(d-1)|V_i|}{(d-1)^{i+1}} = \frac{|V_i|}{(d-1)^i}$$

and so $\frac{V_i}{(d-1)^i}$ is a non-increasing function allowing us to write:

$$\frac{a^2(2\sqrt{d-1}-1)|V_k|}{(d-1)^k} \leq \frac{(2\sqrt{d-1}-1)}{k} \sum_{i=1}^{k} \frac{a^2|V_i|}{(d-1)^i}$$

Finally we have

$$A \leq \left((d - 2\sqrt{d-1}) + \frac{(2\sqrt{d-1}-1)}{k}\right)\sum_{i=1}^{k}\frac{a^2|V_i|}{(d-1)^i}$$

$$B \leq \left((d - 2\sqrt{d-1}) + \frac{(2\sqrt{d-1}-1)}{k}\right)\sum_{i=1}^{k}\frac{b^2|V_i|}{(d-1)^i}$$ using exactly the same procedur

$$\langle \mathcal{L}\psi, \psi\rangle \leq A + B \leq \left((d - 2\sqrt{d-1}) + \frac{(2\sqrt{d-1}-1)}{k}\right)\left(\sum_{i=1}^{k}\frac{a^2|V_i|}{(d-1)^i} + \sum_{i=1}^{k}\frac{b^2|V_i|}{(d-1)^i}\right) \tag{2.9}$$

Now substituting (2.9) and (2.5) in (2.6) immediately yields the desired result. ■

Corollary 2.1.1. *Let $G = (V, E, ep)$ be a d-regular graph, M_G be its adjacency matrix and let μ denote its largest non-trivial eigenvalue, then*

$$\mu_1 \geq 2\sqrt{d-1} - \frac{2\sqrt{d}-1}{k} \tag{2.10}$$

where λ is as in Theorem 2.1

Remark 5. We say largest non-trivial eigenvalue because the all-ones vector is always an eigenvector to an adjacency matrix and in the case of a d-regular graph it is easy to check that it corresponds to an eigenvalue of d. Further we have:

$$\mathcal{L}\mathbf{1} = (D - M_G)\mathbf{1} = (dI - M_G)\mathbf{1} = (dI)\mathbf{1} - M_G\mathbf{1} = d - d = 0$$

So this trivial eigenvalue of M_G corresponds to the trivial eigenvalue of $\mathcal{L}$!

Proof. We will show that for a d-regular graph $\lambda = d - \mu$.
Begin by recalling that $\mathcal{L} = D - M_G$ where D is the degree matrix of the graph defined as a a diagonal matrix with the degree of each vertex along the diagonal. In the case of a d-regular graph this reduces to $\mathcal{L} = dI - M_G$ where I is the identity matrix.
Let $\psi \in \mathbb{R}^{|V|}$ be an arbitrary eigenvector of M_G corresponding to an eigenvalue α, then

$$\mathcal{L}\psi = (dI - M_G)\psi = (dI)\psi - M_G\psi = d\psi - \alpha_\psi = (d - \alpha)\psi$$

Hence ψ is also an eigenvector of the Laplacian $\mathcal{L}$ with eigenvalue $d - \alpha$. Now we are interested in the smallest positive eigenvalue λ of $\mathcal{L}$. To do this we want to minimize $d - \alpha$ but keep it positive, which is clearly what the largest non-trivial eigenvalue μ yields.
Now we have $d - \mu = \lambda \geq d - 2\sqrt{d-1} + \frac{2\sqrt{d-1}-1}{k}$ by Theorem 2.1 giving us $\mu \geq 2\sqrt{d-1} - \frac{2\sqrt{d-1}-1}{k}$ ■

In particular for every $\epsilon > 0$, every sufficiently large d-regular graphs has $\mu \geq 2\sqrt{d-1} - \epsilon$. We need the sufficiently large condition because small ϵ imply large k which means we need to have edges at leas $2k+2$ apart, which requires a large number of vertices. This is however what happens when one wants to construct infinite families of expander graphs, where we require that the number of vertices grow. Ideally in the case of expander graphs, one would want to create families whose adjacency matrices have non-trivial eigenvalues as small as possible, however Theorem 2.1 and in particular Corollary 2.1.1; tell us that the best we can do is $\mu \geq 2\sqrt{d-1}$

References

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