Automatic Control Laboratory
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Signal and System Theory II
4. Semester, BSc

Solutions

Exercise 1

1	2	3	4	5	Exercise
5	3	5	6	6	25 Points

1. The horizontal displacement of the rod can be derived as a function of the angular displacement of the rod $\theta(t)$:

$$h(t) = \frac{l}{2}\sin(\theta(t)) \quad \textbf{(1 pt)}.$$

The force exerted by the spring is given by multiplying the spring constant k with the horizontal displacement $h(t)$ as follows:

$$F_{\text{spring}}(t) = k\,h(t) = \frac{kl}{2}\sin(\theta(t)) \quad \textbf{(1 pt)}.$$

The force exerted by the damper onto the rod is the damper constant d multiplied by the horizontal velocity $v(t)$ at which the rod is moving:

$$\begin{aligned} F_{\text{damper}}(t) = d\,v(t) &= d\,\frac{dh(t)}{dt} \quad \textbf{(1 pt)} \\ &= \frac{dl}{2}\cos(\theta(t))\,\dot{\theta}(t) \quad \textbf{(1 pt)}. \end{aligned}$$

The resulting total force acting on the pendulum $F(t)$ is the summation of the spring and damper force:

$$F(t) = \frac{kl}{2}\sin(\theta(t)) + \frac{dl}{2}\cos(\theta(t))\,\dot{\theta}(t) \quad \textbf{(1 pt)}.$$

(5 pts total)

2.
   - The system is nonlinear: i) The system dynamics have a $\sin(\theta(t))$ and $\cos(\theta(t))$ term in them which are nonlinear functions and ii) The dynamics have a second order derivative term in them $\ddot{\theta}(t)$ which is nonlinear. **(1pt)**
   - It is a second order system due to the angular acceleration term $ml^2\ddot{\theta}(t)$. **(1pt)**
   - The system is autonomous as no input is defined, i.e, $u(t) = 0$. **(1pt)**

   Grading: **1 pt** for each question if answer correct and at least one correct justification, **(3 pts total)**.

3. An equilibrium point of a dynamic system must satisfy the stationarity condition $\dot{\theta}(t) = \ddot{\theta}(t) = 0$ **(1pt)** which leads to the simplified force expression $F(t) = \frac{kl}{2}\sin(\theta(t))$. Therefore, the pendulum dynamics must fulfil the following equation at an equilibrium $\hat{\theta}$:

$$0 = mgl\sin\left(\hat{\theta}\right) - \frac{l}{2}\cos\left(\hat{\theta}\right)F(t) = mgl\sin\left(\hat{\theta}\right) - \frac{l}{2}\cos\left(\hat{\theta}\right)\frac{kl}{2}\sin\left(\hat{\theta}\right)$$
$$= \left(mgl - \frac{kl^2}{4}\cos\left(\hat{\theta}\right)\right)\sin\left(\hat{\theta}\right) \quad \textbf{(1pt)}$$

- We can directly deduce that one equilibrium is at the origin: $\hat{\theta}_1 = 0$ **(1pt)** due to $\sin\left(\hat{\theta}_1\right) = 0 \Rightarrow \hat{\theta}_1 = 0$.
- Additionally, we have that:

$$0 = mgl - \frac{kl^2}{4}\cos\left(\hat{\theta}\right)$$
$$\iff \cos\left(\hat{\theta}\right) = \frac{4mg}{kl}$$

Since we only consider parameters which satisfy $\frac{4mg}{kl} < 1$ the expression above gives two additional equilibrium points $\hat{\theta}_2 = \arccos\left(\frac{4mg}{kl}\right)$ **(1pt)** and $\hat{\theta}_3 = -\arccos\left(\frac{4mg}{kl}\right)$ **(1pt)**.

(5 pts total)

4. We define the state variables $x(t)$ as follows:

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix}$$

The system of two 1st order ODEs $\dot{x}(t)$ is given by:

$$\dot{x}_1(t) = x_2(t),$$

and

$$\dot{x}_2(t) = \frac{g}{l}\sin(x_1(t)) - \frac{1}{2ml}\cos(x_1(t))\left(\frac{kl}{2}\sin(x_1(t)) + \frac{dl}{2}(\cos(x_1(t))x_2(t)\right),$$
$$= \frac{g}{l}\sin(x_1(t)) - \frac{k}{4m}\cos(x_1(t))\sin(x_1(t)) - \frac{d}{4m}\cos(x_1(t))^2 x_2(t).$$

(1pt) for writing out the system of ODES.

Then, performing a Taylor expansion to linearize the system at the origin:

$$\frac{\partial \dot{x}_1(t)}{x_1(t)} = 0 \quad \textbf{(1pt)}, \quad \frac{\partial \dot{x}_1(t)}{x_2(t)} = 1 \quad \textbf{(1pt)},$$

$$\frac{\partial \dot{x}_2(t)}{x_1(t)} = \frac{g}{l}\cos(x_1(t)) + \frac{k}{4m}\sin(x_1(t))^2 - \frac{k}{4m}\cos(x_1(t))^2 + \frac{d}{2m}\sin(x_1(t))\cos(x_1(t))\,x_2(t), \quad \textbf{(1pt)}.$$

or we can use a reduced trigonometric form such that the last term simplifies to: $\frac{d}{2m}\sin(x_1(t))\cos(x_1(t))\,x_2(t) = \frac{d}{2m}\sin(2\,x_1(t))\,x_2(t)$. The last partial derivative is:

$$\frac{\partial \dot{x}_2(t)}{x_2(t)} = -\frac{d}{4m}\cos(x_1(t))^2, \quad \textbf{(1pt)}.$$

Combining all of the above and substituting $\hat{x} = [0,\, 0]^\top$ gives the following $A = \left.\frac{\partial f(x(t))}{\partial x(t)}\right|_{x(t)=\hat{x}}$:

$$A = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} - \frac{k}{4m} & -\frac{d}{4m} \end{bmatrix} \quad \textbf{(1pt)}.$$

For the whole question: **(6 pts total)**.

5. For the parameter values $m = 0.25$ and $l = g$ we get the following A matrix:

$$A = \begin{bmatrix} 0 & 1 \\ 1-k & -d \end{bmatrix} \textbf{(1 pt)}.$$

The characteristic polynomial is: $p(\lambda) = \lambda(\lambda + d) + (k-1) = \lambda^2 + d\,\lambda + (k-1)$ **(1 pt)**. We can see from the characteristic polynomial that the coefficients are nonzero and have the same sign only if: $d > 0$ **(1 pt)** and $k > 1$ **(1 pt)**.

Alternative solution: We compute the eigenvalues as $\lambda_{1,2} = -\frac{d}{2} \pm \sqrt{(\frac{d}{2})^2 - (k-1)}$ and observe that $\lambda_{1,2} < 0$ only if $d > 0$ **(1 pt)** and $\sqrt{(\frac{d}{2})^2 - (k-1)} < \frac{d}{2}$ which holds if $k > 1$ **(1 pt)**.

Thus, the linearised system (3) is asymptotically stable for $d > 0, k > 1$ **(1 pt)**. Consequently, the nonlinear system (1)-(2) is locally asymptotically around the origin for $d > 0, k > 1$ **(1 pt)**.

For the whole question: **(6 pts total)**.

Exercise 2

1	2	3	4	5	6	7	Exercise
3	4	4	4	3	3	4	25 Points

1. The matrix A has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = \alpha$ (**1p**). Therefore, as $\lambda_1 > 0$ the system is always unstable (**1p**). The stability does not depend on the value of α (**1p**).

2. The observability matrix is

$$(\mathbf{1p}) \quad \mathcal{Q} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} \alpha & -1 \\ \alpha & 0 \end{bmatrix},$$

which should be full rank for observability (**1p**). The matrix is full rank when $\alpha \neq 0$ (**1p**). Therefore the system is observable for $\alpha \neq 0$ (**1p**).

3. The controllability matrix is

$$(\mathbf{1p}) \quad \mathcal{P} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & \alpha \end{bmatrix},$$

which should be full rank for controllability (**1p**). The matrix is full rank when $\alpha \neq 2$ (**1p**). Therefore the system is controllable for $\alpha \neq 2$ (**1p**).

4. If the system is controllable (i.e. for $\alpha \neq 2$), it is possible to find an input driving the system from $x(0)$ to $x(t)$ in any given finite time t (**1p**). If the system is uncontrollable (i.e. for $\alpha = 2$), the reachable subspace is the Range($\mathcal{P}$) = Span$\{\begin{bmatrix} 1 & 1 \end{bmatrix}^T\}$ (**1p**). Both $x(0)$ and $x(1)$ belong to the reachable subspace of the system (**1p**). Therefore, it is possible to find an input to steer the system from $x(0)$ to $x(1)$ for any value of α. (**1p**).

5. We compute the closed loop matrices as

$$A_1 + BK = \begin{bmatrix} 1-\alpha & 0 \\ -\alpha & \alpha - 1 \end{bmatrix},$$

and

$$A_2 + BK = \begin{bmatrix} 1-\alpha & 0 \\ -\alpha & \alpha \end{bmatrix},$$

which are asymptotically stable if their eigenvalues are negative (**1p.**). Both matrices are lower triangular so the eigenvalues are $\lambda_{11} = 1-\alpha$, $\lambda_{12} = \alpha - 1$, $\lambda_{21} = 1 - \alpha$, $\lambda_{22} = \alpha$ (**1p.**). It is not possible to find a value of α making all the eigenvalues simultaneously negative, therefore no controller of the given structure can stabilize both systems (**1p.**).

6. For $\alpha = 1$, the closed loop matrices are

$$A_1 + BK = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix},$$

and

$$A_2 + BK = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}.$$

The systems are at equilibrium when $\dot{x}(t) = (A + BK)x(t) = 0$ (**1p.**). The first system is at equilibrium for all points $x = \begin{bmatrix} 0 & \gamma \end{bmatrix}^T$, and the second for all points $x = \begin{bmatrix} \gamma & \gamma \end{bmatrix}^T$ (**1p.**). Therefore, both systems are at equilibrium at $\hat{x} = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$. Alternatively, it is possible to indicate that the origin is an equilibrium point for any linear system. (**1p.**).

7. The closed loop system (A_2+BK) has a positive eigenvalue and is therefore unstable (**1p.**). It will thus diverge from $x_\epsilon(0)$ (**1p.**). The closed loop system $(A_1 + BK)$ has repeated eigenvalues equal to zero and its stability cannot be directly determined by eigenvalues inspection. However, by observing the values of the closed loop matrix, it appears that $(A_1 + BK)$ will also diverge. $\dot{x} = (A_1 + BK)x$ depends exclusively on x_1, and $\dot{x}_1 = 0$ for any x. When initializing at $x_\epsilon(0)$, we get $\dot{x} = \begin{bmatrix} 0 & -\epsilon \end{bmatrix}^T$ at all times, which makes x_2 diverge (**1p.**). Therefore, $(A_1 + BK)$ is also unstable (**1p.**).

Exercise 3

1	2(a)	2(b)	2(c)	3(a)	3(b)	Exercise
4	4	3	3	5	6	25 Points

1. The transfer function can be obtained as

$$\begin{aligned} G(s) &= C(sI-A)^{-1}B \quad \textbf{(1pt)} \\ &= \begin{bmatrix} \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} s+3 & 2 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ &= \frac{1}{s(s+3)+2} \begin{bmatrix} \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} s & -2 \\ 1 & s+3 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \textbf{(1pt)} \\ &= \frac{1}{s^2+3s+2} \begin{bmatrix} \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} 2s \\ 2 \end{bmatrix} \quad \textbf{(1pt)} \\ &= \frac{s-2}{(s+1)(s+2)}. \quad \textbf{(1pt)} \end{aligned}$$

2. (a) The output in Laplace domain $Y(s)$ is given by

$$\begin{aligned} Y(s) &= G(s)U(s), \\ U(s) &= kE(s), \\ E(s) &= \bar{R}(s) - Y(s), \end{aligned}$$

where $Y(s) = \mathcal{L}[y(t)]$, $U(s) = \mathcal{L}[u(t)]$, $E(s) = \mathcal{L}[e(t)]$, $\bar{R}(s) = \mathcal{L}[\bar{r}(t)]$. We immediately have

$$\begin{aligned} & Y(s) = kG(s)[\bar{R}(s) - Y(s)] \quad \textbf{(1pt)} \\ \Longrightarrow\ & Y(s)[1+kG(s)] = kG(s)\bar{R}(s) \\ \Longrightarrow\ & Y(s) = \frac{kG(s)}{1+kG(s)}\bar{R}(s). \quad \textbf{(1pt)} \end{aligned}$$

Since $T_1(s) = Y(s)/\bar{R}(s)$, we have

$$\begin{aligned} T_1(s) &= \frac{kG(s)}{1+kG(s)} \quad \textbf{(1pt)} \\ &= \frac{k(s-2)}{s^2+s(3+k)+2-2k}. \quad \textbf{(1pt)} \end{aligned}$$

(b) To guarantee asymptotic stability, we require the poles of the system to have negative real part **(1pt)**. The poles of the system are the roots of the denominator of $T_1(s)$, which in this case is a second order polynomial. We know that the roots of a second order polynomial have negative real part if and only if the coefficients of the polynomial are strictly positive **(1pt)**. This means that we require

$$3+k>0, \quad 2-2k>0 \iff k>-3, \quad k<1 \iff k \in (-3,1). \quad \textbf{(1pt)}$$

(c) The Laplace transform of the output to a step input is

$$Y(s) = \frac{T_1(s)}{s} = \frac{k(s-2)}{s(s^2 + s(3+k) + 2 - 2k)}. \quad \textbf{(1pt)}$$

Since T_1 is asymptotically stable, we can apply the final-value theorem, which yields

$$\begin{aligned}\lim_{t\to\infty} y(t) &= \lim_{s\to 0} sY(s), \quad \textbf{(1pt)}\\ &= \lim_{s\to 0} \frac{k(s-2)}{s^2 + s(3+k) + 2 - 2k},\\ &= \frac{-2k}{2-2k}. \quad \textbf{(1pt)}\end{aligned}$$

3. (a) We know from task 1 that $Y(s) = T_1\bar{R}(s)$, moreover $\bar{R}(s) = K_{\text{ff}}(s)R(s)$. Therefore:

$$Y(s) = T_1(s)K_{\text{ff}}(s)R(s) \implies T_2(s) = \frac{Y(s)}{R(s)} = T_1(s)K_{\text{ff}}(s). \quad \textbf{(1pt)} \tag{1}$$

Since $k = 1$, we have

$$T_1 = \frac{s-2}{s^2+4s} = \frac{s-2}{s(s+4)}.$$

Replacing in eq. (1) we obtain

$$T_2(s) = \frac{s-2}{s(s+4)}\frac{s}{s-2} = \frac{1}{s+4}. \quad \textbf{(1pt)}$$

Next, we have

$$\left.\begin{aligned}Y(s) &= G(s)U(s),\\ U(s) &= K(s)E(s) = E(s),\\ E(s) &= \bar{R}(s) - Y(s),\\ \bar{R}(s) &= K_{\text{ff}}(s)R(s).\end{aligned}\right\} \quad \textbf{(1pt)}$$

Rearranging:

$$U(s) = K_{\text{ff}}(s)R(s) - G(s)U(s) \implies U(s) = \frac{K_{\text{ff}}(s)}{1+G(s)}R(s), \quad \textbf{(1pt)}$$

and since $T_3(s) = \frac{U(s)}{R(s)}$ we have

$$\begin{aligned}T_3(s) &= \frac{K_{\text{ff}}(s)}{1+G(s)}\\ &= \frac{(s+1)(s+2)}{(s-2)(s+4)}. \quad \textbf{(1pt)}\end{aligned}$$

Alternatively, one can observe that

$$\begin{aligned}
T_2(s) &= \frac{Y(s)}{R(s)},\\
Y(s) &= G(s)U(s),\\
T_2(s) &= \frac{G(s)U(s)}{R(s)} = G(s)T_3(s),\\
T_3(s) &= \frac{T_2(s)}{G(s)} = \frac{(s+1)(s+2)}{(s-2)(s+4)}.
\end{aligned}$$

(b) The Laplace transform of the step response of the input is

$$U(s) = \frac{T_3(s)}{s} = \frac{(s+1)(s+2)}{s(s-2)(s+4)}. \quad \textbf{(1pt)}$$

We first need to rewrite the transfer function as

$$U(s) = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+4}. \quad \textbf{(1pt)}$$

Expanding and equating the coefficients yields

$$\begin{aligned}
&A(s-2)(s+4) + Bs(s+4) + Cs(s-2) = (s+1)(s+2) \quad \textbf{(1pt)}\\
\iff &s^2(A+B+C) + s(2A+4B-2C) - 8A = s^2 + 3s + 2\\
\iff &A+B+C = 1, \quad 2A+4B-2C = 3, \quad -8A = 2,\\
\iff &-0.25+B+C = 1, \quad -0.5+4B-2C = 3, \quad A = -0.25,\\
\iff &C = -B+1.25, \quad -0.5+4B-2C = 3, \quad A = -0.25,\\
\iff &C = -B+1.25, \quad -0.5+4B-2(-B+1.25) = 3, \quad A = -0.25,\\
\iff &C = 0.25, \quad B = 1, \quad A = -0.25. \quad \textbf{(1pt)}
\end{aligned}$$

We conclude that

$$U(s) = -\frac{0.25}{s} + \frac{1}{s-2} + \frac{0.25}{s+4},$$

and taking the inverse Laplace transform yields

$$u(t) = \mathcal{L}^{-1}[U(s)] = -0.25 + 0.25\mathrm{e}^{-4t} + \mathrm{e}^{2t}. \quad \textbf{(1pt)}$$

The step response diverges to ∞, meaning that the input applied to the system will grow infinitely large and that such an input cannot be applied to a real system **(1pt)**.

Exercise 4

1	2	3	4(a)	4(b)	Exercise
3	7	6	6	3	25 Points

1. To compute the equilibrium points, we set the system's dynamics to zero. Hence, we have $\dot{x}_1 = 0$, yielding $x_2 = 0$. Combining it with $\dot{x}_2 = 0$ we obtain the following equation
$$x_1(x_1^2 - 2) = 0 \rightarrow x_1 = 0, 2, -2. \tag{2}$$
We can therefore conclude that the three equilibrium points of the system are $(x_1, x_2) = \{(0,0), (2,0), (-2,0)\}$. **(1 p. each)**.

2. We intend to make use of the La Salle's theorem to answer the question; hence, we begin by checking the properties of the function V:

   i V is differentiable

   ii The derivative of V along the system trajectories is
$$\begin{aligned}\dot{V}(x(t)) &= x_1(x_1^2-2)\dot{x}_1 + x_2\dot{x}_2 \\ &= -x_2^2\left(x_1-2\right)^2 - x_1x_2\left(x_1^2-2\right) + x_1x_2\left(x_1^2-2\right) \\ &= -x_2^2\left(x_1-2\right)^2 \\ &\leq 0\ \forall x \in S \textbf{(1 p.)}\end{aligned} \tag{3}$$

   Next, consider an arbitrary K and the compact invariant set $S = \{x(t) \in \mathbb{R}^2 | V(x) \leq K\}$. We compute the set $\bar{S} = \{x \in S : \dot{V}(x(t)) = 0\}$. From (ii), $\dot{V}(x(t)) = 0$ is satisfied on the lines $x_2 = 0$ or $x_1 = 2$, hence $\bar{S} = \{x \in S \mid \exists \alpha \in \mathbb{R} : x = (\alpha, 0) \text{ or } x = (2, \alpha)\}$, for any $\alpha_1, \alpha_2 \in \mathbb{R}$ **(1 p.)**. We now need to find the largest invariant set M contained in the set $\bar{S}$. The largest invariant sets M within these lines are defined by $\dot{x}_2(t) = 0$ (on the line $x_2 = 0$) and $\dot{x}_1(t) = 0$ (on the line $x_1 = 2$). But we have
$$\left.\begin{aligned} x_2 &= 0 \\ \dot{x}_2 &= 0 \end{aligned}\right\} \implies x_1 = 0, 2, -2\textbf{(1 p.)}, \qquad \left.\begin{aligned} x_1 &= 2 \\ \dot{x}_1 &= 0 \end{aligned}\right\} \implies x_2 = 0\textbf{(1 p.)}.$$
   Therefore, $M = \{(0,0), (-2,0), (2,0)\}$ **(1 p.)**. Hence, by La Salle's theorem, every state trajectory starting in S tends asymptotically to one of the three equilibrium points of the system **(1 p.)**. Since K is arbitrary, for all initial conditions x_0 there exists K such that $x_0 \in S = \{x(t) \in \mathbb{R}^2 \mid V(x) \leq K\}$. Hence, all trajectories of the system converge to one of the equilibria **(1 p.)**.

3. The Jacobian of the system is
$$\frac{\partial f}{\partial x}(x) = \begin{bmatrix} 0 & 1 \\ -2x_2\left(x_1-2\right) - \left(3x_1^2-4\right) & -(x_1-2)^2 \end{bmatrix} \textbf{(1 p.)}.$$
   Evaluated at $x_1 = x_2 = 0$, we get
$$A = \frac{\partial f}{\partial x}((0,0)) = \begin{bmatrix} 0 & 1 \\ 4 & -4 \end{bmatrix} \textbf{(1 p.)}.$$

We now compute the eigenvalue of A as

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ 4 & -4-\lambda \end{bmatrix},$$

which returns $p(\lambda) = \lambda^2 + 4\lambda - 4$ as characteristic polynomial **(1 p.)**. Hence, $\lambda_{1,2} = -2 \pm \frac{\sqrt{32}}{2}$ **(1 p.)**. Since we have one positive eigenvalue **(1 p.)**, we conclude by Lyapunov indirect method that the origin is an unstable equilibrium of our system **(1 p.)**.

Alternative solution: Given the characteristic polynomial $p(\lambda) = \lambda^2 + 4\lambda - 4$, we can directly conclude that the origin is an unstable equilibrium by noting that the coefficients of the binomial do not have the same sign **(3 p. in total)**.

4. (a) We begin by evaluating $U(x(t))$ at $(x_1, x_2) = (\pm 2, 0)$. It holds $U(\pm 2, 0) = 0$ **(1 p.)**.

Next, by graphical inspections of the Figures 4 and 5 and by recalling the hint, we notice that $U(x) > 0$ for all $x \in S_1$ with $x \neq (2, 0)$ **(1 p.)** and similarly $U(x) > 0$ for all $x \in S_2$ with $x \neq (-2, 0)$ **(1 p.)**.

Additionally, we have that $\dot{U} = \dot{V} = -x_2^2 (x_1 - 2)^2$ **(1 p.)** and hence $\dot{U} < 0$ for all $x \neq (\pm 2, 0), x \neq (0, 0)$ **(1 p.)**.

Hence, $U(x(t))$ is a valid Lyapunov function for our system and from Lyapunov's direct method, we can conclude that $(x_1, x_2) = (\pm 2, 0)$ are locally asympotically stable **(1 p.)**.

Alternative solution 1: We can conclude about the local asymptotic stability of $\hat{x} = (\pm 2, 0)$ by invoking previous answers. Namely, we from Task 2 we know that every state trajectory converges to $(0, 0) \cup (\pm 2, 0)$. From Task 3 we know that $(0, 0)$ is unstable. Hence, we must converge to $(\pm 2, 0)$, so they must be locally asymptotically stable. **(6 p. in total)**

Alternative solution 2: We can conclude by using linearization as follows. As from Task 3, the Jacobian of the system is

$$\frac{\partial f}{\partial x}(x) = \begin{bmatrix} 0 & 1 \\ -2x_2 (x_1 - 2) - \left(3x_1^2 - 4\right) & -(x_1 - 2)^2 \end{bmatrix}.$$

Evaluated at $\hat{x} = (-2, 0)$, we get

$$A_1 = \begin{bmatrix} 0 & 1 \\ -8 & -16 \end{bmatrix}.$$

The characteristic polynomial of A_1 is $p_1(\lambda) = \lambda^2 + 16\lambda + 8 = 0$, which returns $\lambda_{1,2} = -8 \pm \sqrt{56}$. Since both eigenvalues are negative, we can conclude that $\hat{x} = (-2, 0)$ is locally asympotically stable. Evaluated at $\hat{x} = (2, 0)$, we get

$$A_2 = \begin{bmatrix} 0 & 1 \\ -8 & 0 \end{bmatrix}$$

The characteristic polynomial of A_2 is $p_2(\lambda) = \lambda^2 + 8 = 0$, hence linearization is inconclusive in this case. However, we can conclude that also $\hat{x} = (2, 0)$ is locally asymptotic stable by invoking previous answers, as explained in Alternative solution 1. **(6 p. in total)**

(b) From the previous answers we know that $(0, 0)$ is an unstable equilibrium, while $(\pm 2, 0)$ are locally asympotically stable. Furthermore, from Figure 5 and the fact that $\dot{U} < 0$ for all $x \neq (\pm 2, 0), x \neq (0, 0)$ **(1 p.)**, we conclude that the domain of attraction of $\hat{x} = (2, 0)$ is $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0\}$ **(1 p.)**, while the domain of attraction of $\hat{x} = (-2, 0)$ is $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 < 0\}$ **(1 p.)**.