Automatic Control Laboratory
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D-ITET
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Signal and System Theory II

This sheet is provided to you for ease of reference only.
***Do not* write your solutions here.**

Exercise 1

1	2	3	4	Exercise
5	7	7	6	25 Points

Consider the following linear, time invariant system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t)$$

where $x(t) \in \mathbb{R}^2$ is the state, $u(t) \in \mathbb{R}$ is the input, and $y(t) \in \mathbb{R}$ is the output of the system.

1. Is the system controllable? Is it observable?
2. Determine the reachable and the unobservable subspace.
3. Consider the state feedback $u(t) = -\begin{bmatrix} k_1 & k_2 \end{bmatrix} x(t)$. Compute the coefficients k_1, k_2, so that the poles of the closed loop system are both at -1.
4. Consider the piecewise constant input

   $$u(t) = \begin{cases} a_1 \text{ if } 0 \leq t < 1 \\ a_2 \text{ if } t \geq 1 \end{cases}$$

   Determine the values for a_1 and a_2 to steer the system from $x(0) = (1, 0)$ to $x(2) = (0, 2)$.

Exercise 2

1	2	3	4	Exercise
5	6	6	8	25 Points

Consider the following continuous time system:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 2 & -5 \\ 0 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(t) \\ y(t) &= \begin{bmatrix} 3 & -2 \end{bmatrix} x(t) \end{aligned}$$

where $x(t) \in \mathbb{R}^2$, $y(t) \in \mathbb{R}$, $u(t) \in \mathbb{R}$ is a controlled input to the system and $w(t) \in \mathbb{R}$ is an uncontrolled disturbance input to the system.

1. Assume that $u(t) = 0$ and $w(t) = 0$ for all $t \in \mathbb{R}_+$. Is the system linear? Is it time invariant? Is it stable?

2. Compute the transfer function $G(s)$ from the controlled input $U(s)$ to the output $Y(s)$. Compute the transfer function $H(s)$ from the uncontrolled disturbance input $W(s)$ to the output $Y(s)$.

3. Consider the case $w(t) = 0$ for all $t \geq 0$. Is it true that you can design a state feedback of the form $u(t) = Kx(t)$ for some $k \in \mathbb{R}^{1\times 2}$ such that the state converges to zero from any initial condition? Justify your answer.

4. Consider the state feedback $u(t) = \begin{bmatrix} -3 & 0 \end{bmatrix} x(t)$. Recompute the transfer function from the uncontrolled disturbance input $W(s)$ to the output $Y(s)$. Hence determine the steady state value of the output $(\lim_{t\to\infty} y(t))$ when a step disturbance is applied to the system (Hint: Final Value Theorem).

Exercise 3

1	2	3	Exercise
8	10	7	25 Points

1. Consider the circuit in Figure 1 with $v_{in}(t)$ as input and $v_{out}(t)$ as output. Assume that the operational amplifier is ideal. Derive the state space representation and the transfer function of this system.

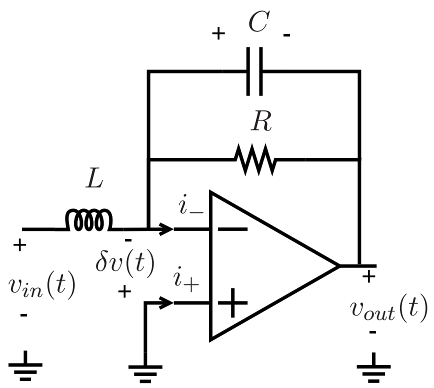

Figure 1: Circuit 1

2. A more realistic model of an operational amplifier is a transfer function of the form

$$V_{out}(s) = \frac{K}{s+1}\delta V(s)$$

and $i_+(t) = i_-(t) = 0$. Augment the state from part 1 and derive a new state space representation for the circuit in Figure 1. Assume that $v_{out}(0) = 0$

3. Derive the transfer function for the system in Figure 2, assuming that the operational amplifier is ideal. Can we put this sytem in state space form? Justify your answer.

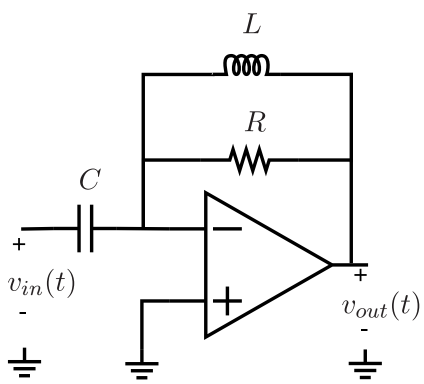

Figure 2: Circuit 2

Exercise 4

1	2	3	4	Exercise
6	7	7	5	25 Points

Consider the continuous time linear system:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $x(0) = x_0$. Let $0 < T < \infty$, define $x_k = x(kT)$, $k = 0, 1, \ldots,$ and assume that $u(t) = u_k \in \mathbb{R}^m$ is constant for all $t \in [kT, (k + 1)\,T)$.

1. Show that
$$x_{k+1} = \hat{A}x_k + \hat{B}u_k \tag{2}$$
and derive expressions for $\hat{A} \in \mathbb{R}^{n\times n}$ and $\hat{B} \in \mathbb{R}^{n\times m}$ in terms of A, B and T.

2. Assume that A is diagonalizable, with complex eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$. Show that $\hat{A}$ is also diagonalizable and derive an expression for its eigenvalues $\left\{\hat{\lambda}_1, \ldots, \hat{\lambda}_n\right\}$.

3. Assume again that A is diagonalizable. Show that the continuous time system (1) is asymptotically stable if and only if the discrete time system (2) is also asymptotically stable.

4. Assume now that $u_k = 0$ for all $k = 0, 1, \ldots,$ and A is diagonalizable. Your friend from EPFL claims that she was able to find $A \in \mathbb{R}^{n\times n}$ and $0 < T < \infty$ such that the response of the sampled system (2) satisfies $x_k = 0$ for all $k \geq n$. Do you believe her? Justify your answer.