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Signal and System Theory II 4. Semester, BSc

Solutions

Exercise 1

1	2	3	4	5	Exercise
5	5	3	9	3	25 Points

- 1. With $u_{c_1}(t)$ we denote the the voltage of the high-pass filter capacitor C_1 . Then, the dynamics are given by $v_{in}(t) = u_{c_1}(t) + R_1 C_1 \frac{d}{dt} u_{c_1}(t)$ (**1p.**) and $v_{out}(t) = R_1 C_1 \frac{d}{dt} u_{c_1}(t)$ (**1p.**). After Applying Laplace transform, we have $v_{in}(s) = u_{c_1}(s) + R_1 C_1 (su_{c_1}(s) - u_{C_1}(0))$ (**1p.**) and $v_{out}(s) = R_1 C_1 (su_{c_1}(s) - u_{C_1}(0))$ (**1p.**) yielding to transfer function $G_{\text{HP}}(s) = \frac{sR_1C_1}{sR_1C_{1+1}}$ (**1p.**).
- 2. By defining the low-pass filter capacitor voltage as $v_{c_2}(t)$, we have that $v_{in}(t) = R_2 C_2 \frac{d}{dt} u_{c_2}(t) + u_{c_2}(t)$ and $v_{out}(t) = u_{c_2}(t)$ (**1p.** for both equations). The state of the system is the capacitor voltage $u_{c_2}(t)$, the input of the system is $v_{in}(t)$ and output $v_{out}(t)$ (**1p.** for the state and input and output). Hence $A = -1/(R_2C_2)$, $B = 1/(R_2C_2)$, C = 1 and D = 0 (**1p.** for all matrices correct). The transfer function can be obtained using $G(s) = C(sI A)^{-1}B$ (**1p.**), yielding $G_{\text{LP}}(s) = \frac{1}{sR_2C_2+1}$ (**1p.**).
- 3. The band-pass filter is obtained by connecting the output of low-pass filter to the input of the high-pass filter (**1p.** give a point for connecting high-pass filter output to low-pass filter input). The block diagram is given in Figure 1 (**1p.**). The equivalent transfer function is obtained by $G_{eq} = G_{HP}G_{LP}$ (**1p.**)

$$v_{in}(t)$$
 $G_{\rm HP}(s)$ $G_{\rm LP}(s)$ $v_{out}(t)$

Figure 1: Block diagram

4. First we calculate $R_1C_1 = \frac{1}{5\pi}10^{-3}$ and $R_2C_2 = \frac{1}{10\pi}10^{-3}$ (**1p.** for both expressions). Next, from the equivalent transfer function $G_{eq}(s)$ we have that the magnitude characteristic is rising with 20dB/dec till $\omega = 5\pi 10^3 \text{rad/s}$ (**1p.**), after which is constant till $\omega = 10\pi 10^3 \text{rad/s}$ (**1p.**) with the amplitude of $20 \log 1 = 0 \text{dB}$ (**1p.**), after which is decreasing with 20dB/dec (**1p.**). Since the transfer function has a zero at $\omega = 0$ the phase plot starts from $\pi/2$ (**1p.**). The two poles drop the characteristic for π ending up at $-\pi/2$ (**1p.**). Sketch of the magnitude characteristic is illustrated in Figure 2 a) (**1p.**) and in Figure 2 b) is the sketch of the phase characteristic (**1p.**).



Figure 2: Bode plot sketch: a) magnitude characteristic, b) phase characteristic

5. It can be easily seen that the frequency range that gets passed through the filter is $\omega \in [R_1C_1, R_2C_2] = [5, 10]\pi 10^3 \text{rad/s}$. Hence $f \in [2.5, 5]$ kHz (**1p.**). Since 1kHz< 2.5kHz and 6kHz> 5kHz, those values are not passed through the filter (**1p.**), while 2.5kHz< 3kHz< 5kHz, hence it is passed through the filter (**1p.**).

Exercise 2

1(a)	1(b)	1(c)	2(a)	2(b)	2(c)	2(d)	3 (a)	3(b)	3(c)	Exercise
3	3	3	2	4	1	3	2	2	2	25 Points

1. (a) The controllability matrix is $C = [B \ AB]$

$$(\mathbf{1p}) \quad \mathcal{C} = \begin{bmatrix} 1 & -1 + ab \\ b & a - b \end{bmatrix},$$

which should be full rank for controllability (1p). The characteristic polynominal is

$$p(\mathcal{C}) = (a - b) - (-b + ab^2) = a - ab^2 = a(1 - b^2).$$

therefore the system is controllable for $a \neq 0$, $b \neq \pm 1$ (1p).

(b) The observability matrix is

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix}.$$

We compute

$$(\mathbf{1p}) \quad \mathcal{O} = \begin{bmatrix} c & 1\\ a-c & ac-1 \end{bmatrix},$$

which should be full rank for observability (1p). The characteristic polynomial is

$$p(\mathcal{C}) = (ac^2 - c) - (a - c) = ac^2 - a = a(c^2 - 1).$$

therefore the system is controllable for $a \neq 0$, $c \neq \pm 1$ (1p).

(c) For the given values we have

$$A + LC = \begin{bmatrix} -1 & 1\\ 1 & -1 \end{bmatrix} + \begin{bmatrix} \ell_1\\ \ell_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 + \ell_1\\ 1 & -1 + \ell_2. \end{bmatrix}$$

The eigenvalues are given by the characteristic polynomial of $(A + LC - \lambda I)$. Resulting in

$$p(A + LC - \lambda I) = (-1 - \lambda)(-1 - \lambda + \ell_2) - (1 + \ell_1) = \lambda^2 + 2\lambda - \ell_2\lambda - \ell_1 - \ell_2.$$

Writing the characteristic polynomial is (1p). The desired poles are at $\lambda_1 =$ $\lambda_2 = -2$ so that we want

$$p(A + LC - \lambda I) = (\lambda + 2)^2 = \lambda^2 + 4\lambda + 4 = \lambda^2 + (2 - \ell_2)\lambda - (\ell_1 + \ell_2).$$

Taking $\ell_1 = -2$, $\ell_2 = -2$ gives the desired result. (2p, one each gain) 4

- 2. Consider the assignment a = -0.5, b = 0, c = 0 with zero input (u(t) = 0).
 - (a) We need the eigenvalues of A.

$$A - \lambda I = \begin{bmatrix} -1 - \lambda & -0.5\\ -0.5 & -1 - \lambda \end{bmatrix}$$
$$p(A - \lambda I) = (-1 - \lambda)^2 - 0.25 = \lambda^2 + 2\lambda + 0.75.$$

The eigenvalues are $\{-0.5, -1.5\}$ (1p), therefore the system with zero input is stable (1p).

- (b) The eigenvalues are distinct, so the matrix is diagonalizable $(\mathbf{1p})$. The columns of W are the eigenvectors of A. The eigenvector for $\lambda_1 = -0.5$ is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $(\mathbf{1p})$, the one for $\lambda_2 = -1.5$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $(\mathbf{1p})$. Therefore $W = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ $(\mathbf{1p})$.
- (c) The A matrix is asymptotically stable, therefore such P always exists (1p).
 (d)

$$\begin{aligned} A^T P + PA &= -I \\ WDW^T P + PWDW^T &= -I \quad (A = WDW^T) \\ DW^T P + W^T PWDW^T &= -W^T \quad \text{left multiply } W^T \ (\mathbf{1p}) \\ DW^T PW + + W^T PWD &= -I \quad \text{right multiply } W \ (\mathbf{1p}) \\ D\tilde{P} + \tilde{P}D &= -I \quad \text{definition of } \tilde{P} \ (\mathbf{1p}) \end{aligned}$$

The last equality shows that \tilde{P} forms the desired Lyapunov function.

- 3. Consider the assignment a = 1, b = 0, c = 0.
 - (a) We need the eigenvalues of A.

$$A - \lambda I = \begin{bmatrix} -1 - \lambda & 1 \\ 1 & -1 - \lambda \end{bmatrix}$$
$$p(A - \lambda I) = (-1 - \lambda)^2 - 1 = \lambda^2 + 2\lambda.$$

The eigenvalues are $\{0, -2\}$ (1p). Therefore the system with zero input is not asymptotically stable due to the pole at zero (1p).

(b) Plugging in the state feedback controller, we have

$$A + BK = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} -1 + k_1 & 1 + k_2 \\ 1 & -1 \end{bmatrix}.$$

With $k_1 = -1$, $k_2 = -1$ we get

$$A + BK = \begin{bmatrix} -2 & 0\\ 1 & -1 \end{bmatrix},$$

$$p(A + BK) = (-2 - \lambda)(-1 - \lambda) - 0 = \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2),$$

$$5$$

therefore the poles are at $\{-1, -2\}$ (1**p**). Since the real part of the poles are negative, the closed-loop system is asymptotically stable (1**p**).

(c) Following the closed loop form in the previous part, with $k_1 = 2$, $k_2 = -2$ we get

$$\begin{split} A+BK &= \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \\ p(A+BK) &= (1-\lambda)(-1-\lambda) - (-1) = \lambda^2 - 1 + 1 = \lambda^2 \end{split}$$

therefore, we have the poles at $\{0, 0\}$.

Since the real part of the poles are zero, the closed-loop system is not asymptotically stable (1p).

Since the closed-loop matrix is non-diagonalizable, the stability in the case of zero real parts cannot be determined from the eigenvalues alone (1p).

Exercise 3	1(a)	1(b)	1(c)	1(d)	2(a)	2(b)	2(c)	Exercise
	4	3	3	3	3	4	5	25 Points

1. The transfer function of an LTI system is given by:

$$G(s) = C(sI - A)^{-1}B + D,$$

$$sI - A = \begin{bmatrix} s + 100 & -1 \\ 0 & s + 2 \end{bmatrix}$$
$$(sI - A)^{-1} = \frac{1}{(s + 100)(s + 2)} \begin{bmatrix} s + 2 & 1 \\ 0 & s + 100 \end{bmatrix}$$
$$G(s) = [(sI - A)^{-1}]_{12} + 0 = \frac{1}{(s + 100)(s + 2)}$$

• System (a)

$$G(s) = C(sI - A)^{-1}B + D$$

$$sI - A = \begin{bmatrix} s + 100 & -1 \\ 0 & s - 2 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{(s + 100)(s - 2)} \begin{bmatrix} s - 2 & 1 \\ 0 & s + 100 \end{bmatrix}$$

$$G(s) = [(sI - A)^{-1}]_{12} = \frac{1}{(s + 100)(s - 2)}$$

(1 pt.)

- $G(0) = \frac{-1}{200} \Rightarrow$ options (ii) and (iv) are valid solutions. (1 pt.) - $G(j\omega) = \frac{1}{-(200+\omega^2)+98j\omega} \Rightarrow$ there is no value of ω for which $200 + \omega^2 = 0$ (1 pt.). Therefore, it cannot be option (iv) and the right answer is option (ii) (1 pt.).

• System (b)

$$G(s) = C(sI - A)^{-1}B + D$$

$$sI - A = \begin{bmatrix} s + 100 & -1 \\ 0 & s + 2 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{(s + 100)(s + 2)} \begin{bmatrix} s + 2 & 1 \\ 0 & s + 100 \end{bmatrix}$$

$$G(s) = 2[(sI - A)^{-1}]_{12} + 0 = \frac{2}{(s + 100)(s + 2)}$$

(1 pt.)

The transfer function has twice the magnitude but the same phase as system (1) (1 pt.). So option (v) is the only valid solution. (1 pt.)

• System (c)

$$G(s) = C(sI - A)^{-1}B + D$$

$$sI - A = \begin{bmatrix} s - 100 & -1 \\ 0 & s - 2 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{(s - 100)(s - 2)} \begin{bmatrix} s - 2 & 1 \\ 0 & s - 100 \end{bmatrix}$$

$$G(s) = 2[(sI - A)^{-1}]_{12} + 0 = \frac{1}{(s - 100)(s - 2)}$$

(1 pt.)

- The transfer function has the same magnitude as system (1) but $\angle G_{(c)}(j\omega) = -\angle G(j\omega)$, where $\angle G(j\omega)$ is the phase as that of system (1). (1 pt.)
- So the correct answer is option (iii). (1 pt.)
- System (d)

$$G(s) = C(sI - A)^{-1}B + D$$

$$sI - A = \begin{bmatrix} s + 100 & -1 \\ 0 & s + 2 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{(s + 100)(s + 2)} \begin{bmatrix} s + 2 & 1 \\ 0 & s + 100 \end{bmatrix}$$

$$G(s) = -[(sI - A)^{-1}]_{12} + 0 = \frac{-1}{(s + 100)(s + 2)}$$

(1 pt.)

- The transfer function has the same magnitude as system (1) but $\angle G_{(d)}(j\omega) = \pi + \angle G(j\omega)$, where $\angle G(j\omega)$ is the phase as that of system (1). (1 pt.) - So the correct answer is option (iv). (1 pt.)

- 2. (a) Let the Laplace transform of y(t) be given by Y(s).

$$\mathcal{L}\{y(t)\} = Y(s) = \int_0^\infty y(t)e^{-st}dt$$

Now, let us perform a variable change $\hat{t} = t - 1 \Rightarrow t = \hat{t} + 1$ and $d\hat{t} = dt$.

$$\mathcal{L}\{y(t-1)\} = \int_{0}^{\infty} y(t-1)e^{-st}dt, \qquad (t = \hat{t} + 1)$$

$$= \int_{-1}^{\infty} y(\hat{t})e^{-s(\hat{t}+1)}d\hat{t}, \qquad (t = 0 \Rightarrow \hat{t} = -1) \qquad (1 \text{ pt.})$$

$$= e^{-s}\int_{-1}^{\infty} y(\hat{t})e^{-s\hat{t}}d\hat{t}, \qquad (y(t) = 0, \quad \forall t < 0)$$

$$= e^{-s}\int_{0}^{\infty} y(\hat{t})e^{-s\hat{t}}d\hat{t}, \qquad (1 \text{ pt.})$$

Therefore, the expression of the transfer function of the system is,

$$G_2(s) = \frac{\mathcal{L}\{y(t-1)\}}{\mathcal{L}\{y(t)\}} = \frac{e^{-s}Y(s)}{8} = e^{-s}.$$
 (1 pt.)

$$G_2(j\omega) = e^{-j\omega}$$

$$|G_2(j\omega)| = 1 = 0 dB$$

$$\angle G_2(j\omega) = -\omega \ (rad) = -57.3\omega^{\circ}$$
(1 pt.)

The bode plot of e^{-s} is shown in Fig. 3. (1 pt.) for the correct magnitude plot, (1 pt.) for the correct values and (1 pt.) for the correct shape.



Figure 3: Bode plot of e^{-s} .

(c) To find the phase margin we need to find the ω at which $|G_3(j\omega)| = 1$,

$$|G_{3}(j\omega)| = \frac{|e^{-0.1j\omega}|}{|0.5 + j\omega|}$$

= $\frac{1}{\sqrt{0.25 + \omega^{2}}}$ (1 pt.)
 $\frac{1}{\sqrt{0.25 + \omega^{2}}} = 1$
 $0.25 + \omega^{2} = 1$
 $\omega = 0.5\sqrt{3}$ (1 pt.)

Next, we compute the phase of $G_3(j\omega)$,

$$\angle G_3(j\omega) = -0.1j\omega - \tan^{-1}\left(\frac{\omega}{0.5}\right) \quad (1 \text{ pt.})$$

$$\angle G_3(j0.5\sqrt{3}) = 57.3 \times (-0.1 \times 0.5\sqrt{3})^\circ - \tan^{-1}\left(\frac{0.5\sqrt{3}}{0.5}\right)^\circ$$

$$= 57.3 \times \left(\frac{-0.1732}{2}\right)^\circ - \tan^{-1}(\sqrt{3})^\circ$$

$$= 57.3 \times (-0.0866)^\circ - 60^\circ$$

$$\approx -65^\circ \quad (1 \text{ pt.})$$
Phase Margin = $180^\circ - 65^\circ$

$$= 115^\circ \quad (1 \text{ pt.})$$

Exercise 4

1	2	3	4	5	Exercise
3	6	8	6	2	25 Points

1. The standard state space form is

$$\dot{x}_1 = x_2$$

 $\dot{x}_2 = -kx_1 - dx_2^3$ (1 p.).

Equilibrium points are obtained by setting system's dynamics to zero. Hence, we have $\dot{x}_1 = 0$, yielding $\hat{x}_2 = 0$ (1 p.). Combining it with $\dot{x}_2 = 0$ we have $\hat{x}_1 = 0$ (1 p.).

2. We compute the Jacobian of the dynamics and we evaluate it at the equilibrium \hat{x}

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\hat{x}) & \frac{\partial f_1}{\partial x_2}(\hat{x}) \\ \frac{\partial f_2}{\partial x_1}(\hat{x}) & \frac{\partial f_2}{\partial x_2}(\hat{x}) \end{bmatrix}$$

=
$$\begin{bmatrix} 0 & 1 \\ -k & -3d\hat{x}_2^2 \end{bmatrix}$$

=
$$\begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix}$$
 (2 p.). (1)

We compute the eigenvalues of the A as follow

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & -1 \\ k & \lambda \end{bmatrix}$$

$$= \lambda^2 + k \stackrel{!}{=} 0 \iff \lambda_{1,2} = \pm i\sqrt{k} \quad (2 \text{ p.}).$$
(2)

The Lyapunov's linearization method is inconclusive since there are imaginary eigenvalues (2 p.).

3. The derivative of Lyapunov function $V(x) = \frac{1}{2} \left(k x_1^2 + x_2^2 \right)$ is

$$\dot{V}(x) = \nabla V(x)f(x)$$

$$= \begin{bmatrix} kx_1 & x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -kx_1 - dx_2^3 \end{bmatrix} (\mathbf{1 p.})$$

$$= kx_1x_2 - kx_1x_2 - dx_2^4$$

$$= -dx_2^4 (\mathbf{1 p.}).$$

On the given open set $S = \mathbb{R}^2$ we have

(a) V(x̂) = 0 (1 p.)
(b) V(x) > 0 for all x ∈ S \ {x̂} (1 p.) 11

(c) $\dot{V}(x) \leq 0$ for all $x \in S$ since d > 0 and $x_2^4 \geq 0$ for all $x_2 \in \mathbb{R}$ (1 p.).

Hence, according to the Lyapunov direct method, the equilibrium \hat{x} is stable (1 p.).

No, the asymptotic stability cannot be concluded via the direct Lyapunov method because $\dot{V}(x) = 0$ for $(x_1, 0)$ with $x_1 \in \mathbb{R}$ (1 p.). Hence $\frac{d}{dt}V(x) < 0$ for all $x \in S \setminus {\hat{x}}$ does not hold (1 p.).

4. $\dot{V}(x(t)) = 0$ for $(x_1, 0)$ with $x_1 \in \mathbb{R}$. Therefore

$$\bar{S} = \left\{ x \in \mathbb{R}^2 \mid x_1 \in [-\ell_1, \ell_1], \, x_2 = 0 \right\} \,.$$

To derive the largest invariant set in \overline{S} we must have a look at the system equations. Indeed, from the system equations we see that in order for the trajectories to be confined to the line where $x_2 = 0$ we need $x_1 = 0$, or the system would diverge from the $x_2 = 0$ line (2 p.). Therefore, the equilibrium $\hat{x} = (0,0)$ is the largest invariant set in \overline{S} (1 p.).

Since S is bounded and closed and $\dot{V}(x(t)) \leq 0$ for all $x \in S$, we can invoke LaSalle's theorem (1 p.). According to LaSalle's theorem, all trajectories starting in S tend to \hat{x} for $t \to \infty$. We can therefore conclude that \hat{x} is locally asymptotically stable (1 p.). In addition, since $||x(t)|| \to \infty \implies V(x(t)) \to \infty$, following the hint we can conclude that \hat{x} is globally asymptotically stable (1 p.).

5. By looking at the phase plane plot in Fig. 6 we can conclude that the equilibrium \hat{x} is unstable (1 p.). From the phase portrait we also see that from any initial state, the system will enter a limit cycle, which is marked in red in the figure (1 p.).