Examining Persistent Homology from Three Different Theoretical Perspectives.

Developing the theoretical foundations of persistent homology through the lens of algebra, quiver theory and category theory.

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Abstract

Persistent modules can be interpreted as graded modules, as quiver representations or even as functors from a poset. Each of these viewpoints has proven useful in different settings. We study persistent homology from each of these viewpoints, highlighting the important results in each setting. We present the necessary preliminaries for each theory, keeping the document accessible to an undergraduate reader. Using algebra, we prove the existence of barcodes for single parameter persistence. Then, using quiver theory we show the existence of barcodes for zig-zag persistence and the non-existence for multiparameter persistence. Along the way, we present a detailed proof of Gabriel's theorem - one of the foundational results in quiver theory. Finally, using category theory we discuss the generalized rank invariant, which is an incomplete invariant for multiparameter persistence.

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Chapter 1

Introduction

Over the past two decades, topological data analysis (TDA) has emerged as a powerful tool to analyze the shape of data. Persistent homology forms the foundation of TDA and a large body of work has been devoted to its study.

This thesis is an exposition that studies persistent homology from three theoretical viewpoints: algebra, quiver theory, and category theory. These viewpoints emerged to tackle different problems as persistence theory grew more complex.

We begin with the algebraic viewpoint, which was also historically the first to be introduced. In particular, we follow [ZC05], to prove the existence of barcodes in the single parameter case. The proof primarily hinges on commutative algebra. In contrast to the other viewpoints of quiver and category theory, it is relatively transparent and provides an insight into how persistence modules work.

Although the algebraic viewpoint laid the foundation for persistent homology, it was not sufficient to deal with extensions to the theory such as zig-zag and multiparameter persistence. Interpreting persistence modules as quiver representations led to an elegant proof of the existence of barcodes for zig-zag modules, and the disappointing non-existence of barcodes for multiparameter persistence. We follow [Oud15] to provide an extremely detailed discussion of the important results in quiver theory and their implications for persistence modules.

Due to the importance of multiparameter persistence modules in practical applications, recent work has been focused on constructing incomplete invariants for these modules. In [KM21], the authors construct the generalized rank invariant by interpreting persistence modules as functors from a poset to the category of vector spaces.

Since each viewpoint is useful in a different setting, they have for the most part been presented separately. One aim of this thesis is to present all of these viewpoints together in a coherent manner.

Finally, the thesis aims to serve as an introduction to persistent homology, which is accessible to an undergraduate reader. To that end, we provide the required background knowledge for each viewpoint in the first three chapters.

At the same time, readers who are familiar with classical persistence but seek an introduction to multiparameter persistence and the generalized rank invariant, can obtain the necessary background in category theory from chapter 2 and then skip ahead to chapter 10. Similarly, readers interested in quiver theoretic details of persistent homology can focus solely on those chapters.

1.1 Acknowledgements

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Chapter 2

Category Theory and Posets

The language of category theory will allow us to phrase many definitions and results throughout the document in a precise and succinct manner. In addition, the category of posets and functors from this category serve as the foundation for the *generalized rank invariant* which we will introduce in chapter 10. In this section, we will introduce the relevant definitions and theorems from category theory, drawing heavily on the standard text [Lan78] as a reference.

2.1 Some basic definitions

Definition 2.1 (Category). *A* category $\mathcal{C}$ *is composed of:*

1. *a collection* $ob\,\mathcal{C}$ *of* objects*;*
2. *for every pair of objects* X, Y*, a collection* $Hom(X, Y)$ *of* morphisms *from* $f : X \to Y$*;*
3. *for each pair of morphisms* $f \in Hom(X, Y)$*,* $g \in Hom(Y, Z)$*, a* composition $g \circ f \in Hom(X, Z)$*;*
4. *for every object* $X \in \mathcal{C}$*, an* identity morphism $id_X \in Hom(X, X)$*;*

such that the following axioms hold:

1. associativity of composition*: for objects* X, Y, W, Z *and morphisms* $f \in Hom(X, Y)$*,* $g \in Hom(X, Y)$*,* $h \in Hom(X, Y)$*,* $h \circ (g \circ f) = (h \circ g) \circ f$*;*
2. identity law*: let* X, Y *be objects and* $f \in Hom(X, Y)$*, then* $id_Y \circ f = f = f \circ id_X$*.*

One often suppresses $ob\,\mathcal{C}$ *in notation and simply writes* $X \in \mathcal{C}$ *to denote an object of* $\mathcal{C}$*.*

Remark 2.1. In general, the collection of objects $\mathrm{ob}\,\mathcal{C}$ need not be a set. Categories where the $\mathrm{ob}\,\mathcal{C}$ form a set are called *small categories*. Most categories one encounters in the wild, such as the categories of sets, groups, vector spaces and persistence modules are small. For our purposes, we will mean a small category when we refer to a category.

Most mathematical objects are examples of categories.

Example 1 (Category of vector spaces). Denote by **Vect** the category of vector spaces, whose objects are vector spaces, morphisms are linear maps, composition is the familiar composition from linear algebra, and the identity matrix plays the role of an identity morphism. Basic linear algebra tells us that the associativity and identity axioms hold.

More subtly, one can define the category **vect** of *finite dimensional vector spaces*, whose objects are finite dimensional vector spaces, morphisms are linear maps, composition is the familiar composition from linear algebra and the identity matrix is the identity morphism. It is clear that these too fulfill the required axioms. **vect** is a *subcategory* of **Vect**.

Given two categories, we would like to map from one category to the other, while preserving certain structural information from the source category. Functors are these maps.

Definition 2.2 (Functors). *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ assigns to*

1. *an object $X \in \mathcal{C}$, an object $\mathcal{F}_X \in \mathcal{D}$;*
2. *a morphism $f : X \to Y$ of $\mathcal{C}$, a morphism $\mathcal{F}(f) : \mathcal{F}_X \to \mathcal{F}_Y$ of $\mathcal{D}$;*

such that

1. *$\mathcal{F}(id_X) = id_{\mathcal{F}_X}$, for every object $X \in \mathcal{C}$;*
2. *$\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$, for every $f \in Hom(X, Y)$ $g \in Hom(Y, Z)$ and $X, Y, Z \in \mathcal{C}$.*

We will encounter many functors throughout this document. Functors may be used to establish an equivalence of categories, meaning that two categories have similar mathematical structures and that theorems established in one category can be translated into the other. This is an extremely powerful notion and one which will allow us to impose well-known results from commutative algebra on persistence modules.

Definition 2.3 (Equivalance of categories). *A functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ establishes an equivalence of categories if and only if it is* full, faithful and essentially isomorphic*. A functor $\mathcal{F}$ is full, if for every pair of objects $X, Y \in \mathcal{C}$, the induced map*

$$\begin{aligned} Hom_{\mathcal{C}}(X, Y) &\to Hom_{\mathcal{D}}(\mathcal{F}_X, \mathcal{F}_Y) \\ f : X \to Y &\mapsto \mathcal{F}(f) : \mathcal{F}_X \to \mathcal{F}_Y \end{aligned}$$

is surjective*. If the same induced map is* injective *for every pair of objects, then the functor is* faithful*. Finally $\mathcal{F}$ is* essentially surjective*, if for every object $D \in \mathcal{D}$, there exists an object $C \in \mathcal{C}$ such that $\mathcal{F}_C \cong D$.*

Remark 2.2. Often an equivalence of categories is defined with forward and backward functors such that their composition is naturally isomorphic to the identity, analogous to the definition of a bijective map. Definition 2.3 is then proven to be an equivalent condition, which one most often uses to check that a functor is indeed an equivalence of categories [1].

[1] see Chapter 4, Sec. 4 of [Lan78] for details.

$$\mathbb{Z}$$

$$\begin{tikzcd}
\dots \arrow[r] & 1 \arrow[r] & 2 \arrow[r] \arrow[d, squiggly, "\text{apply } \mathcal{T}"] & 3 \arrow[r] & \dots \\
\dots \arrow[r] & \mathcal{T}_1 \arrow[r] & \mathcal{T}_2 \arrow[r] & \mathcal{T}_3 \arrow[r] & \dots
\end{tikzcd}$$

$$\mathcal{C}$$

Figure 2.1: A diagram of shape $\mathbb{Z}$ in $\mathcal{C}$.

2.2 Limits and Colimits

Definition 2.4 (Diagrams, indexing categories)**.** *Let $\mathcal{J}$ and $\mathcal{C}$ be categories. A* diagram of shape $\mathcal{J}$ in $\mathcal{C}$ *is simply a functor $\mathcal{T} : \mathcal{J} \to \mathcal{C}$ and $\mathcal{J}$ is called an* index category*.*

Diagrams are an attempt to generalize the notion of an indexing family. A diagram of shape $\mathcal{J}$ in $\mathcal{C}$ indexes objects in $\mathcal{C}$ in a manner patterned on $\mathcal{J}$. This is best understood with an example.

Example 2 (Diagrams of shape $\mathbb{Z}$)*.* The integers $\mathbb{Z}$ can be thought of as a category, whose objects are simply the integers themselves with a unique morphism between $x, y \in \mathbb{Z}$ if and only if $x \leq y$. Given a pair of morphisms $f : x \to y\,, g : y \to z$, we know that $x \leq y$ and $y \leq z$, since morphisms exist only under this condition and are unique in this case. Hence, we also have $x \leq z$ and a unique morphism $x \to z$ which defines the composition $g \circ f$. The identity morphism is the unique morphism $x \to x$ (since $x \leq x$). It is clear that the axioms from Definition 2.1 hold.

Now let $\mathcal{C}$ be an arbitrary category, then a functor $\mathcal{T} : \mathbb{Z} \to \mathcal{C}$ assigns to each integer an object in $\mathcal{C}$ and to each morphism between integers a morphism between their corresponding objects in $\mathcal{C}$. Figure 2.1 depicts the motivation for the phrase "diagram of shape $\mathbb{Z}$ in $\mathcal{C}$".

We will usually use a *poset* as the indexing category and **vect** as the target category. We now want to generalize the notion of a limit.

Definition 2.5 (Cone)**.** *Let $\mathcal{T} : \mathcal{J} \to \mathcal{C}$ be a diagram. A* cone *over $\mathcal{T}$ is an object $L \in \mathcal{C}$ together with a family of morphisms $\pi_x : L \to \mathcal{T}_x$ for each object $x \in \mathcal{J}$, such that Figure 2.2 commutes for every morphism $f : x \to y$ in $\mathcal{J}$.*

$$\begin{tikzcd}
\mathcal{T}_x \arrow[rr, "\mathcal{T}(f)"] & & \mathcal{T}_y \\
& L \arrow[ul, "\pi_x"] \arrow[ur, "\pi_y"'] &
\end{tikzcd}$$

Figure 2.2: A cone over a functor $\mathcal{T}$.

Definition 2.6 (Limit)**.** *Let $\mathcal{T} : \mathcal{J} \to \mathcal{C}$ be a diagram. A* limit $\varprojlim \mathcal{T}$ *of $\mathcal{T}$ is a* cone *over $\mathcal{T}$ with the additional property: if there exists another cone L, then there is a* unique *morphism $u : L \to \varprojlim \mathcal{T}$ such that Figure 2.3 commutes for every $x \in \mathcal{J}$.*

$$\begin{tikzcd}
\mathcal{T}_x \arrow[rr, "\mathcal{T}(f)"] & & \mathcal{T}_y \\
& \varprojlim \mathcal{T} \arrow[ul, "\pi_x"'] \arrow[ur, "\pi_y"] & \\
& L \arrow[u, "u"'] \arrow[uul, "\varphi_y"] \arrow[uur, "\varphi_x"'] &
\end{tikzcd}$$

Figure 2.3: The universal property of a limit.

Remark 2.3. A limit need not always exist, but if a limit does exist it is unique up to isomorphism. The uniqueness follows directly from the definition. Any limit L is also a cone and hence by Definition 2.6, there exists a unique morphism $u : L \to \varprojlim \mathcal{T}$. On the other hand, since L is a limit and $\varprojlim \mathcal{T}$ is a cone, there exists a unique morphism $u' : \varprojlim \mathcal{T} \to L$. Thus, we will often speak of *the limit*.

Categories where a limit always exist are called *complete categories*

Cocones and colimits are simply duals of cones and limits respectively, but we spell out the definitions below.

Definition 2.7 (Cocone, Colimit)**.** *Let $\mathcal{T} \to \mathcal{C}$ be a diagram. A cocone is an object $C \in \mathcal{C}$ together with a family of morphisms $i_x : \mathcal{T}_x \to C$ for every object $x \in \mathcal{J}$, such that Figure 2.4 commutes for every morphism $f : x \to y$ in $\mathcal{J}$.*

$$\begin{tikzcd}
& C & \\
\mathcal{T}_x \arrow[ur, "i_x"] \arrow[rr, "\mathcal{T}(f)"'] & & \mathcal{T}_y \arrow[ul, "i_y"']
\end{tikzcd}$$

Figure 2.4: A cocone over a functor $\mathcal{T}$.

A colimit $\varinjlim \mathcal{T}$ is a cocone with the additional property, that if there exists another cocone C, then there exists a unique morphism $v : \varinjlim \mathcal{T} \to C$, such that Figure 2.5 commutes for every $x \in \mathcal{J}$.

Just as with the limit, the colimit need not always exist but when it does it is unique up to isomorphism. Categories in which the colimit always exists, are called *cocomplete*.

$$\begin{tikzcd}
& C & \\
& \varinjlim \mathcal{T} \arrow[u, "v"] & \\
\mathcal{T}_x \arrow[uur, "i'_x"] \arrow[ur, "i_x"'] \arrow[rr, "\mathcal{T}(f)"'] & & \mathcal{T}_y \arrow[uul, "i'_y"'] \arrow[ul, "i_y"]
\end{tikzcd}$$

Figure 2.5: The universal property of a colimit.

2.3 Posets

Definition 2.8 (Posets, comparable elements)**.** *A poset is a set $\boldsymbol{P}$ endowed with a partial order relation $\leq$ which satisfies the following axioms:*

1. *Reflexivity: $a \leq a \ \forall a \in \boldsymbol{P}$;*
2. *Antisymmetry: $a \leq b$ and $b \leq a \implies a = b$;*
3. *Transitivity: $a \leq b$ and $b \leq c \implies a \leq c$.*

If $a \leq b$ or *$b \leq a$, we say that a and b are* comparable*.*

A poset $\boldsymbol{P}$ can be interpreted as a category, whose objects are simply the elements of $\boldsymbol{P}$ with a unique morphism between $a, b \in \boldsymbol{P}$ if and only if $a \leq b$. We will denote this category by $\boldsymbol{P}$.

We have already encountered an example of a poset in Example 2. Other standard examples include $\mathbb{R}$ and $\mathbb{Z}^n$, $n > 0$. In fact, posets such as $\mathbb{Z}$ or $\mathbb{Z}^2$ are examples of classes of posets which fulfill certain desirable properties such as connectivity and local finiteness.

Definition 2.9 (Connected poset)**.** *A poset $\boldsymbol{P}$ is* connected *if for every $p, q \in \boldsymbol{P}$, there exists a sequence*

$$p = p_0, p_1, \ldots, p_n = q$$

such that p_i and p_{i+1} are comparable *for all i.*

Definition 2.10 (Locally finite posets)**.** *A poset $\boldsymbol{P}$ is* locally finite *if for all $p \leq q \in P$, the set $[p, q] = \{r \in P | p \leq r \leq q\}$ is finite.*

With the help of an induced graph, one can also define path-connectedness for arbitrary posets.

Definition 2.11. *Let $\boldsymbol{P}$ be a poset and $p, q \in P$. If $p \leq q$ and $[p, q] = \emptyset$, then we say that q* covers *p. If q covers p* or *p covers q, we write $p \asymp q$.*

Definition 2.12 (Hasse Diagram, path connected posets)**.** *The* Hasse diagram *$Hasse(\boldsymbol{P})$ of a poset $\boldsymbol{P}$ is a graph, whose vertices are the elements of $\boldsymbol{P}$. Two vertices p and q are adjacent if and only if $p \asymp q$.*

P *is* path connected *if Hasse(**P**) is path connected. A subposet **Q** $\subset$ **P** is path connected if the subgraph of Hasse(**P**) induced by the elements of **Q** is path connected. The set of path connected subposets is denoted by **Con**(**P**) $= \{\boldsymbol{Q} \subset P | \ \boldsymbol{Q}$ path connected$\}$.*

Proposition 2.1. *If **P** is connected and locally finite, then **P** is path connected.*

Proof. Let **P** be a connected, locally finite poset, Hasse(**P**) be its Hasse diagram, and $p, q \in \mathbf{P}$ be arbitrary. Since **P** is connected, there exists a sequence of comparable elements $p = p_0, p_1, \dots, p_n = q$. Due to local finiteness, we claim that in fact $p_i \asymp p_{i+1} \, \forall i$, hence yielding a path between p_i and p_{i+1} for all i .

Assume this is not true, so $[p_i, p_{i+1}] \neq \emptyset$ for some i. By Definition 2.10, $[p_i, p_{i+1}]$ is finite. Let $[p_i, p_{i+1}] = \{r_1, \dots, r_m\}$. We will induct over m to construct a sequence of elements which cover each other. For $m = 1$, we can simply add r_1 to the sequence and obtain $p = p_0, \dots, p_i, r, p_{i+1}, \dots p_n = q$ with $p_i \asymp r \asymp q$. Now assume the statement for $m-1$ and consider $[p_i, p_{i+1}] = \{r_1, \dots, r_m\}$. Let r be a minimal element of $\{r_1, \dots, r_m\}$. While r might not be unique, there must exist a minimal element. Since r is minimal, $[p, r] = \{l \in \mathbf{P} | p \leq l \leq r\} = \emptyset$ and hence $p \asymp r$. We now add r to the sequence to obtain $p_0, \dots, p_i, r, p_{i+1}, \dots, p_n$ and apply the inductive assumption to $[r, p_{i+1}]$. Repeating this finite process for any $[p_i, p_{i+1}] \neq \emptyset$, we obtain the desired sequence, where $p_i \asymp p_{i+1} \, \forall i$.

We can now establish a path from p to q by concatenating the paths. ■

Remark 2.4. The converse of Proposition 2.1 does not hold. Consider the simple 3 element poset $\mathbf{P} = \{a \leq b \leq c\}$ and $\mathbf{Q} = \{a, c\}$. **Q** is connected since $a \leq c$ and these are the only two elements in **Q**. However, as seen in Figure 2.6, the subgraph of Hasse(**P**) induced by **Q** is clearly not path connected and hence **Q** is not path connected.

$$a \text{ ——— } b \text{ ——— } c$$

$$\mathbf{Q} \subset \mathbf{P}$$

Figure 2.6: The Hasse diagram of a connected, but not path connected subposet.

We can also extend the definition of an interval to this setting.

Definition 2.13 (Intervals)**.** *A subset I is* convex*, if $r, t \in I$ and $s \in \boldsymbol{P}$ with $r \leq s \leq t$, then $s \in I$.*

*Let **P** be a poset, then a subset $I \subset \boldsymbol{P}$ is an* interval*, if I is connected and convex. We denote by **Int**(**P**), the collection of all intervals of **P**.*

Remark 2.5. If **P** is locally finite, then due to the connectedness of intervals and Proposition 2.1, **Int**(**P**) $\subset$ **Con**(**P**).

Remark 2.6. **Int**(**P**) and **Con**(**P**) can themselves be regarded as posets with partial order being imposed by inclusion.

Cones and cocones over functors from *connected posets* satisfy an important property. Combining Figure 2.2 and Figure 2.4, one might expect a map from cones to cocones. In the case of functors from connected posets, this map exists and is canonical. It will be the foundation for the generalized rank invariant.

$$
\begin{tikzcd}
 & & C & & \\
\mathcal{F}_p \arrow[urr, "i_p"] \arrow[r, leftrightarrow, "f_0"] & \mathcal{F}_{p_1} \arrow[ur, "i_{p_1}"] \arrow[r, leftrightarrow, "f_1"] & \mathcal{F}_{p_2} \arrow[u, "i_{p_2}"] \arrow[r, leftrightarrow, "f_2"] & \dots \arrow[ul] \arrow[r, leftrightarrow, "f_{n-1}"] & \mathcal{F}_q \arrow[ull, "i_q"'] \\
 & & L \arrow[ull, "\pi_p"] \arrow[ul, "\pi_{p_1}"] \arrow[u, "\pi_{p_2}"] \arrow[ur] \arrow[urr, "\pi_q"'] & &
\end{tikzcd}
$$

Figure 2.7: The canonical map from a cone to a cocone in a connected poset indexed diagram.

Theorem 2.1 (Canonical map from cones to cocones). Let $\mathbf{P}$ be a connected poset and $\mathcal{F} : \mathbf{P} \to \mathcal{C}$ be a functor. Let L be a cone and C be a cocone over $\mathcal{F}$. Then for any $p, q \in \mathbf{P}$, we have $i_p \circ \pi_p = i_q \circ \pi_q$.

Proof. Let $p, q \in \mathbf{P}$ be arbitrary. Since $\mathbf{P}$ is connected, there exists a sequence $p = p_0, \dots, p_n = q$ such that p_j and p_{j+1} are comparable for all j. Since p_j and p_{j+1} are comparable, there exists a unique morphism either $p_j \to p_{j+1}$ (if $p_j \leq p_{j+1}$) or $p_{j+1} \to p_j$ (if $p_j \geq p_{j+1}$). Denote by f_j, the morphism in $\mathcal{C}$ induced by this morphism via $\mathcal{F}$. Combined with the maps from L to each p_j and from each p_j to C defined in Definition 2.5 and Definition 2.7, we obtain the commutative diagram depicted in Figure 2.7. Diagram chasing now yields the required result.

We begin on the left hand side of the diagram. If $f_0 : \mathcal{F}_p \to \mathcal{F}_{p_1}$, then by commutativity we have

$$
i_p \circ \pi_p = (i_1 \circ f_0) \circ \pi_p = i_1 \circ (f_0 \circ \pi_p) = i_1 \circ \pi_{p_1} .
$$

On the other hand, if $f_0 : \mathcal{F}_{p_1} \to \mathcal{F}_p$, then

$$
i_p \circ \pi_p = i_p \circ (f_0 \circ \pi_{p_1}) = (i_p \circ f_0) \circ \pi_{p_1} = i_{p_1} \circ \pi_{p_1} .
$$

Proceeding similarly for $j = 2, \dots, n-1$, we have $i_{p_j} \circ \pi_{p_j} = i_{p_{j+1}} \circ \pi_{p_{j+1}}$, finishing the proof. ■

Remark 2.7. If a functor $\mathcal{F} : \mathbf{P} \to \mathcal{C}$ possesses a limit and colimit, then by Theorem 2.1 there exists a canonical map $\psi_F : \varprojlim \mathcal{F} \to \varinjlim \mathcal{F}$.

Chapter 3

Algebra

We will begin by examining persistent homology through the lens of commutative algebra. The language of commutative algebra gives us an intuitive understanding of classical persistence and the algorithms used in practice. In this section, we develop the required algebraic machinery. A standard reference for most of the material presented here is [CL21].

3.1 Graded rings and modules

Definition 3.1 (Graded ring)**.** *A graded ring is a ring R equipped with a direct sum decomposition $R \cong \bigoplus_{i\in\mathbb{Z}} R_i$, such that*

$$r_n r_m \in R_{n+m} \forall r_n \in R_n, r_m \in R_m\,.$$

Elements such as $r_n \in R_n$, that lie completely in one grade are called homogeneous.

An example of a graded ring of particular importance to us is the polynomial ring with standard grading.

Example 3 (Polynomial ring with standard grading)*.* Let R be a ring and $R[t]$ be its polynomial ring. We can equip $R[t]$ with *standard grading* by writing it as a direct sum of the homogeneous polynomials:

$$R[t] = \bigoplus_{i\in\mathbb{Z}} Rt^i\,.$$

Since $at^n \cdot bt^m = abt^{n+m} \in Rt^{n+m}\, \forall a, b \in R$, this is indeed a grading.

Definition 3.2 (Graded module)**.** *A graded module M over a graded ring R is an R-module endowed with a direct sum decomposition $M \cong \bigoplus_{i\in\mathbb{Z}} M_i$, such that*

$$r_n m_m \in M_{n+m} \forall r_n \in R_n, m_m \in M_m\,.$$

Finitely generated graded R-modules form a category $\mathbf{Grad}_R$, which we will show to be equivalent to the category of finite type persistence modules.

Definition 3.3 (Category of finitely generated graded modules)**.** *The category $\boldsymbol{Grad}_R$ over a graded ring R has as objects finitely generated graded R-modules. A morphism $f : M \to N$ in $\boldsymbol{Grad}_R$ is an R-linear map, such that $f(M^i) \subset N^i$ for every i.*

Since graded R-modules are simply R-modules with additional structure, the standard structure theorem for finitely generated graded modules over a principle ideal domain (PID) R has a graded analogue. This classification theorem will be central in proving the existence of persistence barcodes in classical persistence.

Theorem 3.1 (Structure theorem). Let M be a finitely generated module over a graded principle ideal domain R, then

$$M \cong \bigoplus_{i=1}^{n} \sum^{\alpha_i} M \oplus \bigoplus_{j=1}^{m} \frac{\sum_j^{\gamma} M}{r_j M} \,,$$

where $r_j \in R$ are *homogeneous* elements, such that $r_j | r_{j+1}$, $\alpha_i, \gamma_j \in \mathbb{Z}$ and $\sum^{\alpha}$ denotes an α upward shift in the gradation.

Note the striking similarity to the familiar theorem for the non-graded case. The first term in the sum above is analogous to the free portion in the non-graded case and the second term to the torsional part. In fact, the proof of the theorem (which can be found in [Hun80]) is also similar to the non-graded case, except that one must now keep track of the grading at every step.

Chapter 4

Quiver Theory

We begin by stating some basic definitions in quiver theory.

Definition 4.1 (Quiver)**.** *A quiver is a quadruple* $\mathtt{Q} = (Q_0, Q_1, h, t)$*, where*

1. Q_0 *is the set of* vertices*;*
2. Q_1 *is the set of* arrows*;*
3. $h, t : Q_1 \to Q_0$ *are* head/tail *maps that assign to each arrow a head and tail vertex respectively.*

Visually, a quiver is simply a directed graph*. Denote by* $\bar{\mathtt{Q}}$*, the* undirected *graph determined by* $\mathtt{Q}$

Remark 4.1. In general, the vertex or arrow set may not be finite. We will deal exclusively with finite quivers, which means that both Q_0 and Q_1 are finite.

The underlying graph $\bar{\mathtt{Q}}$ of a quiver, allows us to use graph theoretic language to describe a quiver $\mathtt{Q}$. For example $\mathtt{Q}$ is *connected/disconnected* if $\bar{\mathtt{Q}}$ is connected/disconnected.

4.1 Quiver representations

Definition 4.2. *A representation* $\mathbb{V}$ *of a quiver* $\mathtt{Q}$ *over a field* k *assigns to each vertex* $i \in Q_0$ *a* k*-vector space* V_i *and to each arrow* $a \in Q_1$ *a* k*-linear map* $v_a : V_{t(a)} \to V_{h(a)}$*.*

Definition 4.3 (Dimension)**.** *The* dimension vector *of a representation* $\mathbb{V}$ *of a* finite quiver $\mathtt{Q}$ *is a vector* $\boldsymbol{dim}\mathbb{V} = (\dim V_1, \ldots, \dim V_n)^T$*. The* dimension *of* $\mathbb{V}$ *is* $\dim \mathbb{V} = \|\boldsymbol{dim}\mathbb{V}\|_1 = \sum_i^n \dim V_i$*.*

Remark 4.2. Importantly, the maps v_a are completely arbitrary and must not fulfill any commutativity conditions. Consider, for example the quiver $\mathtt{Q}$ with the following underlying graph:

$$\begin{tikzcd} 1 \arrow[rr, "a"'] \arrow[dr, "c"] & & 2 \arrow[dl, "b"'] \\ & 3 & \end{tikzcd}$$

The graph misleadingly suggests that the maps commute, but for an arbitrary representation $v_c \neq v_b \circ v_a$.

Further, the associated vector spaces V_i might be infinite dimensional. A representation $\mathbb{V}$ is finite if $\dim \mathbb{V}$ is finite.

Definition 4.4 (Subrepresentation)**.** *$\mathbb{W} = (W_i, w_a)$ is a* subrepresentation *of $\mathbb{V} = (V_i, v_a)$ if $W_i \subset V_i$ is a subspace for every vertex $i \in Q_0$ and $w_a = v_a\big|_{W_{t(a)}}$, for every arrow $a \in Q_1$.*

4.1.1 The category $\mathbf{rep}_k\mathtt{Q}$

Fix a quiver $\mathtt{Q}$ and a base field k.

Definition 4.5 (Morphisms between quiver representations)**.** *Let $\mathbb{V}$ and $\mathbb{W}$ be two representations of $\mathtt{Q}$. Then, a morphism $\phi : \mathbb{V} \to \mathbb{W}$ is a family of linear maps $\phi_i : V_i \to W_i, i \in Q_0$, such that the following diagram commutes for every arrow $a \in Q_1$:*

$$\begin{tikzcd}
V_{t(a)} \arrow[r, "v_a"] \arrow[d, "\phi_{t(a)}"] & W_{h(a)} \arrow[d, "\phi_{h(a)}"] \\
W_{t(a)} \arrow[r, "w_a"] & W_{h(a)}
\end{tikzcd}$$

The composition of two morphisms $\phi : \mathbb{V} \to \mathbb{W}$ and $\psi : \mathbb{W} \to \mathbb{Z}$ is carried out pointwise, that is

$$(\psi \circ \phi)_i = \psi_i \circ \phi_i\,.$$

The identity morphism is also defined pointwise $(1_{\mathbb{V}})_i = 1_{V_i} \forall i \in Q_0$.

Clearly with the above definitions of morphisms, the representations of $\mathtt{Q}$ over k denoted by $\mathbf{Rep}_k(\mathtt{Q})$ form a category.

We will deal exclusively with the *subcategory* $\mathbf{rep}_k(\mathtt{Q})$ of *finite dimensional representations.*

We are introducing quiver theory with the hope of using it to derive an elegant classification theorem for various classes of persistence modules. In order to do so, it would be nice to only have to deal with indecomposable representations (or indeed isomorphism classes of indecomposable representations). The following proposition tells us that every representation can be expressed as a direct sum of indecomposable representations.

Proposition 4.1. *$\boldsymbol{rep}_k(\mathtt{Q})$ is a Krull-Schmidt category.*

Proof. We need to show that every $\mathbb{V} \in \mathbf{rep}_k(\mathtt{Q})$ can be written as $V \cong \mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_m$, where $\mathbb{V}_1, \ldots, \mathbb{V}_m$ are indecomposable. Further, this decomposition is unique up to isomorphism and permutation of the indecomposable representations. We proceed by induction on $n = \dim \mathbb{V}$. If $\mathbb{V}$ is indecomposable to begin with, then we are done. Otherwise, we can write $\mathbb{V} = \mathbb{V}_1 \oplus \mathbb{V}_2$. Now, if both $\mathbb{V}_1$ and $\mathbb{V}_2$ are indecomposable then we are done. Otherwise, at least one of them, say $\mathbb{V}_1$, is decomposable, however $\dim \mathbb{V}_1 < n$, so we can apply the inductive hypothesis and conclude (since the base case $n = 1$ is trivial).

This decomposition is unique up to isomorphism and permutation of the summands. More formally, if there is another decomposition $\mathbb{V} \cong \mathbb{W}_1 \oplus \cdots \oplus \mathbb{W}_s$, where W_j is indecomposable and non-zero, then $m = s$ and there exists a permutation σ, such that $\mathbb{V}_i \cong \mathbb{W}_{\sigma(i)} \, \forall 1 \leq i \leq m$. Consequently,

$$\mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_m \cong \mathbb{W}_1 \oplus \cdots \oplus \mathbb{W}_s \, .$$

Suppose $s \geq m$, then we can write $m = s + k$, $k \geq 0$ and

$$\mathbb{V}_1 \oplus \cdots \oplus \mathbb{V}_s \cong \mathbb{W}_1 \oplus \cdots \oplus \mathbb{W}_s \oplus \cdots \oplus \mathbb{W}_{s+k} \, .$$

Since $\mathbb{V}_i$ is indecomposable for all $1 \leq i \leq s$, we must have $V_i \cong \mathbb{W}_j$, $1 \leq j \leq s + k$. In other words, each V_i must be isomorphic to precisely one of the $\mathbb{W}_j$. But then the fact that the two decompositions are isomorphic forces $k = 0$. ■

Chapter 5

A Short Review of Simplicial Homology

Persistent homology can be thought of as the extension of simplicial homology to filtered simplicial complexes, so we begin with a brief review of simplicial homology. For more details on (simplicial) homology, see [Hat02].

5.1 The Big Picture

Broadly speaking, homology assigns algebraic invariants to topological spaces. More specifically, n^{th} homology is a functor $H_n : \mathbf{Top} \to \mathbf{Ab}$, which means that:

1. a topological space X is assigned an abelian group $H_n(X)$;
2. A continuous map $f : X \to Y$ is assigned a group homomorphism $H_n(f) : H_n(X) \to H_n(Y)$ such that $H_n(\mathrm{id}) = \mathrm{id}_{H_n(X)}$, and $H_n(g \circ f) = H_n(g) \circ H_n(f)$ for all maps $X \xrightarrow{f} Y \xrightarrow{g} Z$.

In addition, homology is homotopy invariant, that is given two spaces X and Y which are homotopy equivalent to each other, their homology groups $H_n(X) \cong H_n(Y)$ are isomorphic.

Further, homology (and in particular simplicial homology) is efficiently computable, making it well suited to our goal of studying the shape of data.

5.2 Simplicial Complexes

There is a large range of homology theories (such as singular, cellular, and Morse homology to name a few). We will work with simplicial homology, because computing simplicial homology boils down to a slight modification of Gaussian elimination, making computation extremely efficient. Simplicial homology is defined on simplicial complexes which we now introduce.

Definition 5.1 (Simplicial complexes)**.** *A* simplicial complex *is a set K of* vertices *together with a collection S of subsets of K, called* simplices *such that*

1. *every vertex is a simplex, $\{v\} \in S \ \ \forall v \in K$;*

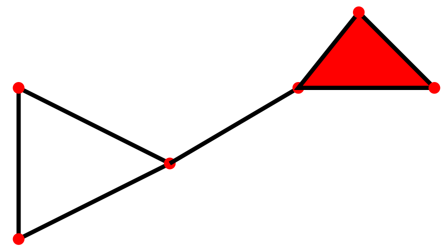


Figure 5.1: A simplicial complex with 0-, 1- and 2-simplices.

2. *let $\sigma \in S$ and $\tau \subset \sigma$, then $\tau \in S$.*

The simplex τ is called a face *of σ. A simplicial complex (L, R) is a subcomplex of (K, S) if $L \subset K$ and $R \subset S$.*

Notation. We will often just write K to denote a simplicial complex and $[v_0, \dots, v_n], v_i \in K$ for the simplex formed by $\{v_0, \dots, v_n\}$.

Formalism. Simplicial complexes form a category denoted by **Simp**, where the objects are simplicial complexes and a morphism $f : (K, S) \to (L, R)$ between two simplicial complexes is a map between the vertex sets K and L, such that

$$\sigma = [v_0, \dots, v_n] \in S \implies f(\sigma) = [f(v_0), \dots, f(v_n)] \in R\,.$$

In other words, simplices in K are mapped to simplices in L.

5.3 Simplicial Homology

We still need a few definitions to be able to define simplicial homology.

Definition 5.2 (Chain groups)**.** *Let K be a simplicial complex, then an n-chain c on K is a linear combination of n-simplices in K,*

$$c = \sum_i n_i \sigma_i \qquad\qquad n_i \in \mathbb{Z}, \sigma_i \in K \;\; n\textit{-simplex}\,.$$

The n^{th} chain group $C_n(K)$ is the free abelian group generated by n-chains on K.

Example 4. Figure 5.2 depicts a simplicial complex with a *one chain* $\sigma_2 + \sigma_3$ highlighted in green and a two chain A in orange. One possible basis for the chain vector spaces is:

$$C_0 = \langle v_0, v_1, v_2, v_3, v_4, v_5, v_6 \rangle$$
$$C_1 = \langle \sigma_1, \sigma_2, \sigma_3, \sigma_3, \sigma_5, \sigma_5, \sigma_7 \rangle$$
$$C_2 = \langle A \rangle$$

Remark 5.1 (Coefficients). We make a choice of coefficients in Definition 5.2 with $n_i \in \mathbb{Z}$. One could also choose $n_i \in R$ for any ring R. We will primarily work with coefficients $n_i \in \mathbb{Z}_2$, though the theory of persistent homology works with coefficients in any field (but as we will see in chapter 7 **not** over general principle ideal domains such as $\mathbb{Z}$!)

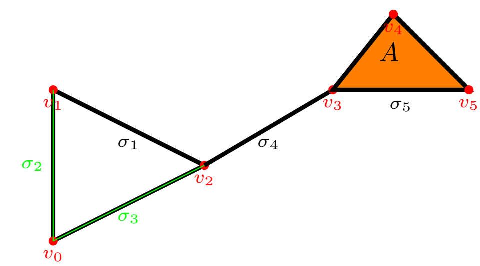


Figure 5.2: A simplical complex with a one chain highlighted in green and a two chain in orange.

Definition 5.3 (Boundary operator)**.** *Let $\sigma \in K$ be an n-simplex spanned by $[v_0, \ldots, v_n]$, then set*

$$\partial_n \sigma = \sum_i (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_n]\,,$$

where $[v_0, \ldots, \hat{v}_i, \ldots, v_n]$ denotes the face of σ obtained by removing v_i. So, $\partial_n \sigma \in C_{n-1}(K)$ as a linear combination of $n-1$-simplices.

We can extend this definition linearly to obtain a homomorphism:

$$\begin{aligned} \partial_n : & C_n \to C_{n-1} \\ & \sum_i n_i \sigma_i \mapsto \sum_i n_i \partial_n(\sigma_i)\,. \end{aligned}$$

Example 5. We now explicitly write out the boundary matrices from Example 4. We will work in $\mathbb{Z}_2$, where $-1 = 1$, and save ourselves the mess associated with orientations.

$$\begin{aligned} \partial_2 : & C_2 \to C_1 \\ & A \mapsto \sigma_5 + \sigma_6 + \sigma_7 \end{aligned}$$

$$\partial_2 = \begin{array}{c} \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \\ \sigma_7 \end{array} \begin{array}{c} A \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \end{array}$$

Next:

$$\partial_1 : C_1 \to C_0$$

$$\partial_1 = \begin{array}{c} \\ v_0 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 & \sigma_7 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Finally, $\partial_0 = 0$.

We now have a sequence of abelian groups $C_n(K)$ connected by boundary operators:

$$\dots \xrightarrow{\partial_{n+2}} C_{n+1}(K) \xrightarrow{\partial_{n+1}} C_n(K) \xrightarrow{\partial_n} C_{n-1}(K) \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_1} C_0(K)$$

The boundary operators fulfill the following important property.

Proposition 5.1. *Applying the boundary operator twice consecutively yields the zero map, in other words:*

$$\partial_n \partial_{n+1} = 0 \; \forall n \in \mathbb{Z}\,.$$

Thus, n-chains and their boundary operators are an example of *chain complexes.*

Definition 5.4 (Chain complex)**.** *A chain complex $(C_\bullet, \partial)$ is a sequence of abelian groups $C_n, n \in \mathbb{Z}$ together with homomorphisms $\partial_n : C_n \to C_{n-1} \; \forall n$, such that $\partial_n \partial_{n+1} = 0 \; \forall n$*

Formalism (The category of chain complexes). Chain complexes form a category **Comp** whose objects are chain complexes and morphisms are chain maps.

A chain map $f_\bullet : C_\bullet \to C'_\bullet$ is a sequence of group homomorphisms $f_n : C_n \to D_n \; \forall n$, such that $f_{n-1} \circ \partial_n = \partial'_n \circ f_n \; \forall n$. Pictorially, the following diagram commutes:

$$\begin{tikzcd} \dots \arrow[r] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \dots \\ \dots \arrow[r] & C'_n \arrow[r, "\partial'_n"] & C'_{n-1} \arrow[r] & \dots \end{tikzcd}$$

The composition of two chain maps is carried out pointwise. If $C_\bullet \xrightarrow{f_\bullet} C'_\bullet \xrightarrow{g_\bullet} C''_\bullet$ are chain maps, then define $(g \circ f)_\bullet$ by $(g \circ f)_n = g_n \circ f_n$.

Formalism (Functoriality of chain complexes). Definition 5.2 and Definition 5.3 define a functor from **Simp** to **Comp**. It is clear that we get a chain complex for every simplicial complex. Let $\varphi : (K, S) \to (L, R)$ be a morphism of simplicial complexes, then define a chain map

$f_\bullet : C_\bullet(K) \to C_\bullet(L)$ between the corresponding chain complexes as follows:

$$\begin{aligned} f_n : &C_n(K) \to C_n(L) \\ &\sum_i n_i \sigma_i \mapsto \sum_i n_i \varphi(\sigma_i)\,. \end{aligned}$$

To prove that this is a chain map we need to show $f_{n-1} \circ \partial_n = \partial'_n \circ f_n$, but this is a straightforward verification:

$$\begin{aligned} \partial_n f_n(\sigma) &= \partial_n[\varphi(v_0), \ldots, \varphi(v_n)] \\ &= \sum_{i=1}^{n} (-1)^i [\varphi(v_0), \ldots, \hat{\varphi(v_i)}, \ldots, \varphi(v_n)] \\ &= \sum_{i=1}^{n} (-1)^i [\varphi(v_0), \ldots, \varphi(\hat{v}_i), \ldots, \varphi(v_n)] \\ &= \sum_{i=1}^{n} (-1)^i f_{n-1}[v_0, \ldots, \hat{v}_i, \ldots, v_n] \\ &= f_{n-1} \sum_{i=1}^{n} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_n] = f_{n-1}\partial_n(\sigma)\,. \end{aligned}$$

We now use the boundary operator to define homology groups.

Definition 5.5 (Cycle, Boundary and Homology groups)**.** *Denote by $Z_n(K) = \ker \partial_n$ the n^{th} cycle group and by $B_n(K) = Im\partial_{n+1}$ the n^{th} boundary group. Since $\partial_n \partial_{n+1} = 0$, we have $B_n(K) \subset Z_n(K)$ and hence $H_n(K) = Z_n/B_n$ is a well defined quotient group called the n^{th} homology group.*

Intuitively, one might think of H_0 as counting the number of connected components of the space, H_1 as counting the number of loops or "one dimensional holes", for example $H_1(\mathbb{S}^1) = \mathbb{Z}_2$, H_2 as counting the number of "voids", for example $H_2(\mathbb{S}^2) = \mathbb{Z}_2$ and so on.

Remark 5.2. If one works over fields, the chain group is a vector space with basis n-chains and the boundary operators are simply linear maps. Consequently, $Z_n(K)$ and $B_n(K)$ are simply subspaces of $C_n(K)$ obtained as kernels and images of linear maps and $H_n(K)$ is simply a quotient vector space. Computing kernels and images of linear maps is an exercise in linear algebra for which there are efficient algorithms, justifying the use of simplicial homology.

Formalism. Definition 5.5 is sometimes called the algebraic homology functor which is a functor from **Comp** to abelian groups **Ab**. Paying closer attention to our construction of simplicial homology, we note that homology is really the composition of two functors. We began by assigning to a simplical complex K, a chain complex $(C_\bullet, \partial)$. Then, with Definition 5.5 we assigned to the chain complex a homology group (the definition hinged on $\partial_n \partial_{n+1} = 0$, which holds for arbitrary chain complexes).

Example 6. Using the boundary matrices from earlier, computing the homology groups for our running example is simply row reduction (remember we are working over $\mathbb{Z}_2$). Note first that

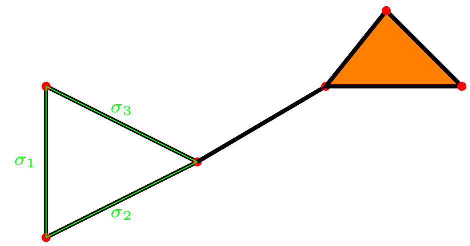


Figure 5.3: Simplicial complex with the basis for H_1 in green.

$H_n = 0\,\forall n \geq 2$, since $\partial_n = 0 \forall n \geq 3$. In Example 4 the kernels and images are:

$$Z_1 = \ker \partial_1 = \ker \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} = \langle \sigma_1 + \sigma_2 + \sigma_3, \sigma_5 + \sigma_6 + \sigma_7 \rangle$$

$$B_1 = \mathrm{Im}\partial_2 = \mathrm{Im} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \langle \sigma_5 + \sigma_6 + \sigma_7 \rangle$$

$$H_1 = Z_1/B_1 \cong \langle \sigma_1 + \sigma_2 + \sigma_3 \rangle \cong \mathbb{Z}_2 \,.$$

Notice how this agrees with the geometric notion that H_1 counts the number of loops or "1 dimensional holes" in a shape. The computation also gives us a basis chain that forms this hole and indeed $\sigma_1 + \sigma_2 + \sigma_3$ forms a loop. Finally note how quotienting out by the boundary removed $\sigma_5 + \sigma_6 + \sigma_7$ which was in Z_1, but does not form a "hole" since it is the boundary of the 2-simplex A.

$$Z_0 = \ker \partial_0 = C_0 = \langle v_0, v_1, v_2, v_3, v_4, v_5, v_6 \rangle$$

$$B_0 = \mathrm{Im} \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} = \langle v_1 + v_2, v_0 + v_1, v_1 + v_3, v_3 + v_4, v_4 + v_6, v_5 + v_6, v_4 + v_5 \rangle$$

$$H_0 = Z_0/B_0 \cong \langle v_0, v_1, v_2, v_3, v_4, v_5, v_6 \rangle / \langle v_1 + v_2, v_0 + v_1, v_1 + v_3, v_3 + v_4, v_4 + v_6, v_5 + v_6, v_4 + v_5 \rangle \cong \mathbb{Z}_2$$

This too corresponds to the geometric notion of counting the number of connected components.

Chapter 6

Introducing Persistent Homology with an Example

Before delving into the various theoretical viewpoints on persistent homology, we present an example that illustrates how persistent homology works intuitively.

As stated earlier, our goal is to measure the "shape of data". Data, for now, will mean *point cloud data*, that is a finite, discrete subset of $\mathbb{R}^n$. The example we will consider is that of a noisy circle, depicted in Figure 6.1a. Our tool should be able to capture the circle as the central feature of the data.

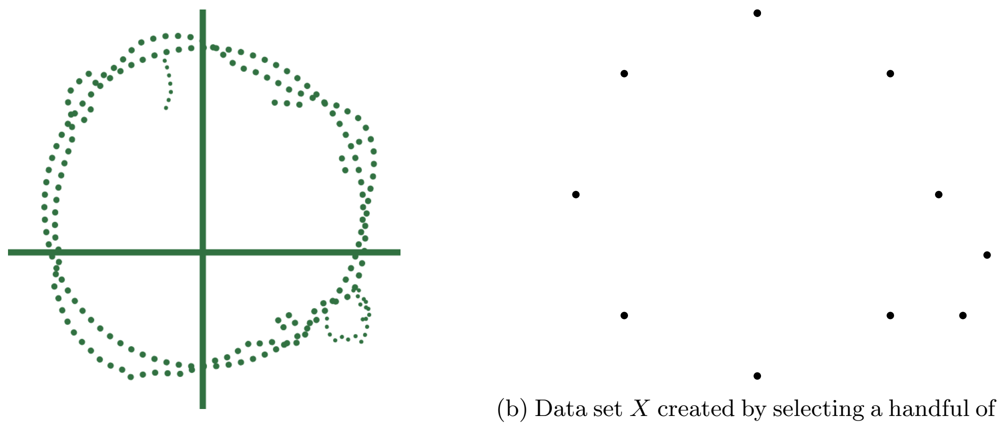
In order to visualize how persistent homology captures the circle and perform some calculations by hand, we use a data set composed of a handful of points selected from Figure 6.1a. This data set is depicted in Figure 6.1b and is denoted by X. A first naive approach might be to consider X as a simplicial complex composed simply of 10 zero simplices. Since the complex lacks higher dimensional simplices, the chain groups are trivial for $n > 0$ and is $C_0(X) = \mathbb{Z}_2^{\#\text{ of points}}$, $n = 0$. Since maps to or from a zero space must be zero, the boundary maps are all zero maps resulting in trivial homology groups H_n for $n > 0$ and $H_0(X) = \mathbb{Z}_2^{\#\text{ of points}}$. Thus, this approach provides no useful information. Persistent homology resolves this issue by extending simplicial homology to filtered simplicial complexes.

6.1 From Point Cloud Data to Filtered Simplicial Complexes

To resolve the problem arising from our discrete data set, we leverage the underlying metric $X \subset \mathbb{R}^n$ to construct a *filtered simplicial complex*, whose homology better captures the underlying shape of the data. Though there are multiple methods to construct this filtration, we describe the simplest construction - the *Vietoris-Rips* complex. We begin by defining filtrations.

Definition 6.1 (Filtration)**.** *A* filtration *of a complex K is a nested sequence of subcomplexes $\emptyset \subset K^0 \subset K^1 \subset \cdots \subset K^m = K$. A filtration is finite, if the sequence $K^0 \subset K^1 \subset \cdots \subset K^m$ is finite.*

Definition 6.2 (Vietoris-Rips Complex)**.** *Let X be a finite subset of $\mathbb{R}^n$ and $\epsilon > 0$. Denote by $VR(X, \epsilon)$ the simplicial complex whose vertices are simply the points in X. A subset $\{x_0, \ldots, x_n\} \subset$*



(a) Point cloud data resembling a noisy circle.

(b) Data set X created by selecting a handful of points from the noisy circle.

Figure 6.1

X spans a simplex in $VR(X, \epsilon)$ if and only if

$$d(x_i, x_j) < \epsilon \, \forall i, j \, .$$

By varying ϵ one obtains a collection of simplicial complexes $\{(VR(X, \epsilon))\}_\epsilon$.

Proposition 6.1. *Let $X \subset \mathbb{R}^n$ be a finite subset, then:*

1. *$VR(X, \epsilon) \to VR(X, \epsilon')$ is a subcomplex $\forall \epsilon \leq \epsilon'$. In other words, $\{(VR(X, \epsilon))\}_\epsilon$ is a filtered simplicial complex.*
2. *$\{(VR(X, \epsilon))\}_\epsilon$ is a finite filtration.*

Proof.

1. Let $0 < \epsilon \leq \epsilon'$. If $\{x_0, \ldots, x_n\} \subset X$ spans a simplex in $VR(X, \epsilon)$, then $d(x_i, x_j) < \epsilon \leq \epsilon' \, \forall i, j$. Thus, per definition $\{x_0, \ldots, x_n\}$ also spans a simplex in $VR(X, \epsilon')$.
2. Since X is finite, there exists some $\epsilon_N > 0$ such that $d(x, y) < \epsilon_N \forall x, y \in X$. Consequently, $VR(X, \epsilon) = VR(X, \epsilon_N) \, \forall \epsilon \geq \epsilon_N$. Due to the finiteness of X, new simplices can enter the filtration only a finite number of times.

■

Thus, given a finite data set X we can construct a finite filtered simplicial complex $\emptyset \subset K^0 \subset K^1 \cdots \subset K^m$. In the case of our example from Figure 6.1b, one obtains the filtration depicted in Figure 6.2, where the shapes filled in blue indicate 2-simplices, while purple indicates 3-simplices.

This example also highlights the reason for constructing a filtration - the isolation of noisy features. Indeed, we could instead just pick one particular ϵ and consider the homology of $VR(X, \epsilon)$. However, it is not clear how one should choose this ϵ. For example, ϵ_1 is not a good choice, since most of the points are still discrete and the central circle is not present. In addition to the central circle not being present in ϵ_2, there is also a smaller circle on the bottom right, owing to the noisy points. This noisy loop is filled in by ϵ_3, ϵ_4, and ϵ_5. Each of these also captures the main loop and may be considered “good choices”, but this is not mathematically rigorous and there is a priori no method in which one can always pick the ”good” choice. Hence, we work with the entire filtration and consider features which *persist* throughout the filtration.

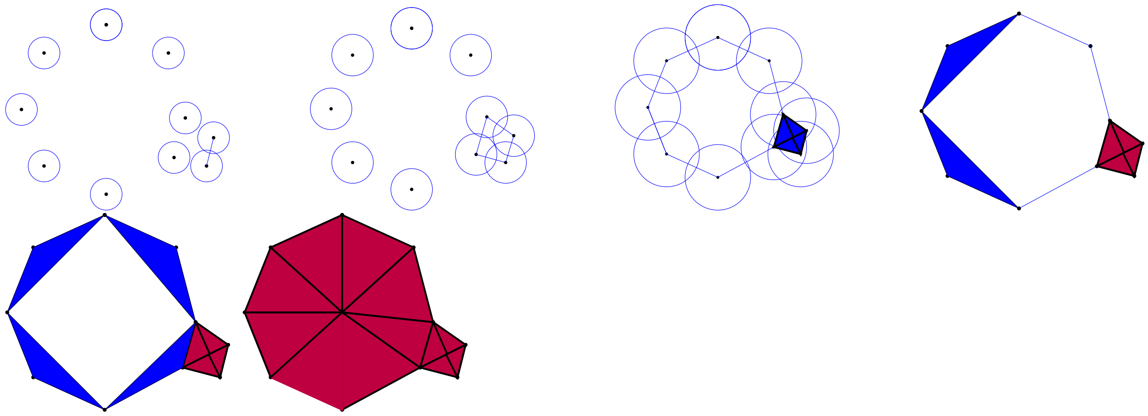


Figure 6.2: A filtration obtained from X by the Vietoris-Rips complexes.

6.1.1 Persistent Homology Groups

Having converted our initial point cloud data to a filtered simplicial complex, we are ready to discuss *persistent homology groups*. Let $\emptyset \subset K^0 \subset K^1 \cdots \subset K^m$ be a finite filtered simplicial complex and $n > 0$. Then, applying the n^{th} homology functor H_n to each of the subcomplexes in the filtration yields a sequence of abelian groups $\{H_n(K_i), \ldots H_n(K^m)\}$. Since H_n is a functor, we also get inclusion induced homomorphisms:

$$H_n(K^0) \to H_n(K^1) \to \cdots \to H_n(K^m)\,.$$

Remark 6.1. This is an example of a *persistence module*.

Observe that $H_n(K^i) = 0\,\forall n > 1$ and $\forall 1 \leq i \leq 6$, since there are no higher dimensional voids in any of the subcomplexes of the filtration. As an example, will now compute $H_0(K^1), H_1(K^1), H_0(K^2)$ and $H_1(K^2)$:

1. Let $i = 1$, then:

$$\begin{aligned} C_0 = \langle v_0, v_1, \ldots, v_9 \rangle &\cong \mathbb{Z}_2^{10} \\ C_1 = \langle e_1 \rangle &\cong \mathbb{Z}_2 \\ \partial_1 : C_1 &\to C_0 \\ e_1 &\mapsto < v_9 - v_8 \rangle \\ \partial_0 &= 0\,. \end{aligned}$$

   Consequently,

$$\begin{aligned} &H_0 = \ker \partial_0 / \mathrm{Im}\partial_1 = \langle v_0, v_1, \ldots, v_{10} \rangle / \langle v_{10} - v_9 \rangle \cong \mathbb{Z}_2^9 \\ &H_1 = \ker \partial_1 / \mathrm{Im}\partial_2 = 0\,. \end{aligned}$$

   Observe that this coincides with the intuitive understanding of H_0 counting connected components, and H_1 counting loops.

2. Now let $i = 2$, then:

$$\begin{aligned}
C_0 = \langle v_0, v_1, \ldots, v_9 \rangle \cong \mathbb{Z}_2^{10} \\
C_1 = \langle e_1, e_2, e_3, e_4 \rangle \cong \mathbb{Z}_2^4 \\
\partial_1 : C_1 \to C_0 \\
e_1 \mapsto < v_9 - v_8 \rangle, e_2 \mapsto \langle v_{10} - v_9 \rangle, e_3 \mapsto \langle v_7 - v_1 0 \rangle, e_4 \mapsto \langle v_8 - v_7 \rangle \\
\partial_0 = 0 \,.
\end{aligned}$$

Hence,

$$\begin{aligned}
&H_0 = \ker \partial_0 / \mathrm{Im} \partial_1 \cong \langle v_0, v_1, \ldots, v_{10} \rangle / \langle v_9 - v_8, v_{10} - v_9, v_7 - v_1 0, v_8 - v_7 \rangle \cong \mathbb{Z}_2^7 \\
&H_1 = \ker \partial_1 / \mathrm{Im} \partial_2 = \ker \partial_1 = \langle e_1 + e_2 + e_3 + e_4 \rangle \cong \mathbb{Z}_2 \,.
\end{aligned}$$

Proceeding with similar calculations one obtains:

$$H_0(K^i) \cong \begin{cases} \mathbb{Z}_2^9 & i = 1 \\ \mathbb{Z}_2^7 & i = 2 \\ \mathbb{Z}_2^1 & i \geq 3 \end{cases}$$

$$H_1(K^i) = \begin{cases} 0 & i = 1 \\ \mathbb{Z}_2^1 & 2 \leq i \leq 5 \\ 0 & i = 6 \end{cases} \,.$$

Although $H_1(K^1) \cong \mathbb{Z}_2 \cong H_1(K^i)$, $2 \leq i \leq 5$, the cycle that generates $H_1(K^1)$ is not the same as the one that generates $H_1(K^2), H_1(K^3), H_1(K^4)$ and $H_1(K^5)$. In particular, the cycle formed by the noisy points on the bottom right generates $H_1(K^1)$. On the other hand, $H_1(K^i)$, $2 \leq 5$ are generated by the cycle formed by the main loop. Having computed the homology for each K^i, we want to now see which cycles *persist* long through the filtration.

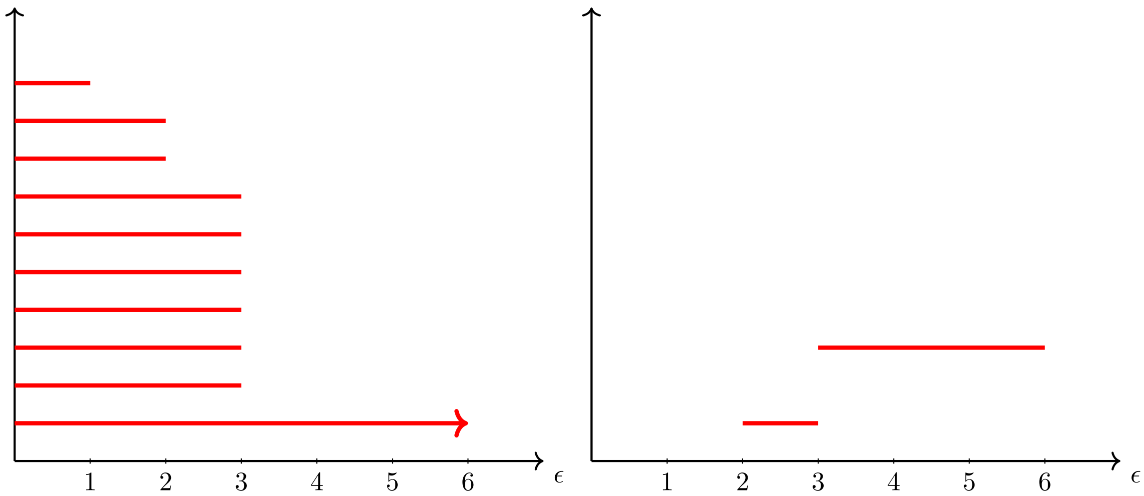
Definition 6.3 (Persistent homology groups)**.** *The p^{th} persistent, n^{th} homology group of K^i is*

$$H_n^p(K^i) = \frac{Z_n(K^i)}{B_n(K^{i+p}) \cap Z_n(K^i)} \,.$$

Remark 6.2. The definition of persistent homology groups given above is the original definition due to Edelsbrunner. Intuitively, $H_n^p(K^i)$ asks the following question: which cycles that were present in K^i (corresponding to $Z_n(K^i)$) are also cycles in K^{i+p}? If a cycle persists in K^{i+p}, then it is not in the boundary B_n^{i+p} and hence would be non trivial in $H_n^p(K^i)$. On the other hand, if the cycle becomes a bounding element in K^{i+p}, then it would be in the boundary group and be quotiented out in $H_n^p(K^i)$.

While this definition is intuitive, it is rather unwieldy to work with when proving theorems or constructing generalizations.

Computing $H_n^p(K^i)$ boils down to finding a common basis across $H_n(K^0) \to H_n(K^1) \to \cdots \to H_n(K^m)$. Using well established theorems from commutative algebra, we can prove the existence of such a basis (at least when working over fields) and construct the critical barcode representations.



(a) The H_0 barcode for X.

(b) The H_1 barcode for X.

6.2 Barcode Representation

Persistent homology groups are often represented as barcodes. We introduce this concept using our familiar example.

Consider the barcode for H_0 depicted in Figure 6.3a. Before K^1, there were 10 discrete points each generating one of the bars in the figure. At K^1, two of these points are connected by the edge e_1, and hence one connected component *dies*, leaving 9 bars alive. In K^2, another two points are connected, reducing the number of connected components to 7. Finally, in K^3 all of the points are connected up, leaving a single connected component that survives forever.

The barcode for H_1 is depicted in Figure 6.3b and shows one loop being formed in K^2 and dying in K^3. This short bar is attributed to noise, which is exactly what we want. The main feature of the central loop is captured by the long bar which begins in K^3 and survives until K^6 where it is filled in.

Chapter 7

Viewpoint 1: Algebra

In this chapter, we formalize the ideas presented in chapter 6 through the lens of algebra from chapter 3. Following [ZC05], we will prove the existence of barcode representations for classical single parameter persistence. The proof involves first establishing a correspondence between *persistence modules* and *graded modules* and then using the classification theorem for graded modules to classify peristence modules. This classification naturally leads to a barcode representation. While other methods, such as that of quiver representations, yield stronger results, the algebraic proof sheds light on the underlying structure of persistence modules and is helpful to gain an intuitive understanding of persistence theory. Throughout this chapter, we denote by R a principle ideal domain, k a field, and by $\emptyset \subset K^0 \subset K^1 \cdots \subset K^m$ a finite filtered simplicial complex.

7.1 Persistence Modules

We begin by reintroducing persistent homology in the language of persistence modules.

Analogous to the case in simplicial homology, where we first constructed chain complexes, we first construct a persistence complex.

Definition 7.1 (Persistence complex)**.** *A persistence complex is a family of chain complexes $\{(C^i_\bullet, \partial)\}_i$ along with chain maps $f^i : C^i_\bullet \to C^{i+1}_\bullet \ \forall i$.*

Every finite filtration $\emptyset \subset K^0 \subset K^1 \cdots \subset K^m$ yields a finite persistence complex. From every K^i one can construct a chain complex $C^i_\bullet$ with Definition 5.2. The chain map $f^i : C^i_\bullet \to C^{i+1}_\bullet$ is simply an inclusion map, since every n-simplex in K^i is included in K^{i+1} and this inclusion naturally extends to n-chains which are just linear combinations of n-simplices. This persistence complex is depicted in Figure 7.1 where moving to the right increases the filtration index while moving down drops down in dimension. Again drawing an analogy to simplicial homology, we now apply the algebraic homology functor to each of the chain complexes $C^i_\bullet = C_\bullet(K^i)$ in the persistence complex to obtain $H^i_n = H_n(K^i)$. By functoriality, the chain maps from the persistence complex induce maps in homology $H^i_n \to H^{i+1}_n$, leading to the definition of *persistence modules*.

Definition 7.2 (Persistence module)**.** *A* persistence module $\mathcal{M}$ *over a ring R is a family of R-modules $\{M^i\}_i$, with R-linear maps $\varphi^i : M^i \to M^{i+1}$. A persistence module is of* finite type *if each M^i is a finitely generated R-module and there exists an m such that $\varphi^i : M^i \to M^{i+1}$ is an isomorphism for all $i \geq m$.*

$$\begin{tikzcd}
 & \cdots \arrow[d, "\partial_{n+2}"] & \cdots \arrow[d, "\partial_{n+2}"] & \cdots \arrow[d, "\partial_{n+2}"] & \cdots \arrow[d, "\partial_{n+2}"] & \\
\cdots \arrow[r] & C^{i-1}_{n+1} \arrow[r, "f^{i-1}_{n+1}"] \arrow[d, "\partial_{n+1}"] & C^{i}_{n+1} \arrow[r, "f^{i}_{n+1}"] \arrow[d, "\partial_{n+1}"] & C^{i+1}_{n+1} \arrow[r, "f^{i+1}_{n+1}"] \arrow[d, "\partial_{n+1}"] & C^{i+2}_{n+1} \arrow[r] \arrow[d, "\partial_{n+1}"] & \cdots \\
\cdots \arrow[r] & C^{i-1}_{n} \arrow[r, "f^{i-1}_{n}"'] \arrow[d, "\partial_{n}"] & C^{i}_{n} \arrow[r, "f^{i}_{n}"'] \arrow[d, "\partial_{n}"] & C^{i+1}_{n} \arrow[r, "f^{i+1}_{n}"'] \arrow[d, "\partial_{n}"] & C^{i+2}_{n} \arrow[r] \arrow[d, "\partial_{n}"] & \cdots \\
\cdots \arrow[r] & C^{i-1}_{n-1} \arrow[r, "f^{i-1}_{n-1}"'] \arrow[d] & C^{i}_{n-1} \arrow[r, "f^{i}_{n-1}"'] \arrow[d] & C^{i+1}_{n-1} \arrow[r, "f^{i+1}_{n-1}"'] \arrow[d] & C^{i+2}_{n-1} \arrow[r] \arrow[d] & \cdots \\
 & \cdots & \cdots & \cdots & \cdots &
\end{tikzcd}$$

Figure 7.1: A persistence complex

Remark 7.1. The persistence modules arising from finite filtered simplicial complexes are always of finite type. Since the simplicial complexes are all finite, there exists some $n > 0$ with $C^i_n = 0\,\forall i$, where $C^i_\bullet$ is the persistence complex associated to the filtration. Hence the persistence module $\mathcal{M}$ obtained from this persistence complex must be of finite type.

7.2 Correspondence and decomposition

Obtaining a barcode representation, involves classifying all persistence modules of finite type. In order to do this, we establish a categorical equivalence between finite type persistence modules and finitely generated graded modules and then exploit the strong classification results for finitely generated graded modules from chapter 3.

Definition 7.3 (Category of finite type persistence modules)**.** *The category $\boldsymbol{Pers}_R$ has as objects finite type persistence modules over a ring R. A morphism $f : \mathcal{M} \to \mathcal{N}$ $\boldsymbol{Pers}$ is a family of R-linear maps $f^i : M_i \to N_i$ such that the following diagram commutes for every i*

$$\begin{tikzcd}
M^i \arrow[r, "\varphi^i"'] \arrow[d, "f^i"] & M^{i+1} \arrow[d, "f^{i+1}"'] \\
N^i \arrow[r, "\psi^i"] & N^{i+1}
\end{tikzcd}$$

We will now establish a correspondence between $\mathbf{Pers}_R$ and $\mathbf{Grad}_{R[t]}$, the category of graded $R[t]-$modules.

Theorem 7.1 (Correspondence). The functor

$$\alpha : \mathbf{Pers}_R \longrightarrow \mathbf{Grad}_{R[t]}$$
$$\mathcal{M} \mapsto \bigoplus_i M^i$$
$$f : \mathcal{M} \to \mathcal{N} \mapsto \alpha(f) : \bigoplus_i M^i \to \bigoplus_i N^i$$

where

$$\alpha(f)(m^0, m^1, m^2, \dots) = (f^0(m^0), f^1(m^1), f^2(m^2), \dots)$$

induces an equivalence of categories.

Proof. α is clearly a functor. In order to prove equivalence, it suffices to show that α is a full, faithful and essentially surjective functor. Let $\mathcal{M}, \mathcal{N} \in \mathbf{Pers}_R$ and consider the induced map

$$\alpha^* : \mathrm{Hom}(\mathcal{M}, \mathcal{N}) \to \mathrm{Hom}(\alpha(\mathcal{M}), \alpha(\mathcal{N}))$$

Full: We need to show that α^* is surjective. Let $f \in \mathrm{Hom}(\bigoplus_i M^i, \bigoplus_i N^i)$, then $f(M_i) \subset N_i$. Define the map

$$f_\# : \mathcal{M} \to \mathcal{N}$$
$$f_\#^i : M^i \to N^i$$
$$m^i \mapsto f(m^i)$$

Clearly, $\alpha(f_\#) = f$.

Faithful: We need to show that α^* is injective, but this is clear since $f \in \ker \alpha* \iff f^i = 0 \forall i \iff f = 0$.

Essentially surjective: Let $M = \bigoplus_i M^i \in \mathbf{Grad}_{R[t]}$, we need to show that there exists $\mathcal{M} \in \mathbf{Pers}_R$ with $\alpha(\mathcal{M}) \cong M$. Define

$$\mathcal{M} = \{M^i\}_i$$
$$\varphi^i : M^i \to M^{i+1}$$
$$m^i \mapsto tm^i$$

Clearly, $\alpha(\mathcal{M}) = M$. ■

Having established this equivalence of categories, the structure theorem for finitely generated $R[t]$ modules delivers a basis for the persistence module. However, Theorem 3.1 only applies when $R[t]$ is a PID. This is the case only when R is a field. This is the reason for Remark 5.2. If this is the case, then applying Theorem 3.1 yields:

$$M \cong \bigoplus_{i=1}^{n} \Sigma^{a_i} F[t] \oplus \bigoplus_{j=1}^{m} \Sigma^{b_j} F[t]/t^{n_j} \,. \tag{7.1}$$

We can now associate the decomposition in Equation 7.1 to an interval decomposition, and hence a barcode.

Definition 7.4 (Barcode)**.** *A* barcode *is a multiset of $\mathbb{Z}$-intervals. Intervals are of the form* $(i, j)\, i, j \in \mathbb{Z} \cup \{\infty\}\;, 0 \leq i < j$

Proposition 7.1. *There exists a bijection between the (finite) set of barcodes and the set of finitely generated graded $F[t]-$modules.*

Proof. Let $I \subset \mathbb{Z}$ be an interval, then we can construct a bijection Ψ as follows:

$$\Psi(I) = \begin{cases} \Sigma^i F[t]/t^{j-i} & \mathcal{I} = (i,j) \text{ where } i,j \text{ finite} \\ \Sigma^i F[t] & \mathcal{I} = (i,\infty) \end{cases}$$

We can also define its inverse. Let $M \cong \bigoplus_{i=1}^{n} \Sigma^{a_i} F[t] \oplus \bigoplus_{j=1}^{m} \Sigma^{b_j} F[t]/t^{n_j}$. Then the corresponding barcode consists of m intervals of the form (i, j) corresponding to the m terms in $\bigoplus_{j=1}^{m} \Sigma^{b_j} F[t]/t^{n_j}$ and n intervals of the form (a_j, ∞) corresponding to the n terms in $\bigoplus_{i=1}^{n} \Sigma^{a_i} F[t]$. We denote the barcode of a persistence module $\mathcal{M}$ by barc($\mathcal{M}$). ∎

Remark 7.2. Combining Theorem 7.1 and Proposition 7.1, we can conclude that each isomorphism class of persistence modules corresponds to one of finitely many barcodes. In other words, the barcode is a *complete invariant* of finite type persistence modules.

7.3 Translating between algebra and geometry

One might think of the graded $F[t]$-module M as the algebraic representation of our filtration. Each element $x \in M$ corresponds to some cycle in the persistence module $\mathcal{M}$. In each gradation, M^i is the algebraic representation of H_n^i in the filtration. An α upward shift in the gradation, denoted by Σ^α, is then the same as moving to H^α in the filtration. Finally, the action of t captures the inclusion induced maps $H_n^i \to H_n^{i+1}$. In other words, applying t to an element $x \in M^i$ is the algebraic equivalent of applying the inclusion induced map to the corresponding cycle $\langle c \rangle \in H_n^i$.

After decomposing M according to Proposition 7.1, each cycle class $\langle c \rangle$ corresponds to a particular $\Sigma^{a_i} F[t]$ or $\Sigma^{b_j} F[t]/d_j F[t]$. How do we switch between the geometric notion of a cycle being born and dying and the decomposition from Proposition 7.1?

In a first step, we consider the geometric notion of a cycle z being born in K^b. On the algebra side the homology class $\langle c \rangle$ represented by $F[t]$ must be shifted b along the gradation so that it only generates elements in the module after this point. Hence, write $\Sigma^b F[t]$ to represent $\langle c \rangle$ in the graded module. If $\langle c \rangle$ never dies (that is it lives in the final K^n), then it is simply represented by $\Sigma^b F[t]$ and forms part of the first summand (or free part) in the decomposition.

But this is only half of the story. We also need a way to algebraically describe the death of a cycle $\langle c \rangle = 0$ in K^d. We achieve this by modding by t^{d-b} and choosing $\Sigma^b F[t]/t^{d-b}$ to denote a cycle born in K^b and dying in K^d. Indeed, consider an element $f \in \Sigma^b F[t]/t^{d-b}$, then applying t $(d - b)$ times to this element will send it to 0 because we are quotienting out by this term. Recall now that applying t is the algebraic equivalent of applying the inclusion induced map $H_n^b \to H_n^{b+1}$ to $\langle c \rangle$. Applying t^{d-b} thus takes us to H_n^d where the cycle dies.

Chapter 8

Zig-Zag and Multiparameter Persistence

The persistent homology we have discussed so far is sometimes called single-parameter persistence. It is the form that arises most directly from the study of data and has proven extremely powerful; however, it does have some weaknesses. These weaknesses led to the development of zig-zag and multiparameter persistence. In this chapter, we introduce and motivate these two concepts.

8.1 Zig-Zag persistence

Introduced in [CS10], zig-zag persistence is a very natural extension of single parameter persistence. Zig-zag persistence modules even have a barcode representation similar to those single-parameter persistence modules, which makes zig-zag persistence particularly powerful.

Zig-zag persistence arises quite often in data analysis. We present one of these instances, topological bootstrapping, as motivation. Suppose that we are given an extremely large data set X, so that single parameter persistence, which involves constructing a VR complex, is computationally infeasible. A naive approach to tackle this issue might be to consider a collection of samples $X_1, \ldots, X_n$ of X and apply single parameter persistence to each one individually. Then, if the barcodes of the samples are similar to each other, one might deduce that the barcodes of the individual samples are a good approximation of the barcode of X. However, this does not distinguish between a single feature appearing n times, once in each X_i, and multiple similar features spread out across the samples.

Instead, consider the union sequence:

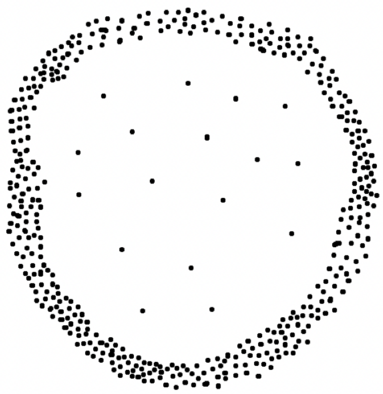
$$X_1 \to X_1 \cup X_2 \leftarrow X_2 \to X_2 \cup X_3 \leftarrow X_3 \to \cdots \leftarrow X_n \,.$$

This sequence gives rise to a family of simplicial complexes:

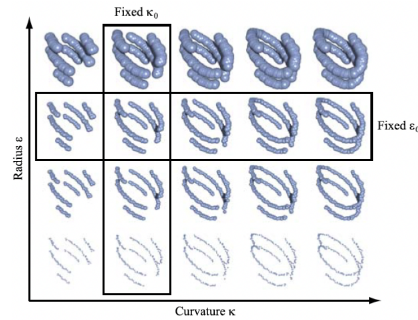
$$K^0 \hookrightarrow K^{0,1} \hookleftarrow K^1 \hookrightarrow K^{1,2} \hookleftarrow K^2 \hookrightarrow \cdots \hookleftarrow K^{n-1,n} \hookleftarrow K_n \,. \tag{8.1}$$

Applying the simplical homology functor to Equation 8.1 yields:

$$H_k^0 \to H_k^{0,1} \leftarrow H_k^1 \to H_k^{1,2} \leftarrow H_k^2 \to \cdots \leftarrow H_k^{n-1,n} \to H_k^n \,. \tag{8.2}$$



(a) A circle with noisy points in the center.



(b) A data set filtered by curvature and radius.

Figure 8.1

Notice that the inclusion induced maps in Equation 8.2 solve the problem we mentioned earlier.

Equation 8.2 is a prime example of a *zig-zag module*

Definition 8.1 (Zig-zag module)**.** *Let k be a field. Then, a* zig-zag module $\mathbb{V}$ *over k is a family of vector spaces $\{V_i | 1 \leq i \leq n\}$ connected by linear maps $V_i \to V_{i+1}$ or $V_i \leftarrow V_{i+1}$ for each i. Visually,*

$$V_1 \longleftrightarrow V_2 \longleftrightarrow V_3 \longleftrightarrow \cdots \longleftrightarrow V_1$$

where $\longleftrightarrow$ denotes a map that can be in either direction.

8.2 Multiparameter persistence

Multiparameter persistence was introduced in [CZ09] as another extension to persistence theory.

One severe drawback of single parameter persistence is its limitation to data sets parameterized by a single variable. However, as one might imagine, many data sets one encounters in the wild are multidimensional. Applying single parameter persistence to a multidimensional data set requires choosing one dimension and ignoring the others, losing valuable information in the process. One example of a multidimensional data set, taken from [CZ09], is depicted in Figure 8.1a.

There is another less obvious application of multiparameter persistence. Consider Figure 8.1a taken from [Car14]. Visually, we see that the data might be sampled from a circle. However, due to the presence of noisy points in the middle of the circle and on the outside, the barcode obtained by single parameter persistence will not detect the main loop feature. One way we might tackle this problem is to focus on the densest $p\%$ of the points and discard the rest. With the right *choice* of p, we can remove the noisy central points and hence capture the circle. But how do we choose p? Ideally, we do not want to make a choice, just as with the radius r of the balls in the Vietoris-Rips complex $VR(X, r)$. This is where multiparameter persistence comes in. In this example, we may build a bifiltration, filtering both by p and by r.

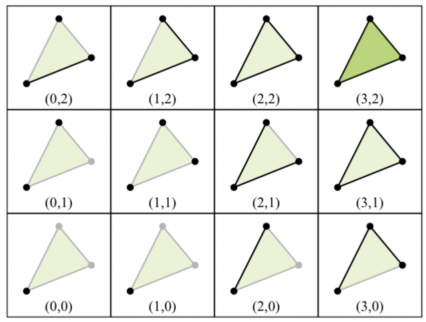


Figure 8.2: A bifiltration of a triangle.

Definition 8.2 (Multiparameter persistence module)**.** *A multiparameter persistence module* $\mathbb{V}$ *over a field* k *is a collection of vector spaces* $\{V_\mu | \mu \in \mathbb{N}^n, n \geq 0\}$ *together with linear maps* $\phi_{\mu,\nu} : V_\mu \to V_\nu$ *for all* $\mu \leq \nu$*, such that* $\phi_{\mu,\upsilon} = \phi_{\nu,\upsilon} \circ \phi_{\mu,\nu} \, \forall \, \mu \leq \nu \leq \upsilon$*.*

Note that $\leq$ is the partial order on $\mathbb{N}^n$, defined by $\mu \leq \nu \iff \mu_i \leq \nu_i \, \forall \, 0 \leq i \leq n$.

We end this section by constructing a 2-parameter persistence module for the bifiltration of a triangle depicted in Figure 8.2. Obtaining such a multifiltered simplicial complex from multidimensional data is straightforward.

Recalling that 0 dimensional homology amounts to counting the number of connected components, the 2-parameter persistence module obtained by attaching the 0 dimensional homology groups of each element of the bifiltration is:

$$\begin{tikzcd}
k^2 \arrow[r] & k \arrow[r] & k \arrow[r] & k \\
k^2 \arrow[r] \arrow[u] & k^3 \arrow[r] \arrow[u] & k \arrow[r] \arrow[u] & k \arrow[u] \\
k \arrow[r] \arrow[u] & k \arrow[r] \arrow[u] & k \arrow[r] \arrow[u] & k \arrow[u]
\end{tikzcd}$$

Chapter 9

Viewpoint 2: Quiver Representations

In earlier chapters, we saw that for classical persistent homology, commutative algebra provides us with a barcode representation of persistence modules. We then introduced the idea of zig-zag persistence and remarked that it too has an analogous barcode representation. Although one can prove this constructively, we will follow [Oud15] and present a quiver theoretic proof. This will also deliver a proof of the non-existence of barcodes in the multiparameter case.

9.1 Persistence modules as quiver representations

Classical, zig-zag, and multiparameter persistence are all examples of quiver representations. Classical and zig-zag persistence are representations of A_n quivers, which are depicted in Figure 9.1a. In particular, classical persistence modules are representations of A_n quivers, whose arrows are all forward facing while zig-zag persistence modules are presentations of A_n quivers where arrows can be in either direction. Multiparameter persistence modules are representations of quivers of the form:

$$
\begin{tikzcd}
 & \cdots \arrow[d, dash] & \cdots \arrow[d, dash] & \cdots \arrow[d, dash] & \\
\cdots \arrow[r, dash] & \bullet \arrow[r, dash] \arrow[d, dash] & \bullet \arrow[r, dash] \arrow[d, dash] & \bullet \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[d, dash] \\
\cdots \arrow[r, dash] & \bullet \arrow[r, dash] \arrow[d, dash] & \bullet \arrow[r, dash] \arrow[d, dash] & \bullet \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[d, dash] \\
\cdots \arrow[r, dash] & \bullet \arrow[r, dash] \arrow[d, dash] & \bullet \arrow[r, dash] \arrow[d, dash] & \bullet \arrow[r, dash] \arrow[d, dash] & \cdots \\
 & \cdots & \cdots & \cdots &
\end{tikzcd}
$$

We aim to derive a general classification theorem analogous to the one presented in Proposition 7.1. In the language of quiver theory, this amounts to classifying the isomorphism classes of quiver representations.

9.2 Classification of indecomposable representations

By Proposition 4.1, we can focus on classifying indecomposable representations. The central classification theorem is due to Gabriel and deals with *finite type quivers*.

Definition 9.1 (Finite type quivers)**.** *A quiver* $\mathtt{Q}$ *is of* finite type *if it has a finite number of isomorphism classes of indecomposable finite dimensional representations.*

Theorem 9.1 (Gabriel). Let $\mathtt{Q}$ be a finite and connected quiver, then $\mathtt{Q}$ is finite type $\iff$ $\mathtt{Q}$ is Dynkin. We say $\mathtt{Q}$ is Dynkin if $\bar{\mathtt{Q}}$ is one of the diagrams in Figure 9.1. Note that arrows without heads denote arrows that can be in either direction.

Remark 9.1. Note that A_n quivers are Dynkin, so classifying the representations of Dynkin quivers will classify important classes of persistence modules.

Proving this theorem will require a lot of machinery which we now develop.

9.3 Distinguishing Dynkin quivers algebraically

The notion of a quiver being Dynkin is currently a purely visual one. In a first step, we introduce the *Tits form*, a quadratic form on quivers which gives us an algebraic characterization of Dynkin quivers. It will turn out to deliver much more than bargained for. In particular, once we prove Theorem 9.1, the roots of the Tits form on A_n quivers, give us the familiar barcode representations.

Let $\mathtt{Q}$ be a quiver and $\mathbb{V} \in \mathbf{rep}_k(\mathtt{Q})$, then the identification $\mathbb{V} \mapsto \mathbf{dim}\mathbb{V} \in \mathbb{Z}^n$ yields a vector on which we can define bilinear forms.

Definition 9.2 (Euler and Tits forms)**.** *Let* $\mathtt{Q}$ *be a quiver, then its* Euler form *is a bilinear map*

$$\begin{aligned}\langle x, y\rangle_{\mathtt{Q}} : \mathbb{Z}^n \times \mathbb{Z}^n &\to \mathbb{Z}\\ (x, y) &\mapsto \sum_{i \in Q_0} x_i y_i - \sum_{a \in Q_1} x_{t_a} y_{h_a}\,.\end{aligned}$$

Its symmetrization:

$$(x, y)_{\mathtt{Q}} = \langle x, y\rangle_{\mathtt{Q}} + \langle y, x\rangle_{\mathtt{Q}}$$

is called the symmetric Euler form. *Denote by* $q_{\mathtt{Q}}$*, the Tits form, which is the quadratic form associated with the Euler form:*

$$q_{\mathtt{Q}}(x) = \langle x, x\rangle_{\mathtt{Q}} = \frac{1}{2}(x, x)_{\mathtt{Q}}\,.$$

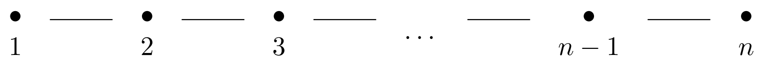
Lemma 9.1 (Formula for the Tits form). We can rewrite the symmetric Euler form using a matrix.

$$(x, y)_{\mathtt{Q}} = x^{\intercal} C_{\mathtt{Q}} y$$

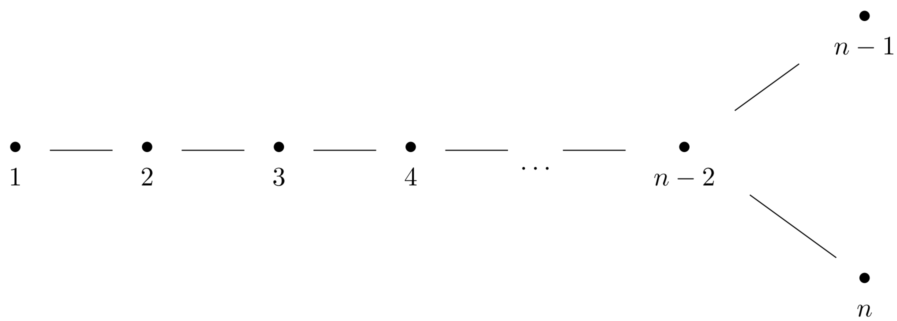
where the entries of $C_{\mathtt{Q}}$ are given by:

$$c_{ij} = \begin{cases} 2 - 2|\{\text{loops at } i\}| & i = j \\ -|\{\text{arrows between } i \text{ and } j\} & i \neq j \end{cases}.$$

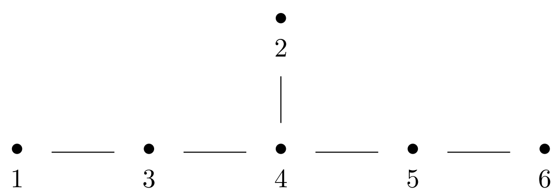
Consequently $q_{\mathtt{Q}}(x) = \frac{1}{2} x^{\intercal} C_{\mathtt{Q}} x$.



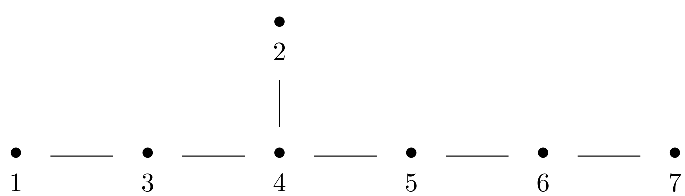
(a) A_n, $n > 0$



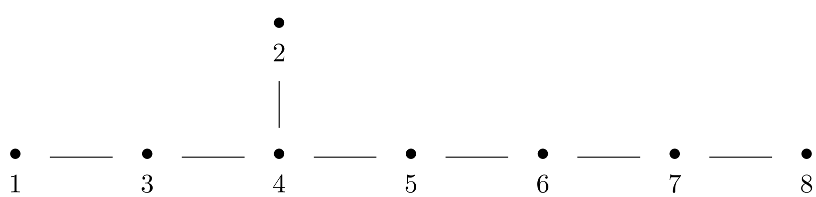
(b) D_n, $n \geq 4$



(c) E_6



(d) E_7



(e) E_8

Figure 9.1: The Dynkin diagrams.

Proof. We begin by writing out the formula:

$$
\begin{aligned}
(x, y)_{\mathtt{Q}} &= \sum_{i \in Q_0} x_i y_i - \sum_{a \in Q_1} x_{t_a} y_{h_a} + \sum_{i \in Q_0} y_i x_i - \sum_{a \in Q_1} y_{t_a} x_{h_a} \\
&= 2 \sum_{i \in Q_0} x_i y_i - \sum_{a \in Q_1} x_{t_a} y_{h_a} + y_{t_a} x_{h_a}
\end{aligned}
$$

The positive term $2\sum_{i \in Q_0} x_i y_i$ is obtained in the matrix form by setting $c_{ij} = 2$. The $-2|\{\text{loops at } i\}|$ is relevant only if there is a loop, that is an arrow $a \in Q_1$ such that $t_a = h_a = i$, $i \in Q_0$. In this case, we have $x_{t_a} y_{h_a} + y_{t_a} x_{h_a} = x_i y_i + y_i x_i = 2x_i y_i$ from the second term which we then need to subtract. Finally, the non-diagonal entries subtract the number of arrows between i and j. Let $a \in Q_1$ with $t_a = i$, $h_a = j$ then by subtracting $|\{\text{arrows between } i \text{ and } j\}|$, we get two negative terms $x_{t_a} y_{h_j} = x_i y_j$ and $y_{t_a} x_{h_j} = y_i x_j$. Note that symmetry is guaranteed since an arrow between i and j is also an arrow between j and i. ■

Example 7. We now calculate $q_{\mathtt{Q}}$ for the all important A_n type quivers. Observe that:

1. A_n-type quivers have no loops.
2. Arrows only exist between adjacent vertices, that is arrows only exist between vertices i and $i + 1$ for $i \in [1, n)$.

The formula from Lemma 9.1 yields:

$$
q_{\mathtt{Q}}(x) = \frac{1}{2} x^{\intercal} C_{\mathtt{Q}} x
$$

$$
c_{ij} = \begin{cases} 2 & i = j \\ -1 & j = 1 + 1 \\ 0 & \text{otherwise} \end{cases}.
$$

Hence with some manipulation:

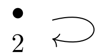
$$
\begin{aligned}
q_{\mathtt{Q}}(x) &= \frac{1}{2}(2\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n-1} x_i x_{i+1} + x_{i+1} x_i) \\
&= \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n-1} x_i x_{i+1} \\
&= \sum_{i=1}^{n-1} \frac{1}{2}(x_i - x_{i+1})^2 + \frac{1}{2}(x_1^2 + x_n^2)\,.
\end{aligned}
$$

Definition 9.2 does not depend on arrow orientations, hinting at their ability to distinguish Dynkin quivers from the rest. This is the content of the next theorem.

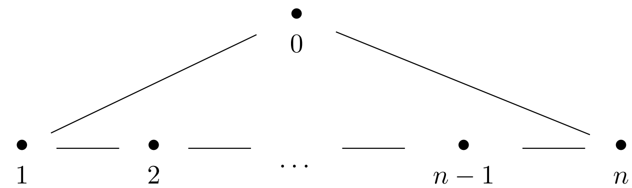
Theorem 9.2. Let $\mathtt{Q}$ be a finite, connected quiver. Then $\mathtt{Q}$ is Dynkin $\iff$ $q_{\mathtt{Q}}$ is positive definite.

In order to prove Theorem 9.2 we need the following definition and lemma.

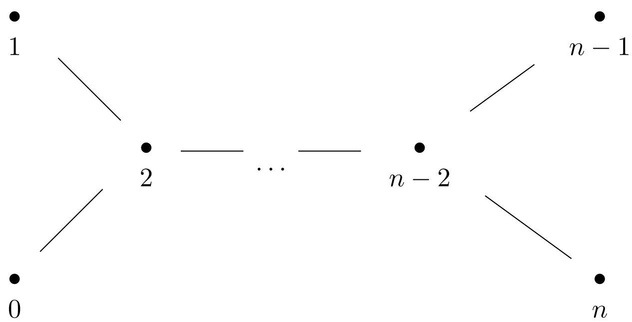
Definition 9.3 (Euclidean quivers)**.** *A quiver* $\mathtt{Q}$ *is* Euclidean *if its underlying graph is one of the diagrams depicted in Figure 9.2.*



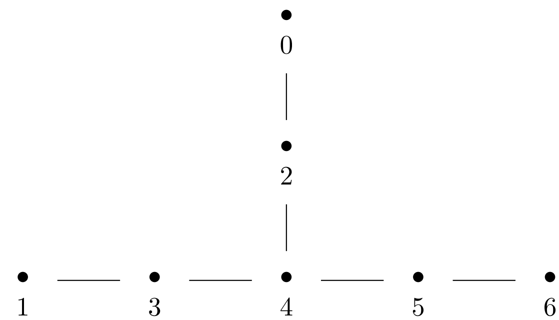
(a) $\tilde{A}_0$



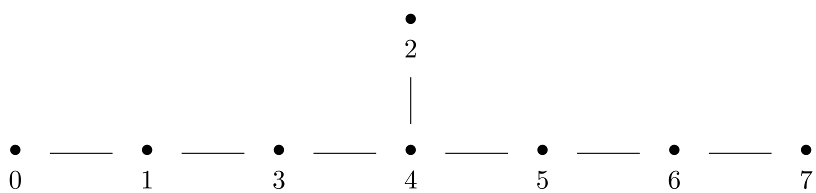
(b) $\tilde{A}_n$, $n > 0$



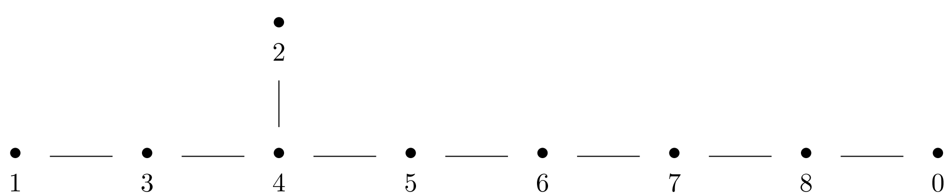
(c) $\tilde{D}_n \, n \geq 4$



(d) $\tilde{E}_6$



(e) $\tilde{E}_7$



(f) $\tilde{E}_8$

Figure 9.2: The Euclidean diagrams.

Lemma 9.2. Let $\mathtt{Q}$ be an Euclidean quiver, then $q_{\mathtt{Q}}$ is positive semidefinite.

Proof. We claim that $q_{\mathtt{Q}}(x) = 0$, where $x = \begin{cases} 1 & \mathtt{Q} = \tilde{A}_0 \\ (1, 1 \ldots, 1, 1)^{\intercal} & \mathtt{Q} = \tilde{A}_n \; n \geq 0 \\ (1, 1, 2, \ldots, 2, 1, 1)^{\intercal} & \mathtt{Q} = \tilde{D}_n, \; n \geq 4 \\ (1, 1, 2, 2, 3, 2, 1)^{\intercal} & \mathtt{Q} = \tilde{E}_6 \\ (1, 2, 2, 3, 4, 3, 2, 1)^{\intercal} & \mathtt{Q} = \tilde{E}_7 \\ (1, 2, 3, 4, 6, 5, 4, 3, 2)^{\intercal} & \mathtt{Q} = \tilde{E}_8 \end{cases}$.

We verify the claim for the first three cases with the last $\tilde{E}_7$ and $\tilde{E}_8$ being similar to $\tilde{E}_6$.

1. $q_{\tilde{\mathtt{A}}_0}(x) = 1^2 - 1^2 = 0$
2. $q_{\tilde{\mathtt{A}}_n}(x) = \sum_{i \in Q_0} x_i^2 - \sum_{a \in Q_1} x_{t_a} x_{h_a} = \sum_{i=0}^{n} x_i - \sum_{i=0}^{n-1} x_i x_{i+1} - x_n x_0 = n - (n-1) - 1 = 0$
3. $q_{\tilde{\mathtt{D}}_n}(x) = x_1 + x_2 + x_n + x_{n-2} + \sum_{i=2}^{n-2} x_i^2 - \sum_{a \in Q_1} x_{t_a} x_{h_a} = 4 + \sum_{i=2}^{n-2} 2^2 - 8 - \sum_{i=2}^{n-3} 2^2 = 0$
4. $q_{\tilde{E}_6}(x) = 1 + 1 + 2^2 + 2^2 + 3^2 + 2^2 + 1 - x_1 x_3 - x_3 x_4 - x_0 x_2 - x_2 x_4 - x_4 x_5 - x_5 x_6 = 24 - 2 - 6 - 2 - 8 - 8 - 2 = 0$.

■

Proof of Theorem 9.2. The " $\Longrightarrow$ " direction is proven by calculating $q_{\mathtt{Q}}$ for each Dynkin quiver. In Example 7, we performed this calculation for A_n quivers, which as a sum of squares, is clearly positive definite. We omit the tedious calculations for the other Dynkin quivers.

We prove " $\Longleftarrow$ " using contraposition by finding an element $x \in \mathbb{Z}^n$ with $q_{\mathtt{Q}}(x) \leq 0$ for an arbitrary non-Dynkin quiver $\mathtt{Q}$. The first step is to observe that the underlying graph $\bar{\mathtt{Q}}$ of every non-Dynkin quiver $\mathtt{Q}$ contains an Euclidean diagram as a subgraph.

Indeed, if $\bar{\mathtt{Q}}$ contains a loop then it must contain $\tilde{A}_0$ as a subgraph. If $\tilde{\mathtt{Q}}$ contains a cycle, that is a path such that the first and last vertices are equal, then it must contain $\tilde{A}_n$ as a subgraph. Next, $\tilde{\mathtt{Q}}$ contains at least two branch vertices, that is two vertices that are each connected to at least 3 distinct vertices, then it must contain $\tilde{D}_n$. Finally, if it contains exactly one branch vertex, then it must contain $\tilde{E}_n$, $6 \leq n \leq 8$.

Let $\mathtt{Q}'$ be the subquiver of $\mathtt{Q}$ corresponding to the Euclidean subgraph of $\bar{\mathtt{Q}}$. By Lemma 9.2, there exists $x' \in \mathbb{Z}^{|Q_0'|}$ with $q_{\mathtt{Q}'} = 0$. We can define $x \in \mathbb{Z}^{|Q_0|}$, with $q_{\mathtt{Q}}(x) \leq 0$ as follows:

1. If $\mathtt{Q}' = \mathtt{Q}$, we can simply set $x = x'$.
2. Suppose $Q_0' = Q_0$ and $Q_1' \subsetneq Q_1$. Since the vertex sets are equal we can set $x = x'$. Then $q_{\mathtt{Q}}(x) < q_{\mathtt{Q}'}(x') = 0$, since the positive term in Definition 9.2 remains the same, while the negative term is larger due to the increase in the number of arrows.
3. Finally suppose $Q_0' \subsetneq Q_0$. Denote by $x'' \in \mathbb{Z}^{|Q_0|}$, the vector with $x_i'' = x_i' \, \forall i \leq |Q_0'|$ and $x_i'' = 0 \, \forall i > |Q_0'|$. Since $\mathtt{Q}$ is connected, there exists a vertex $j \in \mathtt{Q} \setminus \mathtt{Q}'$ which is connected to a vertex $k \in \mathtt{Q}'$ by an arrow b. Set $x = 2x'' + e$, where e is the unit vector with 0's everywhere

except in the j^{th} position.

$$\begin{aligned} q_{\mathtt{Q}}(x) &= \sum_{i\in Q_0'} x_i^2 - \sum_{a\in Q_1'} x_{t_a}x_{h_a} + \sum_{i\in Q_0\setminus Q_0'} x_i^2 - \sum_{a\in Q_1\setminus Q_1'} x_{t_a}x_{h_a} \\ &= \underbrace{\sum_{i\in Q_0'} (2x_i''+e_i)^2 - \sum_{a\in Q_1'} x_{t_a}x_{h_a}}_{\leq 4q_{\mathtt{Q}'}(x')} + \underbrace{\sum_{i\in Q_0\setminus Q_0'} x_i^2}_{0\,\forall i\neq j} - \underbrace{\sum_{a\in Q_1\setminus Q_1'} x_{t_a}x_{h_a}}_{=0\,\forall a\neq b} \\ &\leq 4q_{\mathtt{Q}}'(x') + e_j^2 - x_{t_b}x_{h_b} = 0+1-2<0. \end{aligned}$$

■

We now rephrase Theorem 9.1 using these results.

Theorem 9.3 (Gabriel rephrased). Let $\mathtt{Q}$ be a finite and connected quiver, then $\mathtt{Q}$ is of finite type $\iff$ $q_{\mathtt{Q}}$ is positive definite.

We now consider the roots of $q_{\mathtt{Q}}$, which deliver (at least in the case of Dynkin quivers) the desired characterization of isomorphism classes of indecomposable representations. Moreover, in the case of A_n, the familiar barcode representation is easily recovered.

Definition 9.4 (Roots). *$x \in \mathbb{Z}^n$ is a* root *if $q_{\mathtt{Q}}$ if $q_{\mathtt{Q}}(x) \leq 1$. Denote by $\Phi_{\mathtt{Q}}$, the set of roots of $q_{\mathtt{Q}}$. In particular if $\mathtt{Q}$ is Dynkin, then by Theorem 9.2:*

$$\Phi_{\mathtt{Q}} = \{x \in \mathbb{Z}^n | q_{\mathtt{Q}}(x) \leq 1\} = \{x \in \mathbb{Z}^n | q_{\mathtt{Q}}(x) = 1\}\,.$$

Lemma 9.3. Let $\mathtt{Q}$ be Dynkin, then $\Phi_{\mathtt{Q}}$ is finite.

Proof. $\mathtt{Q}$ Dynkin $\implies$ $q_{\mathtt{Q}}$ positive definite on $\mathbb{Z}^n$ $\implies$ $q_{\mathtt{Q}}$ positive definite on $\mathbb{Q}^n$. By taking limits, $q_{\mathtt{Q}}$ is *positive semidefinite* on $\mathbb{R}^n$. Since $q_{\mathtt{Q}}$ is positive definite on $\mathbb{Q}^n$, it is invertible over $\mathbb{Q}$. Consequently, it is also invertible over $\mathbb{R}$, which can only be the case if $q_{\mathtt{Q}}$ is positive definite on $\mathbb{R}^n$ as well. But this means that the set $\{x \in \mathbb{R}^n | q_{\mathtt{Q}}(x) \leq 1\}$ is an ellipsoid and in particular bounded. The intersection of this bounded set with $\mathbb{Z}^n$ is $\Phi_{\mathtt{Q}}$ and must be finite. ■

We are now in a position to state the classification theorem for Dynkin quivers, which is in some sense a stronger version of the " $\Longleftarrow$ " direction of Theorem 9.1. We will prove this along with the earlier theorem.

Theorem 9.4. Let $\mathtt{Q}$ be Dynkin with $|Q_0| = n$ and $\mathbb{V} \in \mathbf{rep}_k(\mathtt{Q})$, then the map $\mathbb{V} \to \mathbf{dim}\mathbb{V}$ induces a bijection:

$$\text{isomorphism classes of indecomposable representations of } \mathtt{Q} \longleftrightarrow \Phi_{\mathtt{Q}} \cap \mathbb{N}^n\,.$$

We will prove Theorem 9.3 and Theorem 9.4 in the coming sections, but before doing so we apply Theorem 9.4 to A_n quivers to illustrate its significance to constructing barcodes.

Example 8. Using our calculation from Example 7:

$$\begin{aligned} x \in \Phi_{\mathtt{Q}} &\iff q_{\mathtt{Q}}(x) = 1 \\ &\iff \sum_{i=1}^{n-1} \frac{1}{2}(x_i - x_{i+1})^2 + \frac{1}{2}(x_1^2 + x_n^2) = 1\,. \end{aligned}$$

The telescoping nature of the sum means that only vectors of the form $(0, 0, \dots, 1, 1, \dots, 1, 0, 0, \dots, 0)^\intercal$ can be roots. Indeed denoting by b the index of the first 1 and d the index of the last 1, only these terms survive, each contributing $\frac{1}{2}$.

Now by Theorem 9.4, each $x \in \mathbb{Z}^n$ of the above form corresponds to an indecomposable representation of an A_n quiver. We claim that the representation $\mathbb{V}$ corresponding to $x = (0, 0, \dots, 1, 1, \dots, 1, 0, 0, \dots, 0)^\intercal$ has $V_i = \begin{cases} k & i \in [b, d] \\ 0 & \text{otherwise} \end{cases}$.

The spaces cannot be $k^n, n > 1$ because then the representation would be decomposable. Consider for example:

$$\begin{tikzcd} 0 \arrow[r, dash, "0"] & k \arrow[r, dash, "(1\ \ 0)"] & k^2 \arrow[r, dash, "(0\ \ 1)^\intercal"] & k \arrow[r, dash, "0"] & 0 \end{tikzcd}$$

which can be written as the sum of

$$\begin{tikzcd} 0 \arrow[r, dash, "0"] & k \arrow[r, dash, "1"] & k \arrow[r, dash, "0"] & 0 \arrow[r, "0"] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r, dash, "0"'] & 0 \arrow[r, dash, "0"'] & k \arrow[r, dash, "1"'] & k \arrow[r, dash, "0"'] & 0 \end{tikzcd}$$

Maps to or from 0 are forced to be the zero map and hence the representations look like this:

$$\begin{tikzcd} 0 \arrow[r, dash, "0"] & k \arrow[r, dash, "1"] & k \arrow[r, dash, "1"] & k \arrow[r, dash, "0"] & 0 \end{tikzcd}$$

Such representations are of particular importance and are called interval representations.

Definition 9.5. *Let* $\mathtt{Q}$ *be an* A_n *quiver and* $[b, d] \subset \mathbb{Z}$ *be an interval. The interval representation* $\mathbb{I}_{\mathtt{Q}}[b, d]$ *of* $[b, d]$ *assigns* k *to each* $i \in [b, d]$ *and* 0 *to elements outside the interval. The map between two copies of* k *is the identity, while all the other maps are zero maps.*

Remark 9.2. Observe that interval representations are indecomposable. Any decomposition of $\mathbb{I}_{\mathtt{Q}}[b, d]$ would have to decompose one of the identity maps connecting two copies of k, which is impossible.

This means we can associate every indecomposable representation to an interval representation $\mathbb{I}_{[b,d]}$. Since every A_n quiver can be written as a sum of indecomposable representations, we can construct a barcode for every A_n-quiver by putting together the intervals from each of the indecomposable summands.

9.4 Reflection functors

Having developed an algebraic way to distinguish Dynkin quivers from the rest, we now introduce reflection functors which together with the Tits form will pave the way to proving Theorem 9.3.

Definition 9.6 (Sinks, Sources)**.** *Let* $\mathtt{Q}$ *be a finite, connected quiver. A vertex* $i \in Q_0$ *is called a* sink *if all arrows incident to* i *are incoming, more formally* $t_a \neq i \,\forall a \in Q_1$. *Dually, a vertex is a* source *if all arrows are outgoing.*

Definition 9.7 (Reflection at i)**.** *Let* $i \in Q_0$ *be a sink (or a source). Denote by* $s_i\mathtt{Q}$ *the quiver obtained from* $\mathtt{Q}$ *by reversing the direction of all arrows incident (or originating from) to* i.

Example 9. Let $\mathtt{Q}$ be the quiver depicted in the following diagram:

$$\begin{array}{ccccccccc} \bullet & \longrightarrow & \bullet & \longleftarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

Vertices 2 and 5 are sinks and vertices 1 and 3 are sources while 4 is neither a sink nor a source. The corresponding reflections are depicted in Figure 9.3. Although vertex 4 is neither a sink nor a source in $\mathtt{Q}$, it is made a source by s_3 and a sink by s_5.

$$\begin{array}{ccccccccc} \bullet & \longleftarrow & \bullet & \longleftarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

(a) $s_1\mathtt{Q}$

$$\begin{array}{ccccccccc} \bullet & \longleftarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

(b) $s_2\mathtt{Q}$

$$\begin{array}{ccccccccc} \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longleftarrow & \bullet & \longrightarrow & \bullet \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

(c) $s_3\mathtt{Q}$

$$\begin{array}{ccccccccc} \bullet & \longrightarrow & \bullet & \longleftarrow & \bullet & \longrightarrow & \bullet & \longleftarrow & \bullet \\ 1 & & 2 & & 3 & & 4 & & 5 \end{array}$$

(d) $s_5\mathtt{Q}$

Figure 9.3: Reflections of $\mathtt{Q}$

We now define reflection functors which take $\mathbb{V} \in \mathbf{rep}_k(\mathtt{Q})$ to a representation of a reflection $\mathbb{W} \in \mathbf{rep}_k(s_i\mathtt{Q})$, where i is a sink or a source.

Definition 9.8 (Reflection functor)**.** *Let $i \in Q_0$ be a sink and $\mathbb{V} \in \boldsymbol{rep}_k(\mathtt{Q})$, we want to define a functor:*

$$\begin{aligned} \mathcal{R}_i^+ : \boldsymbol{rep}_k(\mathtt{Q}) &\to \boldsymbol{rep}_k(s_i\mathtt{Q}) \\ \mathbb{V} &\mapsto \mathbb{W} = (W_i, w_a)\,. \end{aligned}$$

We first define what happens to the spaces:

$$W_j = \begin{cases} V_j & j \neq i \\ \ker \xi_i \end{cases}$$

where

$$\xi_i : \bigoplus_{a \in Q_1^i} V_{t_a} \to V_i$$
$$(x_{t_a})_{a \in Q_1^i} \mapsto \sum_{a \in Q_1^i} v_{t_a} x_{t_a} \,.$$

By Q_1^i, we mean the set of arrows incident to i, so $\bigoplus_{a \in Q_1^i} V_{t_a}$ is the direct sum of the spaces adjacent to V_i. By $v_{t_a} x_{t_a}$, we mean the linear map v_{t_a} applied to x_{t_a}. Recall that since $v_{t_a} : V_{t_a} \to V_{h_a} = V_i$, ξ_i is indeed a map into V_i.

Next, we need to define the maps w_a. We leave the arrows not incident to i unchanged, so $w_a = v_a \;\; \forall a \notin Q_1^i$. Suppose, on the other hand, $a \in Q_1^i$, then denote by b the reversal of a and define w_b as the composition of the inclusion map and the projection map:

$$W_{t_b} \overset{reversal}{=} W_{h_a} \overset{i \text{ is sink}}{=} W_i \overset{def}{=} \ker \xi_i \hookrightarrow \bigoplus_{c \in Q_1^i} V_{t_c} \overset{\pi_a}{\to} V_{t_a} \overset{W_j = V_j}{=} W_{t_a} \overset{reversal}{=} W_{h_b} \,.$$

*Finally, in order to define a functor, we need to define what happens to morphisms $\phi : \mathbb{V} \to \mathbb{W} \;\; \mathbb{V}, \mathbb{W} \in$ **rep**$_k(\mathtt{Q})$. Define $\mathcal{R}^+\phi = \psi : \mathcal{R}^+\mathbb{V} \to \mathcal{R}^+\mathbb{W}$.*

Again, since the arrows not incident to i are unchanged, we can write $\psi_j = \phi_j \;\; \forall j \neq i$. All that remains is to define:

$$\psi_i : \mathcal{R}^+ V_i = \ker \xi_i \to \ker \zeta_i = \mathcal{R}^+ W_i \,.$$

But since $\ker \xi_i \hookrightarrow \bigoplus_{a \in Q_1^i} V_{t_a}$, we can just restrict the following map to the kernel:

$$\bigoplus_{a \in Q_1^i} \phi_{t_a} : \bigoplus_{a \in Q_1^i} V_{t_a} \to \bigoplus_{a \in Q_1^i} W_{t_a} \,.$$

The fact that $\mathcal{R}^+$ is a functor is easily verified.

By replacing *sink* with *source*, ker with coker and reversing the maps in Definition 9.8, one obtains a dual construction of $\mathcal{R}^- : \mathbf{rep}_k(\mathtt{Q}) \to \mathbf{rep}_k(s_i\mathtt{Q})$ for a source i.

Notation. We will abuse notation and write $(\mathcal{R}^+\mathbb{V})_n = \mathcal{R}^+ V_n$ and similarly for $\mathcal{R}^-$.

We now discuss some important properties of reflection functors.

Denote by $\mathbb{S}_i$ the *simple representation* of a quiver, which has $V_j = 0 \;\; \forall j \neq i$ and $V_i = k$

Proposition 9.1 (Properties of reflection functors)**.** *Let $\mathtt{Q}$ be a finite, connected quiver and $\mathbb{V} \in$ **rep**$_k(\mathtt{Q})$. If $\mathbb{V} \cong \mathbb{U} \oplus \mathbb{W}$, then $\mathcal{R}^+\mathbb{V} \cong \mathcal{R}^+\mathbb{U} \oplus \mathcal{R}^+\mathbb{W}$. Let i be a sink and $\mathbb{V}$ be indecomposable then there are two possibilities:*

1. *$\mathbb{V} \cong \mathbb{S}_i$, then $\mathcal{R}^+\mathbb{V} = 0$.*
2. *$\mathbb{V} \ncong \mathbb{S}_i$, then $\mathcal{R}^+\mathbb{V}$ is indecomposable, $\mathcal{R}^-\mathcal{R}^+\mathbb{V} \cong \mathbb{V}$ and the relationship between the dimension vectors $y =$ **dim**$\mathcal{R}^+\mathbb{V}$ and $x =$ **dim**$\mathbb{V}$ is:*

$$y_j = \begin{cases} x_j & j \neq i \\ -x_i + \sum_{a \in Q_1^i} x_{t_a} & j = i \end{cases} .$$

Similarly, if $\mathbb{V} \cong \mathbb{U} \oplus \mathbb{W}$, then $\mathcal{R}^-\mathbb{V} \cong \mathcal{R}^-\mathbb{U} \oplus \mathcal{R}^-\mathbb{W}$. Let i be a source and $\mathbb{V}$ be indecomposable then there are two possibilities:

1. *$\mathbb{V} \cong \mathbb{S}_i$, then $\mathcal{R}^-\mathbb{V} = 0$.*

2. *$\mathbb{V} \ncong \mathbb{S}_i$, then $\mathcal{R}^-\mathbb{V}$ is indecomposable, $\mathcal{R}^+\mathcal{R}^-\mathbb{V} \cong \mathbb{V}$ and the relationship between the dimension vectors $y = \boldsymbol{dim}\mathcal{R}^-\mathbb{V}$ and $x = \boldsymbol{dim}\mathbb{V}$ is:*

$$y_j = \begin{cases} x_j & j \neq i \\ -x_i + \sum_{a \in Q_1^i} x_{h_a} & j = i \end{cases} .$$

Proof. We will only prove the proposition for $\mathcal{R}^+$, since the proof for $\mathcal{R}^-$ is completely analogous. Let $\mathbb{V}$ be an indecomposable representation of a finite, connected quiver $\mathtt{Q}$ and $i \in Q_0$ be a sink. Suppose $\mathbb{V} \cong \mathbb{S}_i$, then $V_j = 0 \,\forall j \neq i$ and hence the maps into the sink $V_i = k$ are all zero maps. Thus $\mathcal{R}^+V_j = V_j = 0 \,\forall j \neq i$ and $\xi_i : \bigoplus_{a \in Q_1^i} V_{t_a} \to V_i = 0 \to V_i$ yielding $\mathcal{R}^+V_i = \ker \xi_i = 0$.

Suppose now that $\mathbb{V} \ncong \mathbb{S}_i$ and let $y = \mathbf{dim}\mathcal{R}^+\mathbb{V}$, $x = \mathbf{dim}\mathbb{V}$. Clearly, $y_j = x_j \,\forall j \neq i$ since $\mathcal{R}^+V_j = V_j = 0 \,\forall j \neq i$. The case for y_i is a bit trickier. Per definition:

$$\begin{gathered} \xi_i = \bigoplus_{a \in Q_1^i} V_{t_a} \to V_i \\ (x_{a1}, x_{a2}, \dots, x_{am}) \mapsto v_{a1}x_{a1} + \dots + v_{am}x_{am} \\ \ker \xi_i = \ker(\sum_{a \in Q_1^i} v_a) . \end{gathered}$$

Hence, there is an inclusion:

$$\mathcal{R}^+V_i = \ker \xi_i = \ker(\sum_{a \in Q_1^i} v_a) \hookrightarrow \bigoplus_{a \in Q_1^i} V_{t_a} .$$

Further for each $a \in Q_1^i$ we have the commutative diagram in Figure 9.4, where π_a is the standard projection onto the a^{th} coordinate.

$$\begin{tikzcd} \ker(\sum_{a \in Q_1^i} v_a) \arrow[r, "w_a"] \arrow[d, hook, "i"'] & V_a \\ \bigoplus_{a \in Q_1^i} V_{t_a} \arrow[ur, "\pi_a"'] & \end{tikzcd}$$

Figure 9.4

We now consider $\mathcal{R}^-\mathcal{R}^+\mathbb{V}$. Again, the spaces $\mathcal{R}^-\mathcal{R}^+V_j = V_j \,\forall j \neq i$ remain unchanged. Recalling that $\mathcal{R}^+V_i$ is a source, $\mathcal{R}^-\mathcal{R}^+V_j$ is reverted to a sink and is the cokernel of:

$$\chi_i : \ker(\sum_{a \in Q_1^i} v_a) \to \bigoplus_{a \in Q_1^i} V_{t_a} .$$

But this is simply an inclusion, so we have

$$\operatorname{coker}\chi_i = \frac{\bigoplus_{a\in Q_1^i} V_{t_a}}{\ker(\sum_{a\in Q_1^i} v_a)}\,.$$

For every $a \in Q_1^i$ we have the commutative diagram in Figure 9.5, where ρ is the canonical projection and $i_a(x_a) = (0,\ldots,0,x_a,0,\ldots,0)$ is an embedding.

$$\begin{tikzcd}
\frac{\bigoplus_{a\in Q_1^i} V_{t_a}}{\ker(\sum_{a\in Q_1^i} v_a)} & & V_a \arrow[ll, "z_a"'] \arrow[dll, "i_a"'] \\
\bigoplus_{a\in Q_1^i} V_{t_a} \arrow[u, "\rho"'] & &
\end{tikzcd}$$

Figure 9.5

Combining Figure 9.4 and Figure 9.5 we obtain Figure 9.6.

$$\begin{tikzcd}
0 \arrow[r] & \ker(\sum_{a\in Q_1^i} v_a) \arrow[r, hook, "i"] & \bigoplus_{a\in Q_1^i} V_{t_a} \arrow[r, "\rho"] \arrow[d, "\xi_i"] & \frac{\bigoplus_{a\in Q_1^i} V_{t_a}}{\ker(\sum_{a\in Q_1^i} v_a)} \arrow[r] \arrow[dl, dashed, "\varphi"] & 0 \\
& & V_i & &
\end{tikzcd}$$

Figure 9.6

The top row in Figure 9.6 is exact since i is injective, ρ is surjective and $\operatorname{Im} i = \ker(\sum_{a\in Q_1^i} v_a) = \ker\rho$. The first isomorphism theorem tells us:

$$\frac{\bigoplus_{a\in Q_1^i} V_{t_a}}{\ker(\sum_{a\in Q_1^i} v_a)} \cong \operatorname{Im}\sum_{a\in Q_1^i} v_a \hookrightarrow V_i\,.$$

Hence the map φ exists, is injective and the triangle it forms commutes. Setting

$$\phi_j = \begin{cases} \varphi & j = i \\ \mathrm{id}_{V_j} & j \neq i \end{cases}$$

we obtain a morphism $\phi : \mathcal{R}^-\mathcal{R}^+\mathbb{V} \to \mathbb{V}$. If $\operatorname{Im}\sum_{a\in Q_1^i} v_a \cong V_i$, in other words, if ξ_i is surjective, φ is an isomorphism. Consequently, ϕ is an isomorphism. Furthermore, the surjectivity of ξ_i yields

$$\dim \mathcal{R}^-\mathcal{R}^+ V_i = \dim\ker(\sum_{a\in Q_1^i} v_a) - \dim V_i\,,$$

which is precisely the formula for y_i claimed in the theorem.

All that is left to prove is that ξ_i is indeed surjective. Assume the contrary for the sake of contradiction, then $r = \operatorname{coker} \sum_{a\in Q_1^i} v_a = \dim V_i - \dim \operatorname{Im} \sum_{a\in Q_1^i} v_a > 0$. However:

$$\begin{aligned}\dim V_i &= \dim \operatorname{Im} \sum_{a\in Q_1^i} v_a + \dim \operatorname{coker} \sum_{a\in Q_1^i} v_a \\ &= \dim \operatorname{Im} \sum_{a\in Q_1^i} v_a + \dim \frac{\bigoplus_{a\in Q_1^i} V_{t_a}}{\ker(\sum_{a\in Q_1^i} v_a)}.\end{aligned}$$

But then φ must induce an isomorphism $V_i \cong k^r$, which means we can use ϕ to decompose $\mathbb{V}$ as $\mathbb{V} \cong \mathcal{R}^-\mathcal{R}^+\mathbb{V} \bigoplus \mathbb{S}_i^r$, $r > 0$ and arrive at a contradiction. ∎

The formula relating the dimension vectors, leads us to the following critical corollary, which we will use to prove Theorem 9.3.

Corollary 9.1 (Reflection preserves $q_{\mathtt{Q}}(\mathbf{dim}\mathbb{V})$). Let $\mathtt{Q}$ be a finite, connected quiver, $\mathbb{V} \in \mathbf{rep}_k(\mathtt{Q})$ indecomposable, $i \in Q_0$ a sink (or a source), then :

1. If $\mathbb{V} \cong \mathbb{S}_i$, then $q_{s_i\mathtt{Q}}(\mathbf{dim}\mathcal{R}^\pm\mathbb{V}) = 0$.
2. If $\mathbb{V}$ is not a simple representation, then $q_{s_i\mathtt{Q}}(\mathbf{dim}\mathcal{R}^\pm\mathbb{V}) = q_{\mathtt{Q}}(\mathbf{dim}\mathbb{V})$.

Proof. 1. is trivial, since by Proposition 9.1 we have $\mathcal{R}^\pm\mathbb{S}_i = 0$ and so the corresponding dimension vector $\mathcal{R}^\pm\mathbb{S}_i = 0$. Consider the case in which i is a sink in $\mathtt{Q}$ and let $y = \mathbf{dim}\mathcal{R}^+\mathbb{V}$, $x = \mathbf{dim}\mathbb{V}$.

$$\begin{aligned}q_{s_i\mathtt{Q}}(y) &= \sum_{j\in Q_0} y_j^2 - \sum_{a\in Q_1} y_{t_a}y_{h_a} \\ &\overset{\spadesuit}{=} y_i^2 + \sum_{\substack{j\in Q_0\\ j\neq i}} y_j^2 - \sum_{\substack{a\in Q_1\\ a\notin Q_1^i}} y_{t_a}y_{h_a} - \sum_{a\in Q_1^i} y_{t_a}y_i \\ &\overset{\clubsuit}{=} (-x_i + \sum_{a\in Q_1^i} x_{t_a})^2 + \sum_{\substack{j\in Q_0\\ j\neq i}} x_j^2 - \sum_{\substack{a\in Q_1\\ a\notin Q_1^i}} x_{t_a}x_{h_a} - \sum_{a\in Q_1^i} x_{t_a}(-x_i + \sum_{a\in Q_1^i} x_{t_a}) \\ &= x_i^2 - 2x_i \sum_{a\in Q_1^i} x_{t_a} + (\sum_{a\in Q_1^i} x_{t_a})^2 + \sum_{\substack{j\in Q_0\\ j\neq i}} x_j^2 - \sum_{\substack{a\in Q_1\\ a\notin Q_1^i}} x_{t_a}x_{h_a} + x_i \sum_{a\in Q_1^i} x_{t_a} - (\sum_{a\in Q_1^i} x_{t_a})^2 \\ &= \sum_{j\in Q_0} x_j^2 - \sum_{\substack{a\in Q_1\\ a\notin Q_1^i}} x_{t_a}x_{h_a} - 2x_i \sum_{a\in Q_1^i} x_{t_a} + x_i \sum_{a\in Q_1^i} x_{t_a} \\ &\overset{\heartsuit}{=} \sum_{j\in Q_0} x_j^2 - \sum_{a\in Q_1} x_{t_a}x_{h_a} = q_{\mathtt{Q}}(x)\end{aligned}$$

We used the fact that i is a sink to deduce $h_a = i \,\forall a \in Q_1^i$ in ♠. Next, we employed Proposition 9.1 in ♣. Finally, we once again used that i is a sink in ♡ to write $\sum_{a\in Q_1^i} x_i x_{t_a} = \sum_{a\in Q_1^i} x_{h_a}x_{t_a}$.

The proof for a source i is completely analogous. ∎

9.5 Coxeter Functors

The final ingredient that we need to prove Theorem 9.3, are *Coxeter functors*, which are nothing more than the repeated application of reflection functors.

Definition 9.9 (Order on acyclic quivers)**.** *Let* $\mathtt{Q}$ *be an acyclic quiver, that is* $\bar{\mathtt{Q}}$ *has no directed cycles. Then we can impose a partial order on* Q_0 *by ordering vertices such that* $t_a < h_a \forall a \in Q_1$*.*

Remark 9.3. Consider an arbitrary total order on Q_0 compatible with the partial order we just defined and relabel the vertices according to this order. In the following, we will assume that Q_0 has been labeled according to such a total order. The choice of the total order is irrelevant, as long as it is compatible with the partial order in Definition 9.9.

Definition 9.10 (Coxeter functors)**.** *Let* $\mathtt{Q}$ *be an acyclic quiver, then under the total order we just defined:*

$$c^+ = s_1 s_2 \dots s_n$$
$$c^- = s_n s_{n-1} \dots s_1$$

are well-defined operations. Let

$$\mathcal{C}^+ = \mathcal{R}_1^+ \mathcal{R}_2^+ \dots \mathcal{R}_n^+$$
$$\mathcal{C}^- = \mathcal{R}_n^- \mathcal{R}_{n-1}^- \dots \mathcal{R}_1^-$$

be the corresponding functors.

Remark 9.4. Due to the chosen order, $i \in Q_0$ must be a sink in $(s_{i+1} \dots s_n)\mathtt{Q}$ and a source in $(s_1 \dots s_{i-1})\mathtt{Q}$. Along the same lines, one also observes that $c^\pm \mathtt{Q} = \mathtt{Q}$ because every arrow is reversed exactly twice. Importantly, this means that $\mathcal{C}^\pm : \mathbf{rep}_k(\mathtt{Q}) \to \mathbf{rep}_k(\mathtt{Q})$ are endofunctors.

Corollary 9.2. Let $\mathtt{Q}$ be a finite, connected and acyclic quiver and $\mathbb{V} \in \mathbf{rep}_k(\mathtt{Q})$ be indecomposable. Then $\mathcal{C}^\pm\mathbb{V}$ is either indecomposable or the zero representation. If the former holds, then $q_{\mathtt{Q}}(\mathbf{dim}\mathcal{C}^\pm\mathbb{V}) = q_{\mathtt{Q}}(\mathbf{dim}\mathbb{V})$, while if the latter holds we have $q_{\mathtt{Q}}(\mathbf{dim}\mathcal{C}^\pm\mathbb{V}) = 0$.

Proof. Repeated application of Proposition 9.1 yields the first statement and repeated application of Corollary 9.1 yields the second one. ■

9.6 Proving Gabriel's theorem for A_n-type quivers

We are now ready to prove Theorem 9.1. However, before we prove the general case, we prove it for A_n quivers. The basic idea behind the " $\Longleftarrow$ " direction is similar in both cases, but the concrete case of A_n quivers is more intuitive. A reader only interested in the formal proof of the general case may skip ahead to the next section. Our strategy will be to show that $\mathbf{dim}\mathbb{V} \in \Phi_{\mathtt{Q}}$ for an A_n quiver $\mathtt{Q}$. If we do this, we are done since we proved in Example 8, that these roots correspond to interval representations, of which there are only finitely many.

We do this in two steps, first, we show the result for linear quivers L_n and then use reflection functors to reduce an arbitrary A_n quiver to the linear case.

Proposition 9.2. *Let* $\mathtt{Q}$ *be a linear quiver* L_n *and* $\mathbb{V} \in \boldsymbol{rep}_k\mathtt{Q}$*, then applying the coxeter functor* n *times sends* $\mathbb{V}$ *to* 0*, that is* $\boldsymbol{dim}\mathcal{C}^+ \dots \mathcal{C}^+\mathbb{V} = 0$*.*

Proof. Since all the arrows are pointed to the right, it is clear that the only order compatible with the one described in Definition 9.9 is simply the natural order on the integers. Let $\mathbf{dim}\mathbb{V} = (x_1, \dots, x_n)^\intercal$. Let us now calculate $\dim \mathcal{C}^+\mathbb{V}$ by repeatedly applying $\mathcal{R}^+$ and Proposition 9.1. It might be helpful to keep track of the visual representation of the quiver to understand the calculations.

$$\begin{aligned}
\mathbf{dim}\mathcal{R}_n^+\mathbb{V} &= (x_1, x_2, \dots, x_{n-1}, -x_n + \sum_{a \in Q_1^n} x_{t_a}) = (x_1, x_2, \dots, x_{n-1}, -x_n + x_{n-1})^\intercal \text{ or } 0 \\
\mathbf{dim}\mathcal{R}_{n-1}^+\mathcal{R}_n^+\mathbb{V} &= (x_1, x_2, \dots, x_{n-2}, -x_{n-1} + \sum_{a \in Q_1^{n-1}} x_{t_a}, -x_n + x_{n-1})^\intercal \\
&= (x_1, x_2, \dots, x_{n-2}, -x_{n-1} + x_{n-1} + (-x_n + xn - 1), -x_n + x_{n-1})^\intercal \\
&= (x_1, x_2, \dots, x_{n-2}, x_{n-1} - x_n, -x_n + x_{n-1})^\intercal \text{ or } 0 \\
&\dots \\
\mathbf{dim}\mathcal{C}^+\mathbb{V} &= (-x_n, x_1 - x_n, \dots, x_{n-2} - x_n, x_{n-1} - x_n)^\intercal \text{ or } 0
\end{aligned}$$

However, the dimension vector cannot have negative coordinates (vector spaces always have non-negative dimensions), hence by the last line we must have $x_n = 0$ (or the case where the whole vector is 0). If $x_n = 0$, we have $\mathbf{dim}\mathcal{C}^+\mathbb{V} = (0, x_1, \dots, x_{n-2}, x_{n-1})^\intercal$.

Now we can iteratively apply $\mathcal{C}^+$, and by the same calculation the 0 gets pushed one coordinate down after every application of $\mathcal{C}^+$.

$$\mathbf{dim}\mathcal{C}^+\mathcal{C}^+\mathbb{V} = (0, 0, x_1, \dots, x_{n-1}, x_{n-2})^\intercal \text{ or } 0$$

Hence, after at most n applications we are guaranteed $\mathbf{dim}\mathcal{C}^+ \dots \mathcal{C}^+\mathbb{V} = 0$ (of course the 0 might happen much earlier). ∎

We now show how one can use reflection functors to convert an arbitrary A_n quiver to a linear one.

Lemma 9.4 (Converting A_n to L_n). Let $\mathtt{Q}$ be an A_n-quiver, then there exists a sequence of reflections that convert $\mathtt{Q}$ to a linear quiver.

Proof. Let $\mathtt{Q}$ be an A_n quiver with the order from Definition 9.9 imposed on Q_0. Denote by $i_1 < i_2 < \dots < i_r$ the heads of the backward arrows.

The sequence of reflections $s_1 \dots s_{i_1}$ applied to $\mathtt{Q}$ gives us back the same quiver *except* for the fact that *the arrow between* i_1 *and* $i_1 + 1$ *is now a forward arrow.*

Repeat the same sequence of reflections for each of $i_2, \dots, i_r$ and we obtain a linear quiver.

Now consider $\mathbb{V} \in \mathbf{rep}_k(\mathtt{Q})$ indecomposable and the effect of the corresponding sequence of reflection functors

$$\mathcal{R}_1^+ \dots \mathcal{R}_{i_r}^+ \dots \mathcal{R}_1^+ \dots \mathcal{R}_{i_2}^+ \mathcal{R}_1^+ \dots \mathcal{R}_{i_1}^+ \mathbb{V} \,.$$

By Proposition 9.1 this is either an indecomposable representation of a linear quiver or a zero representation. ∎

Remark 9.5. Together the lemma and Proposition 9.2 tell us that

$$\underbrace{\mathcal{C}^+ \dots \mathcal{C}^+}_{ntimes} \mathcal{R}_1^+ \dots \mathcal{R}_{i_r}^+ \dots \mathcal{R}_1^+ \dots \mathcal{R}_{i_2}^+ \mathcal{R}_1^+ \dots \mathcal{R}_{i_1}^+ \mathbb{V} = 0$$

for any indecomposable representation $\mathbb{V} \in \mathbf{rep}_k(\mathtt{Q})$ of a Dynkin quiver $\mathtt{Q}$.

Putting this together with our discussion on root systems, we can prove Gabriel's theorem for A_n quivers.

Theorem 9.5. Let $\mathtt{Q}$ be an A_n-type quiver and $\mathbb{V} \in \mathbf{rep}_k\mathtt{Q}$. Then, $\mathtt{Q}$ is of finite type.

Proof. Let $\mathbb{V} \in \mathbf{rep}_k(\mathtt{Q})$ be indecomposable. By our discussion above, there exists a sequence of reflection functors say $\mathcal{R}_{i_m}^+ \dots \mathcal{R}_{i_1}^+ \mathbb{V} = 0$. Suppose that this sequence is minimal, that is $\mathcal{R}_{i_{m-1}}^+ \dots \mathcal{R}_{i_1}^+ \mathbb{V} = \mathbb{S}_{i_{m-1}}$ and $\mathcal{R}_{i_p}^+ \dots \mathcal{R}_{i_1}^+ \mathbb{V}$ is indecomposable for $p < m$. Note that the step before hitting 0 (assuming minimality), has to be the simple representation by Proposition 9.1.

Now we apply Corollary 9.1:

$$\begin{aligned} q_{\mathtt{Q}}(\mathbf{dim}\mathbb{V}) &= q_{s_{i_1}\mathtt{Q}}(\mathbf{dim}\mathcal{R}_{i_1}^+ \mathbb{V}) \\ &\dots \\ &= q_{s_{i_1} \dots s_{i_{m-1}}\mathtt{Q}}(\mathbf{dim}\mathcal{R}_{i_{m-1}}^+ \dots \mathcal{R}_{i_1}^+ \mathbb{V}) = q_{s_{i_1} \dots s_{i_{m-1}}\mathtt{Q}}(\mathbb{S}_{i_{m-1}}) = 1 \,. \end{aligned}$$

Per Definition 9.4, this means that $\mathbf{dim}\mathbb{V} \in \Phi_{\mathtt{Q}}$ and we calculated all of these roots in Example 8, so:

$$\mathbf{dim}\mathbb{V} = (0, \dots, 0, 1, \dots, 1, 0, \dots, 0)^{\intercal} \,.$$

If we denote by b the index of the first 1 and by d the index of the last 1, we can write:

$$\mathbb{V}_i = \begin{cases} k & b \leq i \leq d \\ 0 & \text{otherwise} \end{cases} .$$

As we noted in the example as well, this forces the maps to be the identity between two k's and 0 everywhere else.

Thus every indecomposable representation of an A_n-quiver is isomorphic to an interval representation $\mathbb{I}_{[b,d]}$. Every isomorphism class of indecomposable representations then contains at least one of these interval representations. There are only finitely many interval representations, so the number of isomorphism classes is finite, which is exactly what we wanted. ∎

From here, proving Theorem 9.4 for A_n-quivers is also not that difficult.

Proof of Theorem 9.4 for A_n-quivers. We have just proven that the map $\mathbb{V} \to \mathbf{dim}\mathbb{V}$ sends indecomposable representations to positive roots of $q_{\mathtt{Q}}$. We need to prove that this map is bijective. Indeed we saw that every representation corresponds to an interval representation. Since any two distinct interval representations have different dimension vectors, they cannot be isomorphic. Thus the map is injective. Further, every interval representation is indecomposable, hence proving surjectivity. ∎

9.7 Proof for general Dynkin quivers

We now prove the theorem in generality. The first step is to show that the coxeter functors, applied often enough, send $\mathbb{V}$ to 0 for any indecomposable representation of a Dynkin quiver. This is not as straightforward as it was for A_n quivers and we will need to introduce the so-called Weyl Group.

Definition 9.11 (Weyl Group of reflection functors)**.** *Let the* Weyl group $W_{\mathtt{Q}}$ *be the subgroup of* $Aut(\mathbb{Z}^n)$ *generated by the reflection functors. Explicitly, the generator automorphisms are precisely the formulas from Proposition 9.1.*

The group $W_{\mathtt{Q}}$ has desirable properties, stemming from the properties of the reflection functors.

Proposition 9.3. *Let* $\mathtt{Q}$ *be a finite, connected quiver. Then:*

1. $W_{\mathtt{Q}}$ *preserves* $q_{\mathtt{Q}}$*. More formally, if we denote by* $s_{i_1},\dots,s_{i_n}$ *the reflections corresponding to an arbitrary* $w = \mathcal{R}^{\pm}_{i_1}\dots\mathcal{R}^{\pm}_{i_n}$*:*

$$q_{s_{i_1},\dots,s_{i_n}\mathtt{Q}}(\mathcal{R}^{\pm}_{i_1}\dots\mathcal{R}^{\pm}_{i_n}x) = q_{\mathtt{Q}}(x) \ \ \forall x \in \mathbb{Z}^n\,.$$

2. *The set of roots* $\Phi_{\mathtt{Q}}$ *is stable under action by* $W_{\mathtt{Q}}$*, that is* $x \in \Phi_{\mathtt{Q}} \implies wx \in \Phi_{s_{i_1},\dots,s_{i_n}\mathtt{Q}}$*.*

3. *If* $\mathtt{Q}$ *is Dynkin, then every element of* $W_{\mathtt{Q}}$ *is a subgroup of the permutation group of* $\Phi_{\mathtt{Q}}$*.*

Proof. 1. We need to show $q_{s_i\mathtt{Q}}(\mathcal{R}^{\pm}_i x) = q_{\mathtt{Q}}(x)$ since these are the generators. Doing so involves a calculation very similar to Corollary 9.1. Let i be a sink, $x \in \mathbb{Z}^n$ and $y = \mathcal{R}^+ x$. Then :

$$\begin{aligned}
q_{s_i\mathtt{Q}}(y) &= \sum_{j\in Q_0} y_j^2 - \sum_{a\in Q_1} y_{t_a}y_{h_a} \\
&\overset{\spadesuit}{=} y_i^2 + \sum_{\substack{j\in Q_0\\ j\neq i}} y_j^2 - \sum_{\substack{a\in Q_1\\ a\notin Q_1^i}} y_{t_a}y_{h_a} - \sum_{a\in Q_1^i} y_{t_a}y_i \\
&\overset{\clubsuit}{=} (-x_i + \sum_{a\in Q_1^i} x_{t_a})^2 + \sum_{\substack{j\in Q_0\\ j\neq i}} x_j^2 - \sum_{\substack{a\in Q_1\\ a\notin Q_1^i}} x_{t_a}x_{h_a} - \sum_{a\in Q_1^i} x_{t_a}(-x_i + \sum_{a\in Q_1^i} x_{t_a}) \\
&= x_i^2 - 2x_i\sum_{a\in Q_1^i} x_{t_a} + (\sum_{a\in Q_1^i} x_{t_a})^2 + \sum_{\substack{j\in Q_0\\ j\neq i}} x_j^2 - \sum_{\substack{a\in Q_1\\ a\notin Q_1^i}} x_{t_a}x_{h_a} + x_i\sum_{a\in Q_1^i} x_{t_a} - (\sum_{a\in Q_1^i} x_{t_a})^2 \\
&= \sum_{j\in Q_0} x_j^2 - \sum_{\substack{a\in Q_1\\ a\notin Q_1^i}} x_{t_a}x_{h_a} - 2x_i\sum_{a\in Q_1^i} x_{t_a} + x_i\sum_{a\in Q_1^i} x_{t_a} \\
&\overset{\heartsuit}{=} \sum_{j\in Q_0} x_j^2 - \sum_{a\in Q_1} x_{t_a}x_{h_a} = q_{\mathtt{Q}}(x)
\end{aligned}$$

The reasoning is also similar. We used the fact that i is a sink to deduce $h_a = i\,\forall a \in Q_1^i$ in $\spadesuit$. Next, we simply substituted the definition of $y = \mathcal{R}^+x$ in $\clubsuit$. Finally, we once again used that i is a sink in $\heartsuit$ to write $\sum_{a\in Q_1^i} x_i x_{t_a} = \sum_{a\in Q_1^i} x_{h_a}x_{t_a}$.

2. This is a trivial consequence of the quadratic form being preserved.

3. By Lemma 9.3, $\Phi_{\mathtt{Q}}$ is finite and since it is stable under any $w \in W_{\mathtt{Q}}$, it must be a permutation. Next $q_{\mathtt{Q}}(\mathbf{dim}\mathbb{S}_i) = 1 \in \Phi_{\mathbb{Z}}^n$ implies that $\Phi_{\mathtt{Q}} \subset \mathbb{Z}^n$ contains the entire basis. Hence we can embed $W_{\mathtt{Q}}$ as a subgroup of the permutation group of $\Phi_{\mathtt{Q}}$. In particular, since $\Phi_{\mathtt{Q}}$ is finite, this implies that $W_{\mathtt{Q}}$ is finite.

■

Now as promised, we prove the generalization of Proposition 9.2 for arbitrary Dynkin quivers. It will follow from a lemma of group theoretic nature.

Lemma 9.5. Let $\mathtt{Q}$ be Dynkin and $x \in \mathbb{Z}^n$ be positive. Let $w \in W_{\mathtt{Q}}$ denote the action corresponding to the coxeter functor $\mathcal{C}^+$. With some abuse of notation, we say $\mathcal{C}^+ x$ is the image of x under w. Then

$$\exists r \in \mathbb{Z} : \underbrace{\mathcal{C}^+ \dots \mathcal{C}^+}_{\text{r times}} x \leq 0\,.$$

In order to prove the lemma, we rephrase it in terms of group theory.

Lemma 9.6 (Lemma 9.5 rephrased). Let $\mathtt{Q}$ be Dynkin, $x \in \Phi_{\mathtt{Q}}$ be a positive root, $W_{\mathtt{Q}}$ be the Weyl group and $\omega \in W_{\mathtt{Q}}$ be arbitrary, then

1. The action of $W_{\mathtt{Q}}$ on $\mathbb{N}^n$ (positive vectors) has no fixed points.

2. $\exists r \in \mathbb{Z}$ finite such that $\omega^r x$ is non-positive.

Proof. 1. Suppose $x \in \mathbb{N}^n$ with $w(x) = x$ for some $w \in W_{\mathtt{Q}}$. Now $w = \mathcal{R}^{\pm}_{i_s} \dots \mathcal{R}^{\pm}_{i_1}$ for some vertices $i_1, i_2, \dots, i_s$. Set $y = w(x)$. Since $\mathcal{R}^{\pm}_{i_r}$ leaves every index of x unchanged except for i_r, we obtain:

$$y_j = \begin{cases} (\mathcal{R}^{\pm}_{i_r} x)_j & j = i_r\, 1 \leq r \leq s \\ x_j & j \neq i_r\, \forall 1 \leq r \leq s \end{cases},$$

where $(\mathcal{R}^{\pm}_{i_r} x)_j = \begin{cases} -x_i + \sum_{a \in Q_1^i} x_{t_a} & \mathcal{R}^+ \\ -x_i + \sum_{a \in Q_1^i} x_{h_a} & \mathcal{R}^- \end{cases}$. Denote by $e_i \in \mathbb{N}^n$, the unit vector with a 1 in the i^{th} position. Note that $e_i = \mathbf{dim}\mathbb{S}_i$ and that these form a basis of $\mathbb{N}^n$. Suppose first that i is a sink. Then:

$$\begin{aligned}
(e_i, x)_{\mathtt{Q}} &= \langle e_i, x \rangle_{\mathtt{Q}} + \langle x, e_i, \rangle_{\mathtt{Q}} \\
&= \sum_{j \in Q_0} e_j x_j - \sum_{a \in Q_1} e_{t_a} x_{h_a} + \sum_{j \in Q_0} x_j e_j - \sum_{a \in Q_1} x_{t_a} e_{h_a} \\
&\overset{\spadesuit}{=} 2x_i - \sum_{\substack{a \in Q_1 \\ t_a = i}} e_i x_{h_a} - \sum_{\substack{a \in Q_1 \\ h_a = i}} x_{t_a} e_i \\
&\overset{\clubsuit}{=} 2x_i - \underbrace{\sum_{\substack{a \in Q_1 \\ t_a = i}} e_i x_{h_a}}_{=0} - \sum_{\substack{a \in Q_1 \\ h_a = i}} x_{t_a}
\end{aligned}$$

where ♠ is due to $e_j = 0 \, \forall j \neq i$, while ♣ is due to the fact that i is a sink and so there are no arrows $a \in Q_1$ with $t_a = i$. Thus we have:

$$(\mathcal{R}_i^+ x)_j = \begin{cases} x_j & j \neq i \\ -x_i + \sum_{a \in Q_1^i} x_{t_a} & j = i \end{cases}$$

$$(\mathcal{R}_i^+ x) = x + \begin{pmatrix} 0 \\ 0 \\ \dots \\ -2x_i + \sum_{a \in Q_1^i} x_{t_a} \\ 0 \\ \dots \\ 0 \end{pmatrix} = x - \begin{pmatrix} 0 \\ 0 \\ \dots \\ (e_i, x)_{\mathtt{Q}} \\ 0 \\ \dots \\ 0 \end{pmatrix} .$$

A similar calculation yields the same formula for a source i.

$$(\mathcal{R}_i^- x) = x - \begin{pmatrix} 0 \\ 0 \\ \dots \\ (e_i, x)_{\mathtt{Q}} \\ 0 \\ \dots \\ 0 \end{pmatrix}$$

We now apply the fixed point condition to obtain:

$$y_{i_r} = \mathcal{R}_{i_r}^{\pm} x = x - \begin{pmatrix} 0 \\ 0 \\ \dots \\ (e_{i_r}, x)_{\mathtt{Q}} \\ 0 \\ \dots \\ 0 \end{pmatrix} = x \; \forall i \leq r \leq s \, .$$

Therefore $(e_{i_r}, x)_{\mathtt{Q}} = 0 \; \forall i \leq r \leq s$. Now since $\mathtt{Q}$ is Dynkin , $q_{\mathtt{Q}} > 0$ and $(-,-)_{\mathtt{Q}}$ is non-degenerate. But the e_i's form a basis, forcing $x = 0$, which contradicts the assumption that x is positive.

2. Since $\mathtt{Q}$ is Dynkin, $W_{\mathtt{Q}}$ is finite. So, $\mathrm{ord}(\omega) = n$ for some finite integer n. Suppose for the sake of contradiction that $x, \omega x, \omega^2 x, \dots, \omega^{n-1} x$ are all positive and consider $y = x + \omega x + \omega^2 x + \cdots + \omega^{n-1} x$. Then applying w yields:

$$\omega y = \omega x + \omega^2 x + \omega^3 x + \cdots + \underbrace{\omega^n x}_{=x} = y \, .$$

This is a contradiction to 1., since x is positive (as the sum of positive vectors by assumption) and a fixed point of $W_{\mathtt{Q}}$.

■

This lemma enables us to elegantly prove that the coxeter functors send an arbitrary indecomposable representation of a Dynkin quiver to the zero representation.

Proposition 9.4. *Let* $\mathtt{Q}$ *be Dynkin and* $\mathbb{V} \in \boldsymbol{rep}_k(\mathtt{Q})$ *be indecomposable. Then, there exists a finite integer* r*, such that:*

$$\underbrace{\mathcal{C}^+ \dots \mathcal{C}^+}_{r\ times} \mathbb{V} = 0\,.$$

Proof. Set $x = \mathbf{dim}\mathbb{V} \in \mathbb{N}^n$ and apply Lemma 9.5 to obtain $r \in \mathbb{Z}$, such that $\underbrace{\mathcal{C}^+ \dots \mathcal{C}^+}_{\text{r times}} \mathbf{dim}\mathbb{V} \leq 0$. By the properties of the dimension vector, this forces the result, since the only representation with 0 dimension vector, is the zero representation. ■

We now present the long-awaited proof of Gabriel's theorem for Dynkin quivers.

Proof of Theorem 9.1. Let $\mathtt{Q}$ be a Dynkin quiver and $\mathbb{V} \in \mathbf{rep}_k(\mathtt{Q})$ be indecomposable. The next steps are almost identical to the proof of Theorem 9.5.

By Proposition 9.4, there exists $i_1, \dots, i_s \in Q_0$ with $\mathcal{R}^+_{i_s} \dots \mathcal{R}^+_{i_1}\mathbb{V} = 0$. Assume that this sequence of reflections is minimal, that is

$$\mathcal{R}^+_{i_r} \dots \mathcal{R}^+_{i_1}\mathbb{V} \text{ indecomposable } \forall r < s\,.$$

In particular by Proposition 9.1:

$$\mathcal{R}^+_{i_{s-1}} \dots \mathcal{R}^+_{i_1}\mathbb{V} \cong \mathbb{S}_{i_{s-1}}\,,$$

and by Corollary 9.1:

$$q_{\mathtt{Q}}(\mathbf{dim}\mathbb{V}) = q_{\mathtt{Q}}(\mathbf{dim}\mathbb{S}_{i_{s-1}}) = 1\,.$$

In other words, $\mathbf{dim}\mathbb{V} \in \Phi_{\mathtt{Q}}$ and $\mathbb{V} \mapsto \mathbf{dim}\mathbb{V}$ is a map from indecomposable representations to positive roots. We want to prove that it induces an injection from the *isomorphism classes of indecomposable representations* to the positive roots. If we prove that this map is injective, then we are done, since for Dynkin quivers $\Phi_{\mathtt{Q}}$ is finite. Suppose $\mathbb{V}, \mathbb{W} \in \mathbf{rep}_k(\mathtt{Q})$ are *indecomposable* with $\mathbf{dim}\mathbb{V} = \mathbf{dim}\mathbb{W}$. Then using the same reflections as earlier:

$$\mathcal{R}^+_{i_{s-1}} \dots \mathcal{R}^+_{i_1}\mathbb{V} \cong \mathbb{S}_{i_{s-1}}$$
$$\mathbf{dim}\mathcal{R}^+_{i_{s-1}} \dots \mathcal{R}^+_{i_1}\mathbb{V} = \mathbf{dim}\mathbb{S}_{i_{s-1}} = e_{i_{s-1}}\,.$$

Now since $\mathbf{dim}\mathbb{V} = \mathbf{dim}\mathbb{W}$, applying the same formula for the reflections on both sides maintains the equality:

$$\mathbf{dim}\mathcal{R}^+_{i_{s-1}} \dots \mathcal{R}^+_{i_1}\mathbb{W} = \mathbf{dim}\mathcal{R}^+_{i_{s-1}} \dots \mathcal{R}^+_{i_1}\mathbb{V} = \mathbf{dim}\mathbb{S}_{i_{s-1}} = e_{i_{s-1}}\,.$$

However, up to isomorphism $\mathbb{S}_{i_{s-1}}$ is the only representation with dimension vector $e_{i_{s-1}}$, giving:

$$\mathcal{R}^+_{i_{s-1}} \dots \mathcal{R}^+_{i_1}\mathbb{W} \cong \mathbb{S}_{i_{s-1}}\,.$$

Apply the mirrored sequence of reflections:

$$\begin{aligned}\mathbb{W} \cong \mathcal{R}^-_{i_1} \dots \mathcal{R}^-_{i_{s-1}} \mathcal{R}^+_{i_{s-1}} \dots \mathcal{R}^+_{i_1} \mathbb{W} &\cong \mathcal{R}^-_{i_1} \dots \mathcal{R}^-_{i_{s-1}} \mathbb{S}_{i_{s-1}} \\ &\cong \mathcal{R}^-_{i_1} \dots \mathcal{R}^-_{i_{s-1}} \mathcal{R}^+_{i_{s-1}} \dots \mathcal{R}^+_{i_1} \mathbb{V} \cong \mathbb{V}\,.\end{aligned}$$

Proving Theorem 9.4 now amounts to proving that the map is surjective. Let $x \in \Phi_{\mathtt{Q}}$. By Lemma 9.5, there exists a sequence of reflections $s_{i_1}, \dots, s_{i_s}$, such that $\mathcal{R}^+_{i_s} \dots \mathcal{R}^+_{i_1} x \leq 0$, where we mean the Weyl action corresponding to $\mathcal{R}^+_{i_s} \dots \mathcal{R}^+_{i_1}$. We can assume, without loss of generality, that this is a minimal such sequence (otherwise we stop one reflection earlier). If the sequence is indeed minimal, then $\mathcal{R}^+_{i_{s-1}} \dots \mathcal{R}^+ i_1$ sends x to $\mathbf{dim}\mathbb{S}_{i_s}$, which is the unit vector with a 1 in the i_s position. We now employ the same trick as earlier and apply the mirrored reflections $\mathcal{R}^+_{i_s} \dots \mathcal{R}^+_{i_1} \mathbb{S}_{i_s}$ to obtain an indecomposable representation which has x as dimension vector. This is due to Proposition 9.1 and Corollary 9.1 using the fact that $\mathbb{S}_{i_s}$ is indecomposable.

We still need to show that $\mathtt{Q}$ is of finite type $\implies$ $\mathtt{Q}$ is Dynkin. We will prove the statement for $\tilde{A}_0, \tilde{A}_n$ and $\tilde{D}_n$, while referring to [Den+08] for the remaining cases. As remarked in the proof of Theorem 9.2, if $\mathtt{Q}$ is not Dynkin, then the underlying graph $\bar{\mathtt{Q}}$ must contain an Euclidean diagram as a subgraph. Now let $\mathtt{Q}'$ denote the restriction of $\mathtt{Q}$ to its Euclidean subgraph and $\mathbb{V}' \in \mathbf{rep}_k(\mathtt{Q}')$ be a representation of the corresponding subquiver. Then we can extend $\mathbb{V}'$ to a representation of $\mathtt{Q}$ by attaching trivial 0 spaces and 0 maps to vertices in $\mathtt{Q}_0 \setminus \mathtt{Q}'_0$. Clearly if $\mathbb{V}'$ is indecomposable to begin with, then its extension $\mathbb{V}$ will remain indecomposable. This means that we can assume, without loss of generality, that $\mathtt{Q}$ is an Euclidean quiver. We need to show that there are an infinite number of pairwise non-isomorphic indecomposable representations for each type of Euclidean quiver. We do this indirectly by constructing a full and faithful functor (see Definition 2.3)$\Psi : \mathrm{Mod}_{k[x]} \to \mathbf{rep}_k(\mathtt{Q})$, where $\mathrm{Mod}_{k[x]}$ denotes the category of finite dimensional $k[x]$-modules, for each type of Euclidean quiver $\mathtt{Q}$.

A $k[x]$-module M is simply an abelian group with an action $\cdot : k[x] \times M \to M$. A k-vector U space is a k-module, with the action of a scalar $x \in k$ being determined by standard scalar multiplication. We can turn U into a $k[x]$ module by endowing it with an endomorphism $f : U \to U$, which defines the action of x by sending an element $u \mapsto x \cdot u$. Thus we can identify a $k[x]$-module M with a pair (U, f), where U is a k-vector space and $f : U \to U$ is an endomorphism. We can take this identification one step further by examining morphisms of finite dimensional $k[x]$-modules. Let $N = (W, g)$ be a finite dimensional $k[x]$-module and $\varphi : M \to N$ be a $k[x]$-linear map. We can interpret $\varphi : U \to W$ as a k-linear map, with the additional property that $\phi \circ f = g \circ f$. This additional property is equivalent to $k[x]$-linearity, since f and g are the actions of x on U and W respectively.

Now, the indecomposable summands of a $k[x]$-module correspond to the Jordan blocks of the decomposition of f. In particular, each indecomposable isomorphism class of $k[x]$-modules corresponds to an unique Jordan block. Since we can construct infinitely many pairwise distinct Jordan blocks, the category $\mathrm{Mod}_{k[x]}$ admits infinitely many isomorphism classes of indecomposables. By providing a fully faithful functor to $\mathbf{rep}_k(\mathtt{Q})$ for each type of Euclidean quiver $\mathtt{Q}$, we are indirectly proving that there are infinitely many indecomposable representations.

The simple case $\mathtt{Q} = \tilde{A}_0$ does not even require this machinery. Indeed, consider first the following indecomposable representation of $\mathtt{Q} = \tilde{A}_0$, where $n > 0$:

$$\begin{tikzcd} k^n \arrow[loop right, "\mathrm{id}"] \end{tikzcd}$$

We obtain an infinite number of indecomposable representations since each $n \in \mathbb{N} > 0$ yields an indecomposable representation.

Next let $\mathtt{Q} = \tilde{A}_n, n \geq 1$. Since reflection functors send isomorphism classes of one indecomposable representation to an isomorphism class of another indecomposable reprepsentation with a reversed arrow, we can assume that $\mathtt{Q}$ is of the form:

$$
\begin{tikzcd}
& & \underset{0}{\bullet} & & \\
\underset{1}{\bullet} \arrow[urr, dash] \arrow[r, dash] & \underset{2}{\bullet} \arrow[r, dash] & \cdots \arrow[r, dash] & \underset{n-1}{\bullet} \arrow[r, dash] & \underset{n}{\bullet} \arrow[ull]
\end{tikzcd}
$$

Indeed, suppose the arrow is in the other direction, that is $0 \to n$. There are two possibilities for the arrow between 0 and 1. Suppose we have $0 \to 1$, then the vertex 0 is a source, and applying $\mathcal{R}_0^-$ yields $n \to 0$. Suppose now that we have $1 \to 0$, and consider the subquiver $\mathtt{S}$ formed by vertices $1, 2, \ldots, n$. Since $\mathtt{S}$ is an A_n-quiver, we can use Lemma 9.4 to convert it to a linear quiver. In particular, we can assume that the arrow $1 \to 2$ is forward facing. Then by $0 \leftarrow 1 \to 2$, vertex 1 is a source and applying $\mathcal{R}_1^-$ yields $0 \to 1 \leftarrow 2$. We can now apply the argument for the earlier case.

Consider the functor:

$$
\begin{aligned}
\Psi :& \mathrm{Mod}_{k[x]} \to \mathbf{rep}_k(\mathtt{Q}) \\
& M = (U, f) \mapsto (V_i, v_i) \\
V_i &= U && 0 \leq i \leq n \\
v_i &= \begin{cases} \mathrm{id}_U & 0 \leq i \leq n-1 \\ f & i = n \end{cases} .
\end{aligned}
$$

Given a $k[x]$-linear map $\varphi : M \to N$, we define $\Psi(\varphi) = \phi : \Psi(M) \to \Psi(N)$ by $\phi_i = \varphi,\ 0 \leq i \leq n$. As we remarked earlier, we can interpret φ as a k-linear map with $\phi \circ f = g \circ \phi$. These are precisely the conditions needed to satisfy Definition 4.5. Visually, we obtain the following representation of $\mathtt{Q}$:

$$
\begin{tikzcd}
& & U & & \\
U \arrow[urr, dash, "id"] \arrow[r, dash, "id"] & U \arrow[r, dash, "id"] & \cdots \arrow[r, dash, "id"] & U \arrow[r, dash, "id"] & U \arrow[ull, "f"']
\end{tikzcd}
$$

We need to prove that the map $\Psi^* : \hom(M, N) \to \hom(\Psi M, \Psi N)$ is injective (faithful) and surjective (full). A $k[x]$-linear map $\varphi : M \to N$ is mapped by Ψ to the morphism $\phi : \Psi M \to \Psi N$ where $\phi_i = \varphi \,\forall i$. Thus $\varphi \in \ker \Psi^* \iff \phi_i = 0 \ \forall i \iff \varphi = 0$, proving injectivity. On the other hand, consider a morphism $\phi : \Psi M \to \Psi N$. The first key observation to make is that $\phi_i = \phi_{i-1},\ 0 < i \leq n$. Indeed per Definition 4.5, we obtain the following commutative diagram for each $1 \leq i \leq n$:

$$
\begin{tikzcd}
(\Psi M)_{i-1} \arrow[r, dash, "\mathrm{id}_U"] \arrow[d, "\phi_{i-1}"'] & (\Psi M)_i \arrow[d, "\phi_i"] \\
(\Psi N)_{i-1} \arrow[r, dash, "\mathrm{id}_W"'] & (\Psi N)_i
\end{tikzcd}
$$

Thus $\phi_i = \phi_{i-1},\ 0 < i \leq n$. In addition, if $i = n$ we also have:

$$\begin{tikzcd}
(\Psi M)_n \arrow[r, "f"] \arrow[d, "\phi_n"'] & (\Psi N)_0 \arrow[d, "\phi_0"] \\
(\Psi N)_n \arrow[r, "g"'] & (\Psi N)_0
\end{tikzcd}$$

Since f and g define the action of x on U and W respectively:

$$\phi_0(x \cdot u) = \phi_0 \circ f(u) = g \circ \phi_n(u) = x \cdot \phi_n(u) = x \cdot \phi_0(u)$$

Finally, due to $\phi_n = \phi_{n-1} = \phi_{n-2} = \cdots = \phi_1 = \phi_0$ and the fact that ϕ_i is k-linear (again by Definition 4.5), we can set $\varphi = \phi_n$ to obtain a $k[x]$-linear morphism with $\Psi(\varphi) = \phi$.

Next, we consider $\mathtt{Q} = \tilde{D}_n\, n \geq 4$. Again ,by repeatedly applying reflection functors, we can assume that the arrows of $\mathtt{Q}$ are of the following form:

$$\begin{tikzcd}
\underset{0}{\bullet} \arrow[dr] & & & & & & \underset{n-1}{\bullet} \arrow[dl] \\
& \underset{2}{\bullet} \arrow[r] & \underset{3}{\bullet} \arrow[r, dash] & \cdots \arrow[r, dash] & \underset{n-3}{\bullet} \arrow[r] & \underset{n-2}{\bullet} & \\
\underset{1}{\bullet} \arrow[ur] & & & & & & \underset{n}{\bullet} \arrow[ul]
\end{tikzcd}$$

Consider the functor

$$\begin{aligned} \Psi :&\mathrm{Mod}_{k[x]} \to \mathbf{rep}_k(\mathtt{Q}) \\ M &= (U, f) \mapsto \mathbb{V} \end{aligned}$$

where $\mathbb{V}$ is the following representation:

$$\begin{tikzcd}
U \arrow[dr, "v_0"] & & & & & & U \arrow[dl, "v_{n-1}"'] \\
& U \oplus U \arrow[r] & U \oplus U \arrow[r, dash, "v"'] & \cdots \arrow[r, dash] & U \oplus U \arrow[r] & U \oplus U & \\
U \arrow[ur, "v_1"'] & & & & & & U \arrow[ul, "v_n"]
\end{tikzcd}$$

and $v_0 = \begin{pmatrix} \mathrm{id}_U \\ 0 \end{pmatrix}$, $v_1 = \begin{pmatrix} 0 \\ \mathrm{id}_U \end{pmatrix}$, $v_{n-1} = \begin{pmatrix} \mathrm{id}_U \\ \mathrm{id}_U \end{pmatrix}$, $v_n = \begin{pmatrix} \mathrm{id}_U \\ f \end{pmatrix}$ and $v = \begin{pmatrix} \mathrm{id}_U & 0 \\ 0 & \mathrm{id}_U \end{pmatrix}$. Next, Ψ sends a $k[x]$-linear map $\varphi : M \to N$ to $\phi : \Psi M \to \Psi N$ with $\phi_i = \begin{cases} \varphi & i \in 0, 1, n, n-1 \\ (\varphi, \varphi) & 2 \leq i \leq n-2 \end{cases}$. We now need to prove that the map $\Psi^* : \hom(M,N) \to \hom(\Psi M, \Psi N)$ is injective and surjective. Both of these properties hold for the same reasons as above, but we will spell out surjectivity which is a bit trickier. Let $\phi : \Psi M \to \Psi N$ be a morphism in $\mathbf{rep}_k(\mathtt{Q})$. Observe first that $\phi_i = \phi_{i-1},\ 0 < i \leq n$. Indeed for $2 < i \leq n-2$ we obtain the commutative diagram:

$$\begin{tikzcd}
U \oplus U \arrow[r, dash, "\begin{pmatrix} \mathrm{id}_U & 0 \\ 0 & \mathrm{id}_U \end{pmatrix}"] \arrow[d, "\phi_{i-1}"'] & U \oplus U \arrow[d, "\phi_i"] \\
W \oplus W \arrow[r, dash, "\begin{pmatrix} \mathrm{id}_W & 0 \\ 0 & \mathrm{id}_W \end{pmatrix}"'] & W \oplus W
\end{tikzcd}$$

Since every ϕ_i is a linear map, we obtain $\phi_i = \phi_{i-1}$, $2 < i \leq n-2$. Next we have $\phi_2 = (\phi_0, \phi_0)$ due to the following diagram:

$$\begin{tikzcd}
U \arrow[r, "\begin{pmatrix} \mathrm{id}_U \\ 0 \end{pmatrix}"] \arrow[d, "\phi_0"'] & U \oplus U \arrow[d, "\phi_2"] \\
W \arrow[r, "\begin{pmatrix} \mathrm{id}_W \\ 0 \end{pmatrix}"'] & W \oplus W
\end{tikzcd}$$

A similar diagram also yields $\phi_2 = (\phi_1, \phi_1)$ and $\phi_{n-2} = (\phi_{n-1}, \phi_{n-1})$. Finally, if $i = n$ we have:

$$\begin{tikzcd}
U \arrow[r, "\begin{pmatrix} \mathrm{id}_U \\ f \end{pmatrix}"] \arrow[d, "\phi_n"'] & U \oplus U \arrow[d, "\phi_{n-2}"] \\
W \arrow[r, "\begin{pmatrix} \mathrm{id}_W \\ g \end{pmatrix}"'] & W \oplus W
\end{tikzcd}$$

Let $u \in U$ be arbitrary, then we have $(\begin{pmatrix} \phi_{n-1}(u) \\ \phi_{n-1}(f(u)) \end{pmatrix}) = \phi_{n-2}(\begin{pmatrix} u \\ f(u) \end{pmatrix}) = \begin{pmatrix} \phi_n(u) \\ g(\phi_n(u)) \end{pmatrix}$. Again, since f and g are the actions of x on U and W respectively, we get:

$$\phi_{n-1}(x \cdot u) = \phi_{n-1} \circ f(u) = g \circ \phi_n(u) = x \cdot \phi_n(u) = x \cdot \phi_{n-1}(u).$$

Hence, setting $\varphi = \phi_n$ yields a morphism of $k[x]$-modules, such that $\Psi\varphi = \phi$.

■

Chapter 10

Viewpoint 3: Category Theory

Since multiparameter persistence modules are representations of *wild quivers*, Gabriel's theorem delivers the unfortunate news that a barcode representation or indeed any complete invariant for multiparameter persistence modules is impossible. However, due to the importance of multiparameter persistence, a large body of work has been devoted to studying various incomplete invariants. We will present one of these invariants - the generalized rank invariant, introduced in [KM21].

10.1 Persistence modules as functors from posets

In the last two chapters, we examined persistence modules by viewing them as finitely generated graded modules and as quiver representations. We now present one final angle from which one might examine persistence modules, namely as functors from general posets.

Definition 10.1 (Persistence modules as functors)**.** *Let $\boldsymbol{P}$ be a poset and **vec** denote the category of finite dimensional vector spaces. A* persistence module *is a functor $\mathcal{M} : \boldsymbol{P} \to \boldsymbol{vec}$.*

This means that $\mathcal{M}$ assigns to each element $p \in \mathbf{P}$, a finite dimensional vector space M_p, and if $p \leq q$, a linear map $\varphi_{\mathcal{M}}(p, q) : M_p \to M_q$.

Example 10 (Single parameter persistence as a functor). Single parameter persistence modules can be viewed as functors $\mathcal{M} : \mathbb{N} \to \mathbf{vec}$.

Remark 10.1. The persistence modules we just defined are sometimes referred to as pointwise finite dimensional (pfd) persistence modules, since the target category **vec** is composed only of finite dimensional vector spaces. A significant portion of the theory holds not only for general vector spaces, but also to a wider class of categories. We will however restrict ourselves to **vec** for the sake of simplicity and since this is the case most often encountered in practice.

The interval representations from chapter 9 are called *interval modules* in this setting.

Definition 10.2 (Interval modules)**.** *Let $\boldsymbol{P}$ be a poset and $I \subset \boldsymbol{P}$ be an interval* [1]. *The associated*

[1] Recall Definition 2.13.

interval module is

$$\mathcal{I}^I : \boldsymbol{P} \to \boldsymbol{vec}$$
$$\mathcal{I}^I_p = \begin{cases} k & p \in I \\ 0 & otherwise \end{cases}$$
$$\varphi_M(p,q) = \begin{cases} 1_k & p,q \in I \text{ and } p \leq q \\ 0 & otherwise \end{cases} .$$

Interval decomposability is defined in a familiar manner.

Definition 10.3. *Let $\boldsymbol{P}$ be a poset. A persistence module $\mathcal{M} : \boldsymbol{P} \to \boldsymbol{vec}$ is* interval decomposable*, if there exists a multiset of intervals barc($\mathcal{M}$) of $\boldsymbol{P}$ such that*

$$\mathcal{M} \cong \bigoplus_{I \in barc(\mathcal{M})} \mathcal{I}^I .$$

We stress again that not every persistence module in this setting is interval decomposable, but if an interval decomposition exists then it is unique up to reordering the summands. In particular, the barcode is a complete invariant of interval decomposable persistence modules.

10.2 The single parameter rank invariant

Before delving into the multiparameter case, we present the rank invariant for the single parameter case. In this case, we can use the rank invariant to recover the barcode. By chapter 7, we know that the barcode is a complete invariant and so since we can recover the barcode from the rank invariant, it too is complete. This result was proven in [CZ09].

Definition 10.4 (Single parameter rank invariant)**.** *Let $\mathcal{M} : \mathbb{N} \to \boldsymbol{vec}$ be a persistence module. The* rank invariant *Rk is a map:*

$$Rk(I) = rank\varphi_{\mathcal{M}}(i,j) \text{ for every interval } I = (i,j) ,$$

where $rank\varphi_{\mathcal{M}}(i,j)$ is simply the rank of the linear map $\varphi_{\mathcal{M}}(i,j)$.

Proposition 10.1. *The barcode barc($\mathcal{M}$) can be recovered from the rank invariant.*

Proof. Let $I = (a,b) \subset \mathbb{N}$ be an interval. We provide an explicit formula for the number of copies of I in barc($\mathcal{M}$) in terms of the rank invariant.

Denote by $C(I)$ the number of intervals in barc($\mathcal{M}$) *containing* I. Intuitively, $\mathrm{Rk}I = \varphi_{\mathcal{M}}(a,b)$ counts the number of features alive between times a and b. However, a feature alive between a and b might have been born before a or die after b. The bar corresponding to a particular feature represents its entire lifespan, which is an interval containing (a,b). Hence,

$$C(I) = \mathrm{Rk}I .$$

.

A standard result from combinatorics, the inclusion-exclusion principle (see [All10]), yields:

$$\# \text{ of copies of } (a,b) = \begin{cases} C((a,b)) - C((a,b+1)) - C((a-1,b)) + C((a-1,b+1)) & 0 \le a < b < \infty \\ C((a,\infty)) - C(a-1,\infty) & 0 \le a < b = \infty \end{cases} .$$

Hence,

$$\# \text{ of copies of } (a,b) = \begin{cases} \mathrm{Rk}((a,b)) - \mathrm{Rk}((a,b+1)) - \mathrm{Rk}((a-1,b)) + \mathrm{Rk}((a-1,b+1)) & 0 \le a < b < \infty \\ \mathrm{Rk}((a,\infty)) - \mathrm{Rk}(a-1,\infty) & 0 \le a < b = \infty \end{cases} . \tag{10.1}$$

■

10.3 The generalized rank invariant

Having seen the completeness of the rank invariant in the single parameter case, we use our discussion in chapter 2 to generalize this notion to arbitrary posets. More specifically, we want to assign an invariant to every path connected subposet $I \in \mathbf{Con}(\mathbf{P})$.

Definition 10.5 (Rank of a persistence module)**.** *Let $\boldsymbol{P}$ be a connected poset and $\mathcal{M} : \boldsymbol{P} \to \boldsymbol{vec}$ be a persistence module. The rank of $\mathcal{M}$ is the rank of the linear map $\varphi_{\mathcal{M}} : \varprojlim \mathcal{M} \to \varinjlim \mathcal{M}$. Recall from Theorem 2.1, that this map exists and is canonical.*

The generalized rank invariant is now obtained by considering the restriction of a persistence module.

Definition 10.6 (Generalized rank invariant)**.** *Let $\boldsymbol{P}$ be a connected poset and $\mathcal{M} : \boldsymbol{P} \to \boldsymbol{vec}$ be a persistence module. The* generalized rank invariant *is a map*

$$Rk : \boldsymbol{Con}(\boldsymbol{P}) \to \mathbb{Z}_{\ge 0} ,$$

which maps $I \in \boldsymbol{Con}(\boldsymbol{P})$ to the rank of the restriction $\mathcal{M}\big|_I$ of $\mathcal{M}$ to I. By $\mathcal{M}\big|_I$, we mean the restriction of the functor $\mathcal{M} : \boldsymbol{P} \to \boldsymbol{vec}$ to the elements of I.

Proposition 10.2 (Definition 10.6 recovers Definition 10.4 in the 1-parameter case)**.** *Let $\mathcal{M} : \mathbb{N} \to \boldsymbol{vec}$ be a persistence module and $I = (a,b)$, $0 \le a \le b$ be an interval. Then the generalized rank invariant of I per Definition 10.6 coincides with the rank invariant of I per Definition 10.4.*

Proof. The generalized rank invariant $\mathrm{Rk}(I)$ is the rank of the map $\varprojlim \mathcal{M}\big|_I \to \varinjlim \mathcal{M}\big|_I$. Proving that $\varprojlim \mathcal{M}\big|_I = \mathcal{M}_a$ and $\varinjlim \mathcal{M}\big|_I = \mathcal{M}_b$ would clearly complete the proof. The first step is to prove that $\mathcal{M}_a$ is a cone. Let $x \in I$ be arbitrary and $f : x \to y$ be a morphism. We need to prove the existence of the commutative diagram depicted in Figure 10.1. Since $x, y \in I$, we have $a \le x$ and $a \le y$. Hence by definition of $\mathbf{P}$ as a category, there exist morphisms $g : a \to x$ and $h : a \to y$. Applying the functor $\mathcal{M}$, we obtain $\pi_x = \mathcal{M}(g)$ and $\pi_y : \mathcal{M}(h)$.

All that remains to be shown is the commutativity. Observe now that $x \le y$, since the morphism $f : x \to y$ can only exist under this condition, yielding $a \le x \le y$. We can compose the morphisms to obtain $f \circ g : a \to y$. Recall, however, that a morphism between objects $a \le y$ in a poset is unique. So, $h = f \circ g$ and $\pi_y = \mathcal{M}(f) \circ \pi_x$ as desired.

$$
\begin{tikzcd}
\mathcal{M}_x \arrow[rr, "\mathcal{M}(f)"] & & \mathcal{M}_y \\
& \mathcal{M}_a \arrow[ul, dashed, "\pi_x"] \arrow[ur, dashed, "\pi_y"'] &
\end{tikzcd}
$$

Figure 10.1

$$
\begin{tikzcd}
\mathcal{M}_x \arrow[rr, "\mathcal{M}(f)"] & & \mathcal{M}_y \\
& \mathcal{M}_a \arrow[ul, "\pi_x"] \arrow[ur, "\pi_y"'] & \\
& C \arrow[uul, "\psi_y"] \arrow[uur, "\psi_x"'] \arrow[u, dashed, "u"'] &
\end{tikzcd}
$$

Figure 10.2

$$
\begin{tikzcd}
\mathcal{M}_a \arrow[rr, "\pi_x"] & & \mathcal{M}_x \\
& C \arrow[ul, "\psi_a"] \arrow[ur, "\psi_x"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{M}_a \arrow[rr, "\pi_y"] & & \mathcal{M}_y \\
& C \arrow[ul, "\psi_a"] \arrow[ur, "\psi_y"'] &
\end{tikzcd}
$$

Figure 10.3

Denote by C, an arbitrary cone over $\mathcal{M}$. To prove that $\mathcal{M}_a$ is a limit, we need to prove the existence of $u : C \to \mathcal{M}_a$ as in Figure 10.2. Since C is a cone, there exists a map $\psi_a : C \to \mathcal{M}_a$, such that the diagrams in Figure 10.3 commute for all $a \leq x \leq y \in I$. Setting $u = \psi_a$ proves $\varprojlim \mathcal{M}\big|_I = \mathcal{M}_a$. In a completely analogous manner, one can show $\varinjlim \mathcal{M}\big|_I = \mathcal{M}_a$, completing the proof. ■

One of the key properties of the single parameter rank invariant is that it is order reversing, that is if $I \subset I'$, then $\mathrm{Rk}(I) \geq \mathrm{Rk}(I')$. Intuitively, this simply means that the number of features that are alive during all of I' is less than or equal to the number of features alive during a sub-time interval I in I'. This property extends to the generalized rank invariant.

Remark 10.2 (Restriction of limits and colimits). Let $\mathbf{P}$ be a poset, $\mathbf{Q} \subset \mathbf{P}$ be a subposet and $\mathcal{F} : \mathbf{P} \to \mathcal{C}$ be a functor. Suppose that $\varprojlim F$ and $\varinjlim F$ exist, then these can be restricted to a cone and cocone respectively over $\mathcal{F}\big|_Q$. Indeed if $x \in \mathbf{Q}$ and $f : x \to y$ is a morphism in $\mathbf{Q}$, then these are also objects and morphisms in $\mathbf{P}$. Moreover, the cone and cocone maps exist since the limit is a cone and colimit a cocone.

Note that $\varprojlim F$ does not restrict to a limit, that is $\varprojlim \mathcal{F}\big|_{\mathbf{Q}} \neq \varprojlim F$. This is because the map $u : C \to \varprojlim F$ in Definition 2.6 might not always be in the restriction. Similarly, $\varinjlim F$ does not restrict to a colimit over $\mathcal{F}\big|_{\mathbf{Q}}$.

Theorem 10.1 (Order reversing). Let $\mathbf{P}$ be a connected poset, $\mathcal{M} : \mathbf{P} \to \mathbf{vec}$ be a persistence module and $I \subset I' \in \mathbf{Con}(\mathbf{P})$. Then, $\mathrm{Rk}I' \leq \mathrm{Rk}I$.

Proof. We need to show that the rank of $\phi : \varprojlim \mathcal{M}\big|_{I'} \to \varinjlim \mathcal{M}\big|_{I'}$ is less than or equal to the rank of $\psi : \varprojlim \mathcal{M}\big|_{I} \to \varinjlim \mathcal{M}\big|_{I}$. By Remark 10.2, $\varprojlim \mathcal{M}\big|_{I'}$ can be restricted to a cone over $\mathcal{M}\big|_{I}$ and $\varinjlim \mathcal{M}\big|_{I'}$ to a cocone. Let $a \in I$ be arbitrary, then by Definition 2.5 and Definition 2.7, we obtain the commutative diagrams diagrams in Figure 10.4. Combining the two diagrams in Figure 10.4

$$
\begin{tikzcd}
\varprojlim \mathcal{M}\big|_{I} \arrow[rr, "\pi_a"] & & \mathcal{M}_a \\
& \varprojlim \mathcal{M}\big|_{I'} \arrow[ul, "u"] \arrow[ur, "\pi'_a"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\varinjlim \mathcal{M}\big|_{I} \arrow[rr, "i_a"] \arrow[dr, "v"'] & & \mathcal{M}_a \arrow[dl, "i'_a"] \\
& \varinjlim \mathcal{M}\big|_{I} &
\end{tikzcd}
$$

Figure 10.4

with the maps ϕ and ψ, we obtain Figure 10.5. The triangles $\spadesuit$ and $\clubsuit$ commute by Figure 10.4,

$$
\begin{tikzcd}
\varprojlim \mathcal{M}_I \arrow[rr, "\phi"] \arrow[dr, "\pi_a"] \arrow[rr, phantom, "\diamondsuit", shift right=6ex] & & \varinjlim \mathcal{M}_I \arrow[dd, "v"] \\
{} \arrow[r, phantom, "\spadesuit"] & \mathcal{M}_a \arrow[ur, "i_a"] \arrow[dr, "i'_a"] \arrow[r, phantom, "\clubsuit"] & {} \\
\varprojlim \mathcal{M}'_I \arrow[uu, "u"] \arrow[ur, "\pi'_a"] \arrow[rr, "\psi"] \arrow[rr, phantom, "\heartsuit", shift left=6ex] & & \varinjlim \mathcal{M}'_I
\end{tikzcd}
$$

Figure 10.5

while $\heartsuit$ and $\diamondsuit$ commute by the properties of the limit and colimit respectively. We complete the proof with a diagram chase:

$$\psi \overset{\heartsuit}{=} i'_a \circ \pi'_a \overset{\spadesuit}{=} i'_a \circ (\pi_a \circ u) \overset{\clubsuit}{=} (v \circ i_a) \circ (\pi_a \circ u) \overset{\text{assoc}}{=} v \circ i_a \circ \pi_a \circ u \overset{\diamondsuit}{=} v \circ \phi \circ u\,.$$

Thus by linear algebra, $\mathrm{rank}\psi \leq \mathrm{rank}\phi$ and consequently, $\mathrm{Rk}I' \leq \mathrm{Rk}I$. ■

In the single parameter case, the rank invariant of an interval (a, b) captured the number of features alive during (a, b). Formally this translates to the number of intervals in the barcode *containing* (a, b). We now want to prove this for the generalized rank invariant of interval decomposable modules.

Theorem 10.2. Let $\mathbf{P}$ be a locally finite, connected poset and $\mathcal{M} : \mathbf{P} \to \mathbf{vec}$ be an interval decomposable persistence module. Then, the rank invariant $\mathrm{Rk}(J)$ of $J \in \mathbf{Con}(\mathbf{P})$ corresponds to the number of intervals $C(J)$ in $\mathrm{barc}(\mathcal{M})$ *containing* J.

Proof (under the extra assumption that $\varprojlim \mathcal{M}\big|_J$ is finite dimensional). Under our assumptions we have:

$$\begin{aligned}
\mathrm{Rk}(J) &\overset{\text{def}}{=} \mathrm{rank}(\varprojlim \mathcal{M}\big|_J \to \varinjlim \mathcal{M}\big|_J) \\
&\overset{\spadesuit}{=} \mathrm{rank}(\varprojlim \bigoplus_{K \in \mathrm{barc}(\mathcal{M})} \mathcal{I}^K\big|_J \to \varinjlim \bigoplus_{K \in \mathrm{barc}(\mathcal{M})} \mathcal{I}^K\big|_J) \\
&\overset{\clubsuit}{=} \mathrm{rank}(\bigoplus_{K \in \mathrm{barc}(\mathcal{M})} \varprojlim \mathcal{I}^K\big|_J \to \bigoplus_{K \in \mathrm{barc}(\mathcal{M})} \varinjlim \mathcal{I}^K\big|_J) \\
&\overset{\heartsuit}{=} \sum_{K \in \mathrm{barc}(\mathcal{M})} \mathrm{rank}(\varprojlim \mathcal{I}^K\big|_J \to \varinjlim \mathcal{I}^K\big|_J)\,.
\end{aligned} \tag{10.2}$$

First, ♠ simply expresses $\mathcal{M}$ in terms of its interval decomposition. Next, since $\varprojlim \mathcal{M}\big|_J$ is finite dimensional, direct sums coincides with direct products. One of the key properties of the limit is that it preserves direct products, and the colimit preserves direct sums (see [Lan78]). By exploiting this property, ♣ exchanges the sum and limit/colimit. Finally, ♡ is due to linear algebra.

Now we claim:

$$\mathrm{rank}(\varprojlim \mathcal{I}^K\big|_J \to \varinjlim \mathcal{I}^K\big|_J) = \begin{cases} 1 & K \subset J \\ 0 & \text{otherwise} \end{cases}.$$

Recall that $\mathcal{I}^K$ has zero spaces everywhere outside the interval K. If $K \not\subseteq J$, then either the limit or the colimit of the restriction $\mathcal{I}^K\big|_J$ is forced to be a zero space and hence the rank is 0. On the other hand, if the entire interval $K \subset J$, then both the limit and colimit are in the restriction $\mathcal{I}^K\big|_J$. Both these spaces are simply the field k with the identity map between them having rank 1.

Thus, the sum in Equation 10.2 amounts to calculating $C(J)$. ■

10.4 Completeness of the Generalized Rank Invariant for Interval Decomposable Modules

Continuing with the process of generalizing concepts from single parameter persistence, we now generalize the idea of persistence diagrams. In single parameter persistence, the barcode and the rank invariant are both complete invariants that can be recovered from each other. We begin with some useful notation, that will enable us to extend Equation 10.1 to this setting.

Definition 10.7 (Neighborhood and perimeter)**.** *Let* $\boldsymbol{P}$ *be a locally finite poset and* $I \in \boldsymbol{Con}(\boldsymbol{P})$*. The* neighborhood *of* I *is*

$$nbd(I) = \{p \in \boldsymbol{P} \setminus I \,|\, \exists q \in \boldsymbol{P} \text{ with } q \asymp p\}\,.$$

The perimeter o_I *of* I *is simply the cardinality of* $nbd(I)$*. If* $I = \{p\}$ *is a singleton, then the neighborhood of* I *coincides with the graph-theoretic neighborhood of the vertex* p *in Hasse*$\boldsymbol{P}$*.*

Example 11 (Neigborhoods in $\mathbb{Z}$). Let $[a, b] \subset \mathbb{Z}$ be an interval, then

$$\text{nbd}(I) = \begin{cases} a-1, b-1 & a, b \text{ finite} \\ a-1 & I = [a, \infty) \\ b+1 & I = (-\infty, b] \\ \emptyset & I = \mathbb{Z} \end{cases}.$$

Consequently,

$$o_I = \begin{cases} 2 & a, b \text{ finite} \\ 1 & I = [a, \infty) \\ 1 & I = (-\infty, b] \\ 0 & I = \mathbb{Z} \end{cases}.$$

Definition 10.8 (Essentially finite posets)**.** *A poset* $\boldsymbol{P}$ *is* essentially finite *if it is* locally finite *and* every $I \in \mathbf{Con}(\mathbf{P})$ has finite perimeter.

The integers $\mathbb{Z}$ are essentially finite as seen in Example 11. While $\mathbb{Z}^d$ is not essentially finite for $d > 1$, clearly any finite subset of $\mathbb{Z}^d$is essentially finite.

Definition 10.9 (n^{th} entourage)**.** *Let* $\boldsymbol{P}$ *be an essentially finite poset,* $I \in \boldsymbol{Con}(\boldsymbol{P})$ *and* $n \in \mathbb{N}$*. The* n^{th} *entourage* I^n *of* I *is*

$$\{J \subset \boldsymbol{P} | I \subset J \subset I \cup nbd(I) \text{ and } \mid J \cap nbd(I) \mid = n\}\,.$$

Intuitively, I^n *is the collection of all subposets obtained by adding* n *points from the neighborhood* $nbd(I)$ *of* I *to* I*.*

Remark 10.3 (Properties of I^n). The following two observations will prove useful:

1. An element of I^1 is a subset that adds 1 point from the neighborhood of I, which amounts to choosing a single point from the neighborhood to add in. I^2 picks 2 points and adds them to I, and so in general $\mid I^n \mid = \begin{cases} \binom{o_I}{n} & n \leq o_I \\ 0 & n > o_I \end{cases}$.
2. Any set in I^n consists of elements of I and its neighborhood and so it is path connected by construction (in $\mathbf{Con}(\mathbf{P})$).

We are now ready to extend Equation 10.1.

Theorem 10.3 (Generalized Rank Invariant and the Barcode). Let $\mathbf{P}$ be a connected, essentially finite poset and $\mathcal{M} : \mathbf{P} \to \mathbf{vec}$ be an interval decomposable persistence module. Then, the number of copies of $I \in \mathbf{Con}(\mathbf{P})$ in $\mathrm{barc}(\mathcal{M})$ is

$$\#(I) = \mathrm{Rk}(I) - \sum_{J \in I^1} \mathrm{Rk} J + \sum_{J \in I^2} \mathrm{Rk}(J) \ldots (-1)^{o_I} \sum_{J \in I^{o_I}} \mathrm{Rk}(J)\,, \tag{10.3}$$

where Rk denotes the generalized rank invariant and o_I the perimeter.

Remark 10.4. To gain an insight into why this might hold, we apply it to the single parameter case, where $\mathbf{P} = \mathbb{Z}$. Let $I = (a, b) \subset \mathbb{Z}$. By Example 11, we know $o_I \leq 2$ and so by Remark 10.3, $I^n = \emptyset \, \forall \, n > 2$. Suppose $a < b$ are both finite, then

$$\begin{aligned} I^1 &= \{(a-1, b), (a, b+1)\} \\ I^2 &= \{(a-1, b+1)\}\,. \end{aligned}$$

Substituting this into Equation 10.3 we obtain:

$$\begin{aligned} \# \text{ of copies of } (a,b) &= \mathrm{Rk}(I) - \sum_{J \in I^1} \mathrm{Rk} J + \sum_{J \in I^2} \mathrm{Rk}(J) \\ &= \mathrm{Rk}((a,b)) - \mathrm{Rk}((a, b+1)) - \mathrm{Rk}((a-1, b)) + \mathrm{Rk}((a-1, b+1))\,. \end{aligned}$$

If $I = (a, \infty)$, then

$$\begin{aligned} I^1 &= \{(a-1, \infty)\} \\ I^2 &= \emptyset\,. \end{aligned}$$

Thus, Equation 10.3 agrees with Equation 10.1.

Proof of Theorem 10.3. Let $I \in \mathbf{Con}(\mathbf{P})$ and $\mathcal{M} \cong \bigoplus_{K \in \mathrm{barc}(\mathcal{M})} \mathcal{I}^K$. We continue mimicking the ideas presented in Proposition 10.1. Denote by $C'(I)$, the number of intervals in $\mathrm{barc}(\mathcal{M})$ which properly contain I, that is $C'(I) = |\{K \in \mathrm{barc}\mathcal{M} | I \subsetneq K\}|$.Then, clearly:

$$\#(I) = C(I) - C'(I)\,.$$

Observe that one can recover the first formula in Proposition 10.1 from the above fomulation. We need only show that $C(I) - C'(I)$ yields Equation 10.3. Recall from Theorem 10.2, that $C(I) = \mathrm{Rk}(I)$.

Intuitively, we would expect $\{K \in \mathrm{barc}\mathcal{M} | I \subsetneq K\}$ to be the union of intervals in the neighborhood of I contained in the barcode. More formally, we want to prove:

$$\{K \in \mathrm{barc}\mathcal{M} | I \subsetneq K\} = \bigcup_{J \in I^1} \{K \in \mathrm{barc}(\mathcal{M}) | J \subset K\}\,.$$

Let $J \in \mathbf{Con}(\mathbf{P})$ such that $I \subsetneq J$. It suffices to show that there exists an element $j \in J \cap \mathrm{nbd}(I)$, since then j would be in some element of I^1. Let $p \in J \setminus I, q \in I$ be arbitrary. Since J is path connected and both points are in J, there exists a sequence

$$p = p_0 p_1 \ldots p_m = q \qquad\qquad p_i \asymp p_{i+1} \, \forall i$$

Visually, there is a path in the subgraph of Hasse($\mathbf{P}$) induced by J from p to q. The path leads from I to $K \setminus I$ with consecutive vertices being adjacent to each other, so some vertex q_i along the way must be in nbd(I). Per construction of the path, q_i is also in J.

Once again using the inclusion exclusion principle (see [All10]):

$$C'(J) = |\bigcup_{J \in I^1} \{K \in \mathrm{barc}(\mathcal{M}) | J \subset K\}| = \sum_{J \in I^1} C(J) - \sum_{J \in I^2} C(J) + \sum_{J \in I^3} C(J) + \cdots + (-1)^{o_I - 1} \sum_{J \in o_I} C(J)\,,$$

where we used $C(J) = |\{K \in \mathrm{barc}(\mathcal{M}) | J \subset K\}|$ and that pairwise intersections correspond to elements of I^2, triple intersections to elements of I^3 and so on. Applying Theorem 10.2 to each of the summands yields:

$$C'(J) = \sum_{J \in I^1} \mathrm{Rk}(J) - \sum_{J \in I^2} \mathrm{Rk}(J) + \sum_{J \in I^3} \mathrm{Rk}(J) + \cdots + (-1)^{o_I - 1} \sum_{J \in o_I} \mathrm{Rk}(J)\,.$$

Finally, computing $C(J) - C'(J)$ yields the desired result. ■

Theorem 10.3 provides a useful criterion for interval decomposability.

Corollary 10.1. Let $\mathbf{P}$ be a connected, essentially finite poset, $\mathcal{M} : \mathbf{P} \to \mathbf{vec}$ be a persistence module. Then if there exists an $I \in \mathbf{Con}(P) \setminus \mathbf{Int}(P)$, such that $\#(P) \neq 0$, then $\mathcal{M}$ is *not interval decomposable*. Furthermore, if there exists $I \in \mathbf{Int}(\mathbf{P})$ such that $\#(I) < 0$, then $\mathcal{M}$ is *not interval decomposable*.

Proof. Let $I \in \mathbf{Con}(\mathbf{P}) \setminus \mathbf{Int}(\mathbf{P})$ and assume $\mathcal{M}$ is interval decomposable, then by Theorem 10.3 $\#(I)$ is the number of copies of I in barc($\mathcal{M}$). However that means the interval decomposition of $\mathcal{M}$ contains $\#(I)$ copies of I, which is not an interval, leading to a contradiction.

Next suppose $I \in \mathbf{Int}(\mathbf{P})$ with $\#(I) < 0$ and assume that $\mathcal{M}$ is interval decomposable. Then again by Theorem 10.3 $\#(I)$ is the number of copies of I in barc($\mathcal{M}$). However this means that the interval decomposition contains a negative number of copies of I, which is a contradiction. ■

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