Whitney Stratifications and Intersection Homology

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May 19, 2023

Contents

1 Introduction

The aim of this exposition is to introduce Whitney stratifications in the context of algebraic geometery. Broadly speaking, stratification theory aims to partition topological spaces into subsets that *glue* togther in a *nice manner*. One of the original motivations was to analyze singular spaces, since a lot of results in algebraic geometery and algebraic topology fail when one considers singular spaces. These include fundamental duality theorems such as Poincaré duality. Consider for example Poincaré duality and how it fails on reasonable spaces such as the pinched torus.

1.1 Poincaré duality fails on singular spaces

To see why Poincaré duality fails on signular spaces, it is convinient to recall the formulation of the theorem in terms of intersections of cycles rather than the modern formulation involving fundamental classes. Indeed, this is the language Poincaré used in his original paper and by Goresky and MacPherson in their introduction of intersection homology in [GM80].

Theorem 1.1 (Poincaré duality)*.* Let M be a an n-manifold and suppose a and b are cycles in M with $\dim a = i$ and $\dim b = j$. Then, the following statements hold:

1. The intersection of $a \cap b$ is a cycle, i.e. $\partial(a \cap b) = 0$.

2. The homology class of $a \cap b$ depends only on the homology classes of a and b, i.e. if a' and b' are homologous to a and b respectively then $a' \cap b'$ is homologous to $a \cap b$. Thus intersection defines a product

$$H_i(M) \times H_j(M) \xrightarrow{\cap} H_{i+j-n}(X).$$

3. *Poincaré duality.* If $i + j = n$,the pairing is *nondegenerate* and there exists an augmentation map

$$H_i(M) \times H_j(M) \xrightarrow{\cap} H_0(X) \xrightarrow{\epsilon} \mathbb{Z}$$

where ϵ simply counts the number of points in the intersection.

Remark 1.1. Strictly speaking, one needs to take the orientation of M into account with positive and negative signs when defining the map ϵ. This will not be relevant for us so we do away with the signs for simplicity.

However, Theorem 1.1 fails for "reasonable" spaces with singularities. Consider first the suspension $S(T)$ of the torus T with singularities at A and B, depicted in Figure 1. Denote by a and b the two 1-cycles of the torus T. In the suspension the corresponding cycles $S(a)$ and $S(b)$ are homeomorphic to $\mathbb{S}^2$, since $S(\mathbb{S}^1)$ is homeomorphic to $\mathbb{S}^2$. $S(a)$ and $S(b)$ form 2-cycles in $S(T)$ but their intersection $S(a) \cap S(b)$ is just a line segment which is *not* a cycle. Thus point 1 of Theorem 1.1 does not hold for $S(T)$.

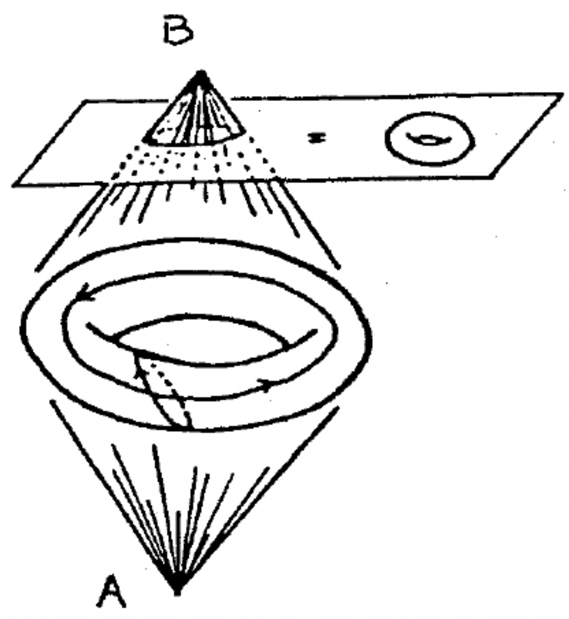

Figure 1: Suspension of a torus (taken from [Bra96])

Even in cases when point 1 holds, point 2 might fail to hold rendering the intersection ill defined. To see this, consider the pinched torus depicted in Figure 2. In this case the cycle b can be shrunk to a point by moving it along the surface to the singularity A. The cycle c is also clearly trivial. Hence, the cycles b and c are homologous, but we have

$$a \cap b = \{*\} \neq \emptyset = a \cap c$$

contradicting point 2.

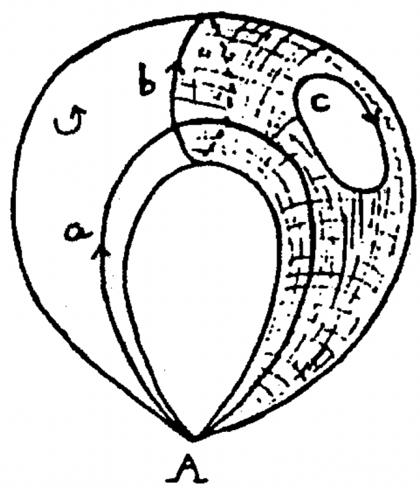

Figure 2: Pinched torus (taken from [Bra96])

Thus the hypotheses excludes a large class of spaces. Whitney stratifications break up singular spaces into smooth chunks in a manner that allows one to draw certain analagous conclusions about the space by analyzing each smooth stratum. The Whitney conditions impose constraints on the strata, that ensure that they fit together in a manner that allows us to draw these conclusions.

2 Definition and first examples

To understand what *nice* means in this setting, let us first define what we mean by a general startification and use an example to see why we might want to impose extra *niceness* conditions.

Definition 2.1 (Stratification)**.** *A* stratification of *a topological space X is a finite filtration*

$$\emptyset \subset F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = X$$

such that

1. *F_i is closed in X for all i.*

2. *The sets $S_i = F_i \setminus F_{i-1}$ are smooth manifolds of dimension i of finitely many connected components and are called the* strata *of X.*

The family $\mathcal{S} = \{S_i\}_i$ forms a partition of X and is referred to as a stratifiction of X.

Remark 2.1. Each stratum is defined as the difference of two closed subsets and is thus locally closed.

Let us now construct a naive stratification of an algebraic set, that exhibits certain *bad* behavior and hence motivates Whitney's condition on strata. The example we consider is Whitney's umbrella, which is the algebraic variety $V \subset \mathbb{R}^3$ given by the zero locus of $f = x^2 - zy^2$. Set $\mathcal{S} = \{X, Y\}$, where X denotes the z-axis and $Y = V \setminus X$. The resultant stratification is depicted in Figure 3.

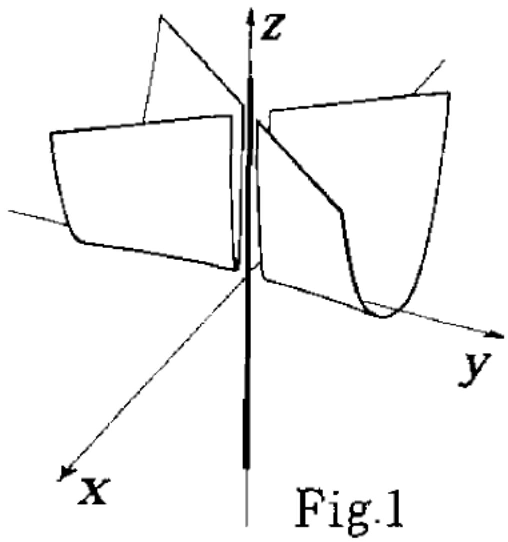

Figure 3: A *bad* stratification (taken from [Gib+76])

To see why this stratification is undesirable, let us examine the intersection of the stratum X with neighborhoods around the origin. Let $\epsilon > 0$, then the intersection of a neighborhood of $(0, 0, -\epsilon)$ in V with X is simply X itself since V lies completely in the region where z is nonnegative. On the other hand, if we examine the intersection of a neighborhood around $(0, 0, 0)$ in V with X we get a self intersecting parabola. Finally the intersection of a neighborhood around $(0, 0, \epsilon)$ in V with X, yields yet another shape. All three situations are depicted in Figure 4, which is also taken from [Gib+76].

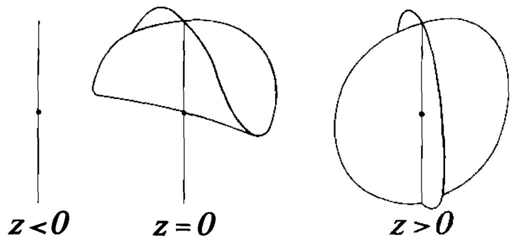

Figure 4: Topology changes along the startum X (from [Gib+76])

Moving about the origin on straum X causes the topology to vary substantially. This is rather undesirable, because one of the points of stratification is to break up the original variety into pieces whose topology we understand and then use this information to draw conclusions about the space as a whole.

Remark 2.2. The stratification we just described was obtained by taking the singular part of V to be one stratum and its complement in V to be the other stratum. One can use this method to inductively stratify an arbitrary algebraic variety V. Indeed, let $\dim V = d$, then we can define a filtration $V = V_d \supset V_{d-1} \supset \dots$ inductively as follows. Begin by setting $V_d = V$, then for $i < d$

$$V_i = \begin{cases} \mathrm{Sing}(V_{i+1}) & \dim V_{i+1} = i + 1 \\ V_{i+1} & \dim V_{i+1} < i = 1 \end{cases}$$

where $\mathrm{Sing}V_{i+1}$ denotes the set of singular points of V_{i+1}.

A stratification $\mathcal{S}$ is obtained by taking the differences $V_i \setminus V_{i-1}$ as the strata. There are only finitely many such differences and each one gives rise to a smooth submanifold of dimension i (or is empty if $V_i = V_{i-1}$).

We can improve this stratification, by further breaking up the z-axis into three strata. This is depicted in Figure 5. We will later see that this stratification is an example of one that satisfies *Whitney's conditions* and known as a *Whitney stratification*. Whitney formalized the concept we

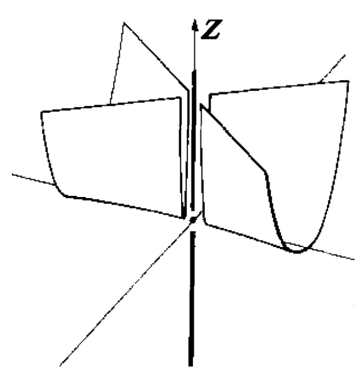

Figure 5: A *Whitney* stratification of Whitney's umbrella (taken from [Gib+76])

just described using the two stratifications into two conditions.

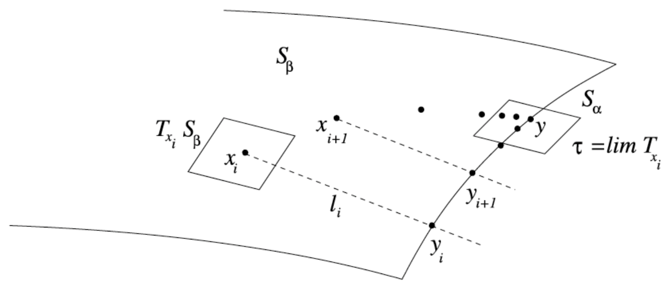

Figure 6: Whitney's Conditions (from [Ban07])

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Definition 2.2 (Whitney's conditions (A) and (B))**.** *Let X and Y be manifolds.*

(A) The pair (X, Y) satisfies the Whitney regularity condition (A) *at $x \in X$ if the following condition holds. Suppose there exist a sequence $\{y_n\}_n \in Y$, such that $\lim_{n\to\infty} y_n = x$. Then, $T_x X \subset \lim_{n\to\infty} T_{y_n} Y$.*

(B) The pair (X, Y) satisfies the Whitney regularity condition (B) *at $x \in X$ if the following condition holds. Suppose there exist sequences $\{x_n\}_n \in X$ and $\{y_n\}_n \in Y$, such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x$. Then, $\lim_{n\to\infty} l_n \subset lim_{n\to\infty} T_{y_n} Y$, where l_n denotes the line spanned by $y_n - x_n$.*

We say that Y is Whitney regular over X, if (X, Y) satisfies condition (B) at every $x \in X$.

Notation 2.1. Denote by $B(X, Y)$ the set $\{x \in X | (X, Y) \text{ does not satisfy condition } (B) \text{ at } x\}$. These are precisely the points in X at which Y fails to be Whitney regular over X and will play a key role in the next section.

Remark 2.3 (Condition (B) $\implies$ (A)). In his widely referenced notes [Mat12] on Whitney's conditions, Mather noted that condition B implies condition A. On the other hand, it is easy to come up with examples to prove that the converse ($A \implies B$) is *false*.

Unless otherwise specfied, we mean condition B when referring to Whitney's conditions.

Definition 2.3 (Whitney stratification)**.** *A stratification $\mathcal{S}$ of a topological space X is a Whitney stratification, if it is* locally finite *and (S_i, S_{i-1}) is Whitney regular for every i.*

The *local finiteness* of $\mathcal{S}$ Definition 2.3, simply means that every point $x \in V$ posseses an open neighborhood $U \subset X$ such that only finitely many $S \in \mathcal{S}$ satisfy $U \cap S \neq \emptyset$. In other words, U meets only finitely many strata. Indeed Whitney's condition weed out the first naive stratification of the Whitney umbrella. It is instructive to work this out completely.

Example 2.1 (Naive stratification of Whitney's umbrella is not regular). We claim that the stratification depicted in Figure 3 fails condition (A) (and hence condition (B)) at the origin. Indeed, along the y-axis we have

$$\nabla V = (2x, -2zy, -y^2) = (0, 0, y^2).$$

If we pick a sequence $\{y_n\}$ with $\lim_{n\to\infty} = 0$, then the sequence of tangent planes $T_{y_n} Y$ converges to the plane $z = 0$ and clearly $T_O X \subsetneq \{z = 0\}$ (recall that X is the z-axis).

3 An intrinsic fomulation of condition (b)

In order to prove Theorem 4.1, which is critical to proving that semialgebraic sets admit Whitney stratification, we need to rephrase condition (B) of Definition 2.2 in intrinsic terms. We will do this using *blow ups*, which is not surprising since we are dealing with singularities after all.

Recall that we can define the blow up of a manifold M along a submanifold U as

$$B_U M = P_{\eta_U} \sqcup (M \setminus U)$$

where P_{η_U} denoting the projective normal bundle of U in M. The projection $\pi : B_U M \to M$ can be obtained by setting it to be the normal projection P_{η_U} on P_{η_U} and the inclusion $M \setminus U \hookrightarrow M$ on $M \setminus U$.

Let M be a manifold, then the blowup of M^2 along its diagonal $\triangle_M = \{(m,m)|m \in M\}$ is

$$B_{\triangle_M} M^2 = P_\eta \sqcup (M^2 \setminus \triangle_M)$$

where P_η is the projective normal bundle of $\triangle_M$ in M^2 (we suppressed the $\triangle_M$ in the notation for convenience). But we can canonically identify the normal bundle η of $\triangle_M$ in M^2 with the tangent bundle TM. Indeed, given $m \in \triangle_M$

$$\eta_m = (T_m M \oplus T_m M)/\triangle_M$$

Finally, we can define a map $T_m M \oplus T_m M \to T_m M$ by sending $v \oplus w \mapsto v - w$. This map induces an isomorphism on eta_m allowing us to identity η_m with $T_m M$ and η with TM. Consequently, we can identify the projective normal bundle P_η with the projective tangent bundle PTM to obtain

$$B_{\triangle_M} M^2 = PTM \sqcup (M^2 \setminus \triangle_M).$$

An element of the blow up may either be a point $(m,n) \in M^2$ with $m \neq n$ or it may be a tangent direction in on M.

To rephrase condition (B) of Definition 2.2 we need to discuss what it means for a sequence of points $(m_i, n_i) \in M^2 \setminus \triangle_M$ to converge to a direction $l \in PTM$. This can only happen if $\lim_{i\to\infty} m_i = \lim_{i\to\infty} n_i$ and the direction from x_i to y_i converges to l. More concretely if $M = \mathbb{R}^n$ the sequence (m_i, n_i) converges to $(x, l) \in \mathbb{R}^n \times \mathbb{R}P^{n-1}$ if $\lim_{i\to\infty} m_i = \lim_{i\to\infty} n_i = x$ and the direction of the line passing through x_i and y_i converges to l. Putting all this together we obtain

Proposition 3.1 (Intrinsic formulation of condition (B))**.** *Let X and Y be submanifolds of a manifold M. The pair (X,Y) satisfies condition (B) at $x \in X$ if and only if the following condition holds. Let $\{x_n\} \in X$ and $\{y_n\} \in Y$ be a sequence of points with $x_n \neq y_n$ for all n. Suppose that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x$, (x_n, y_n) converges to $l \in PTM_x$. Then, $l \subset \lim_{n\to\infty} T_{y_n} Y$.*

Notice that we now have a condition on points of the blow up and in particular on the projective tangent bundle.

4 Algebraic sets admit a Whitney stratification

The Whitney conditions are widely used in many fields, because a large class of objects admit Whitney stratifications. At the highest level of generality, one can prove that every definable set in an o-minimal structure admits a Whitney stratification. A wide range of objects including semialgebraic sets and subanalytic sets are definable on o-minimal structures. Since the focus of this exposition is primarily on stratifications and their application in algebraic geometery, we will limit ourselves to stratifications of semialgebraic sets and refer the reader to lectures 3 and 4 of [Fuk+10] for details on o-minimal structures and stratifications of definable sets.

We recall some properties of semialgebraic sets that are required to prove that they admit a Whitney stratification.

Definition 4.1 (Semialgebraic sets)**.** *Let k be a field, then $S \subset k^n$ is* semialgebraic *if it can be written as a finite union of sets carved out by polynomial inequalites and equalities. In other words S is the finite union of sets of the form*

$$\{x \in k^n | f(x) > 0, \ \ f \in k[x_1, \ldots, x_n]\} \qquad \textit{and}$$
$$\{x \in k^n | f(x) = 0, \ \ f \in k[x_1, \ldots, x_n]\}.$$

Semialgebraic sets have some nice closure properties. We omit the proofs of the next two results for the sake of brevity.

Lemma 4.1 (Closure under unions, intersections and projections)*.* Similar to algebraic varieties, semialgebraic sets are closed under finite unions and intersections. However unlike algebraic varieties, semialgebraic sets are also closed under projection by the Tariski-Seidenberg theorem.

Further, the singularities of a semialgebraic set form a semialgebraic set.

Lemma 4.2*.* Let V be a semialgebraic set. Then the set of singularities $\operatorname{Sing}(V)$ of V is also semialgebraic and has $\dim \operatorname{Sing}(V) < \dim V$.

The following theorem, is perhaps the most involved in this exposition and is critical to constructing Whitney stratifications of semialgebraic sets. It was first proven by Whitney but we follow the more geometeric account provided in [Mat12].

Theorem 4.1 (Whitney)*.* Let X and Y be semialgebraic sets in k^n, then the set $B(X,Y)$ is also semialgebraic and has $\dim B(X,Y) < \dim X$.

Proof. If $x \in B(X,Y)$, then by Proposition 3.1 there exist sequences $\{x_n\} \in X$ and $\{y_n\} \in Y$ with $x_n \neq y_n$ for all n such that $\{(x_n, y_n)\} \in \mathbb{R}^{\not\vDash\ltimes} \setminus \triangle_{\mathbb{R}^{2n}}$ that converge to (x,l) but $l \notin lim_{n\to\infty} T_{y_n} Y$. This just means that $B(X,Y)$ form a subset of PTM. We can project $PTM \to M$ by the standard projection (strictly speaking it is the complement of the set that satisfies the condition, but complements of semialgebraic sets are semialgebraic). By the Tariski-Seidenberg theorem, a projection of a semialgebraic set is semialgebraic, so $B(X,Y)$ is semialgebraic. The statement regarding dimension is rather involved and we refer the interested reader to [Wal75]. ■

Proposition 4.1 (Semialgebraic sets admit Whitney stratifications)**.** *Let V be a semialgebraic set with $\dim V = d$. Then, V admits a Whitney stratification $\mathcal{F}$ composed of finitely many semialgebraic strata.*

Proof. The naive construction we described in Remark 2.2 can be adapted to produce a Whitney stratification. We do this by explicitly removing the points where the Whitney condition fails from each stratum.

Inductively construct a filtration $V = V_d \supset V_{d-1} \supset \dots$ as follows. Begin by setting $V_d = V$ and $V_{d-1} = \mathrm{Sing}(V_d)$. Next assume by induction that filtration has been constructed until V_i, i.e. we have $V = V_d \supset V_{d-1} \supset \cdots \supset V_i$ and wish to construct V_{i-1}. Set

$$V_{i-1} = \begin{cases} V_i & \dim V_i < i \\ \mathrm{closure}(\bigcup_{j=i+1}^{d} W(V_i, V_j \setminus V_{j-1})) & \dim V_i = i \end{cases}$$

where $W(X, Y) = \mathrm{Sing}(X) \cup B(X \setminus \mathrm{Sing}(X), Y \setminus \mathrm{Sing}(Y))$.

By Lemma 4.1, Lemma 4.2, and Theorem 4.1 V_{i-1} is semialgebraic and has $\dim V_{i-1} \leq i - 1$. Moreover, there are only finitely many non-empty differences $F_i = V_i \setminus V_{i-1}$. By construction, (F_i, F_{i-1}) is Whitney regular for all i and thus setting $\mathcal{F} = \{F_i\}$ yields the desired finite Whitney stratification by semialgebraic sets. ■

It is instructive to apply the construction we just described applied to the Whitney umbrella.

Example 4.1 (Applying Proposition 4.1 to Whitney's umbrella). Consider again the algebraic variety $V = Z(x^2 - zy^2)$ with $\dim V = 2$. Following Proposition 4.1, we set $V_2 = V$ and $V_1 = \mathrm{Sing}(V) = z-$ axis. Next, since $\dim V_1 = 1$, we have

$$\begin{aligned} V_0 &= W(V_1, V_2 \setminus V_1) \\ &= \mathrm{Sing}(V_1) \cup B(V_1 \setminus \mathrm{Sing}(V_1), (V_2 \setminus V_1) \setminus \mathrm{Sing}(V_2 \setminus V_1)) \\ &\overset{\heartsuit}{=} B(V_1, (V_2 \setminus V_1)) \\ &\overset{\clubsuit}{=} \{(0, 0, 0)\} \end{aligned}$$

where ($\heartsuit$) is due to $\mathrm{Sing}(V_1) = \mathrm{Sing}(V_2 \setminus V_1) = \emptyset$. This is because V_1 is simply the z-axis and has no singularities, while $V_2 \setminus V_1$ is simply the complement of the z-axis in V and is composed of the two nonsingular connected components depicted in Figure 3. Finally, ($\clubsuit$) is just the result of Example 2.1 where we proved that the complement $V_2 \setminus V_1$ fails to be Whitney regular over V_1 at the origin.

Observe that the resultant stratification

$$\begin{aligned} \mathcal{F} &= \{V_2 \setminus V_1, V_1 \setminus V_0, V_0 \setminus \emptyset\} \\ &= \{V \setminus z\text{-axis}, z\text{-axis} \setminus \mathrm{origin}, \mathrm{origin}\} \end{aligned}$$

is precisely the *good* stratification we obtained in section 2 by following our intuition and removing the origin from the one dimensional stratum.

5 Psuedomanifolds and intersection homology

A key property of Whitney stratified sets is that each straum is locally topologically trivial, in the sense that each point has a neigborhood homeomorphic to an open cone. Recall first the definition of a cone over a space.

Definition 5.1 (Open cone)**.** *The cone*

Definition 5.2 (Psuedomanifold)**.** *A n-psuedomanifold is a n-dimensional topological space with a stratification $\mathcal{S}$ induced by a finite filtration of closed sets*

$$\emptyset \subset F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = X$$

such that the following local triviality condition is fulfilled: every $x \in S_i$ possesses a neighborhood $U_x \subset X$, a stratified space L and a stratification preserving homeomorphism

$$\varphi : U_x \to cone(L) \times R^i$$

L is called the link of the stratum S_i at x. If S_i is connected then the link is independent of x. Furthermore, the strata containing singularities have codimension at least 2.

Remark 5.1. The requirement that the codimension of singular strata be bounded is to ensure the intersection homology groups and the perversity vectors used to define them make sense, but this is out of the scope of this exposition.

Proposition 5.1. *Whitney stratified sets are psuedomanifolds The filtration corresponding to a Whitney stratification of a compelex algebraic variety defines a psuedomanifold*

Proof. Result due to Thom (see [Nic11] for a modern handling) ■

Remark 5.2. Real algebraic varieties may have singular strata of codimension lesser than 2.

We are now ready to give an informal definition intersection homology groups. The idea here is to restrict cycles by forcing them not to hit the singular strata. We do this by assigning to each stratum S_α a perversity $0 \leq p_\alpha$.

Definition 5.3 (Intersection chains)**.** *Let X be a n-dimensional psuedomanifold with a stratification $\mathcal{S}$ and a collection $\bar{p}$ that assigns to each stratum a perversity. Then,*

$$IC_i^{\bar{p}}(X) = \{\sigma \in C_i(X) | \dim(\sigma \cap S_\alpha) \leq i - codim(S_\alpha) + p_\alpha \text{ and } \dim(\partial\sigma \cap S_\alpha) \leq i - 1 - codim(S_\alpha) + p_\alpha\}$$

The first condition places restrictions on intersecting with certain strata, with higher p_α allowing for more cycles to hit the stratum S_α, while the second condition makes the intersection chain complex a chain complex. The intersection homology is the homology of the intersection chain complex and satisfies a version of the Poincare duality.

We will not go into the details here as it requires us to introduce more machinery. Instead, we present the intersection homology groups of the pinched torus. The pinched torus has a single isolated singularity at the pinch point, so we can define a suitable stratification by with one 0-dim stratum comprised of the pinch point and one 2-dim stratum defined as the complement of the pinch point. Consider different perversities for the 0-dim stratum. Let X denote the pinched torus, b denote the 0-homologous 1-cycle, and a the only non trivial cycle (refer to Figure 2). Denote by p the perversity of the zero dimensional stratum, then

$$IC_i^{\bar{p}}(X) = \begin{cases} \langle b \rangle & p = 0 \\ \langle a, b \rangle & p > 0 \end{cases}$$

This is due to the fact that

$$\dim(a \cap S_0) = \{*\} = 0$$
$$\dim(a \cap S_0) = \emptyset = -1$$

and

$$i - \text{codim}(\{*\}) + p = 1 - 2 + p = p - 1.$$

Consequently,

$$IH_i^{\bar{p}}(X) = \begin{cases} 0 & p = 0 \\ \mathbb{Z} & p > 0 \end{cases}.$$

When the perversity is 0 we do not allow any cycles to hit the singular straum and thus the one cycle generated by a does not enter the picture. However, increasing the perversity to 1 allows it back in.

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