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D-ITET
Spring Semester 2022
08.08.2022

Signal and System Theory II

This sheet is provided to you for ease of reference only.
***Do not* write your solutions here.**

Exercise 1

1	**2**	**3**	**4**	**5**	**Exercise**
5	**5**	**3**	**9**	**3**	**25 Points**

While cleaning up your grandparents basement, you found an old speaker which works. However, the sound quality is very "polluted" by low and high frequency noise. After an extensive research, you figured out that you can filter out undesired frequency range by constructing a band-bass filter. One way to build it is by combining high-pass frequency (Fig 1a) and low-pass frequency (Fig 1b) filters. The filters output voltage v_{out} has high/low frequency signals removed from the input voltage v_{in}.

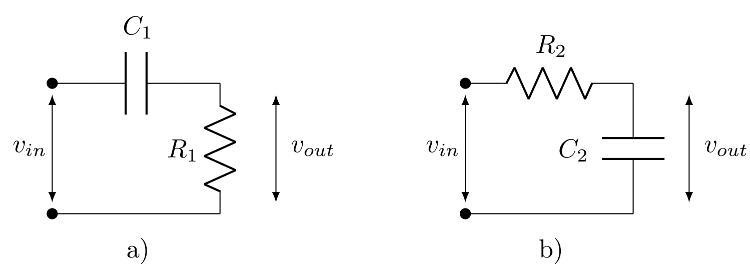

Figure 1: a) high-pass filter and b) low-pass filter

1. Write the dynamics of the high-pass filter in Figure 1 a). Then, use Laplace transform to compute the transfer function $G_{\text{HP}}(s)$ from the input v_{in} to the output v_{out}.

   Hint: Assume that initial capacitor voltage is 0.

2. For the low-pass filter in Figure 1 b) identify inputs, outputs and states of the system. Then, write its state-space model and compute the transfer function of the system $G_{\text{LP}}(s)$.

   Hint: Assume that initial capacitor voltage is 0.

3. A band-pass filter is a series combination of a high-pass and a low-pass filters. Draw the block diagram of a band-pass filter using the high- and low-pass blocks, and using the transfer functions from tasks 1 and 2 verify that the transfer function of the band-pass filter is

$$G_{\text{eq}}(s) = \frac{sR_1C_1}{(sR_1C_1+1)(sR_2C_2+1)}.$$

4. Using $R_1 = 0.4\text{k}\Omega$, $C_1 = \frac{1}{2\pi}\mu\text{F}$, $R_2 = \frac{1}{\pi}\text{k}\Omega$ and $C_2 = 0.1\mu\text{F}$ sketch the Bode plot of the equivalent transfer function $G_{\text{eq}}(s)$.

5. The filter with the components as in task 4 has to be tested. We say that the filter "passes frequencies" if the output signal is not damped, i.e., the magnitude of the input signal has not decreased. If as $v_{in}(t)$ we put sinusoidal signals with frequencies $f_1 = 1\text{kHz}$, $f_2 = 3\text{kHz}$ and 6kHz, which of them will be "passed" through the filter?

Exercise 2

1(a)	1(b)	1(c)	2(a)	2(b)	2(c)	2(d)	3(a)	3(b)	3(c)	Exercise
3	3	3	2	4	1	3	2	2	2	25 Points

Consider the following linear time invariant system

$$
\begin{aligned}
\dot{x}(t) &= \overbrace{\begin{bmatrix} -1 & a \\ a & -1 \end{bmatrix}}^{A} x(t) + \overbrace{\begin{bmatrix} 1 \\ b \end{bmatrix}}^{B} u(t) \\
y(t) &= \underbrace{\begin{bmatrix} c & 1 \end{bmatrix}}_{C} x(t).
\end{aligned}
$$

1. (a) For what values of (a, b) is the system controllable?

   (b) For what values of (a, c) is the system observable?

   (c) Suppose we have $(a, b, c) = (1, 1, 0)$. Can you build a linear observer with matrix $L = [\ell_1\ \ell_2]^T$ such that $A + LC$ has poles on $\lambda_1 = \lambda_2 = -2$?

2. Consider the assignment $a = -0.5$, $b = 0,\ c = 0$ with zero input ($u(t) = 0$).

   (a) Is the system with zero input stable?

   (b) Is the A matrix diagonalizable? If so, give the change of basis W such that $A = WDW^{-1}$ with diagonal D containing the eigenvalues.

   (c) Can you find a $P \succ 0$ such that $A^T P + PA = -I$? Answer without computing the matrix P.

   (d) Suppose we have $P \succ 0$ such that $A^T P + PA = -I$. Show that $V(x) = x^T \tilde{P} x$ with $\tilde{P} = W^T P W$ is a Lyapunov function for the system $\dot{x}(t) = Dx(t)$.
   Hint: For real and symmetric A with $A = WDW^{-1}$, we have $W^T = W^{-1}$. Rewrite the Lyapunov equality and use the change of basis to reformulate the desired result.

3. Consider the assignment $a = 1,\ b = 0,\ c = 0$.

   (a) Is the zero input system asymptotically stable?

   (b) Suppose we have the state feedback controller of the form

   $$u(t) = \begin{bmatrix} k_1 & k_2 \end{bmatrix} x(t).$$

   with $k_1 = -1,\ k_2 = -1$. What are the poles of the closed loop system? Is the closed loop system asymptotically stable?

   (c) Suppose now we have $k_1 = 2,\ k_2 = -2$. Is the closed loop system asymptotically stable? Can you determine stability from the eigenvalues of the closed loop system alone?

Exercise 3

1(a)	1(b)	1(c)	1(d)	2(a)	2(b)	2(c)	Exercise
4	3	3	3	3	4	5	25 Points

1. Consider the LTI system given in its state-space representation:

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1a}$$
$$y(t) = Cx(t) + Du(t), \tag{1b}$$

where,

$$A = \begin{bmatrix} -100 & 1 \\ 0 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D = 0.$$

The Nyquist plot of the transfer function $G(s) = \frac{Y(s)}{U(s)}$ of system (1) is shown in Fig. 2.

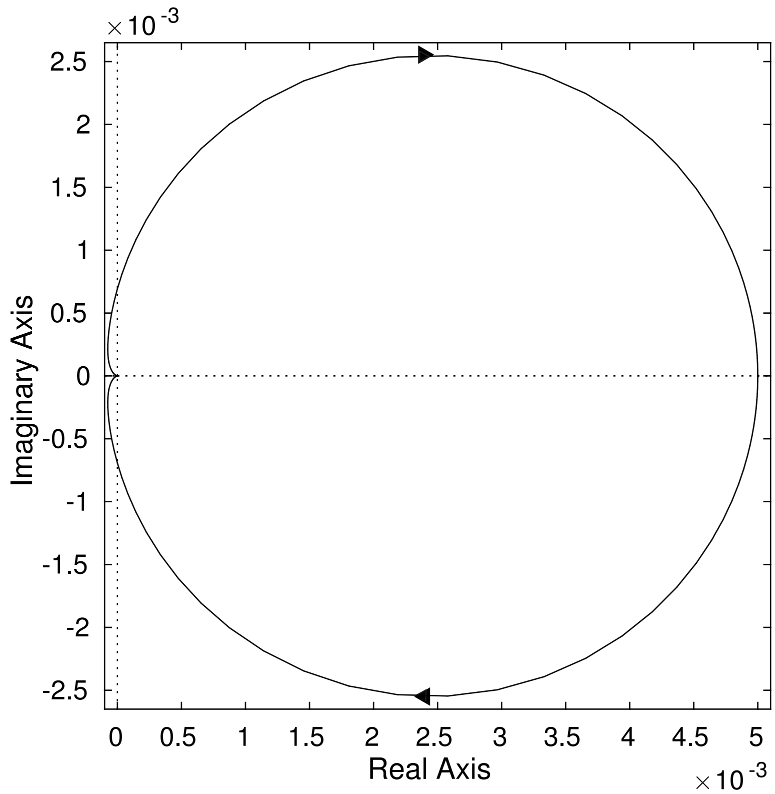

Figure 2: Nyquist plot of $G(s)$.

Given that the transfer function of (1) is

$$G(s) = \frac{1}{(s+100)(s+2)},$$

match the LTI systems (a) - (d) of the form (1) to the corresponding Nyquist plots (i) - (vi), shown in Figs. 3 - 8. Justify your choices with the necessary computations. The Nyquist plot options can be correct for more than one of the systems (a) - (d).

Hint: You can compare the transfer functions of systems (a) - (d) with that of system (1) *to find the correct option.*
NOTE: The dotted lines indicate the x- and y- axes which are the real and imaginary axes respectively.

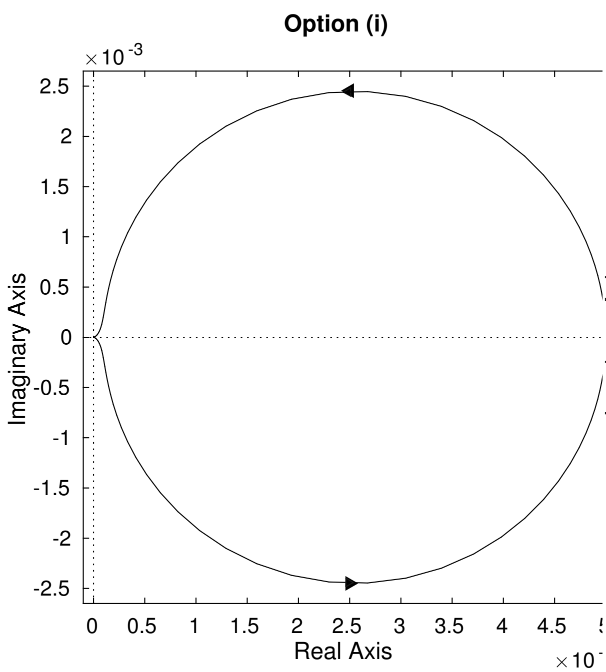

Figure 3: Nyquist plot - option (i)

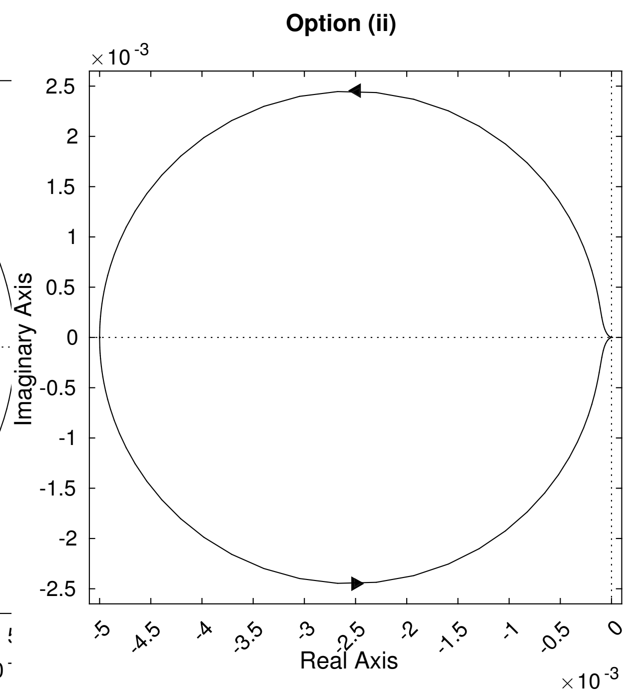

Figure 4: Nyquist plot - option (ii)

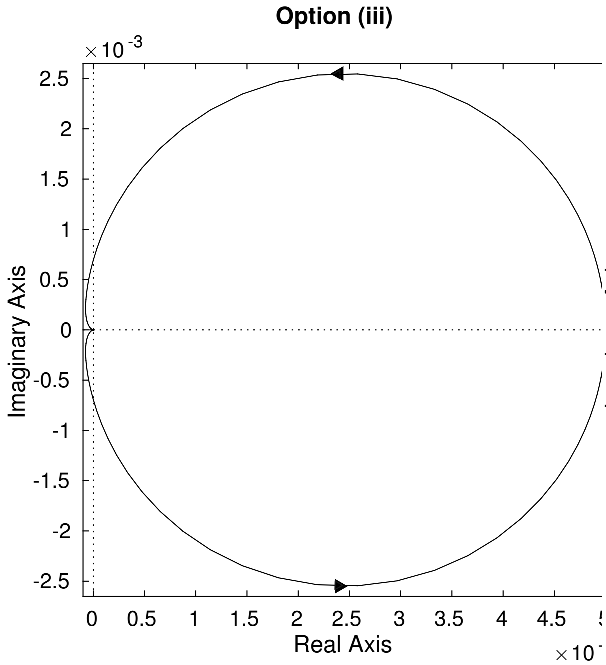

Figure 5: Nyquist plot - option (iii)

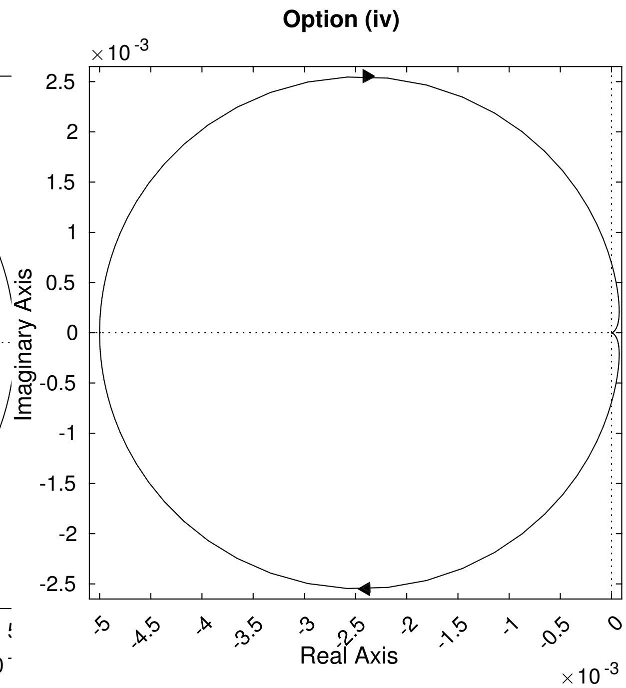

Figure 6: Nyquist plot - option (iv)

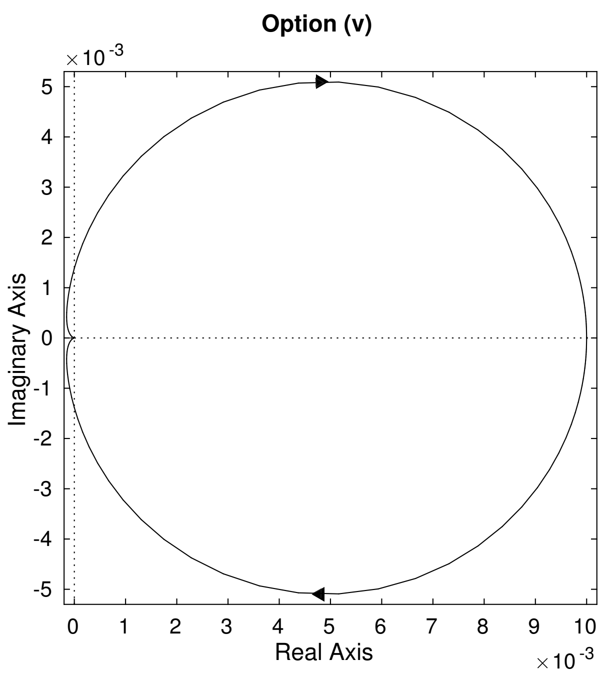

Figure 7: Nyquist plot - option (v)

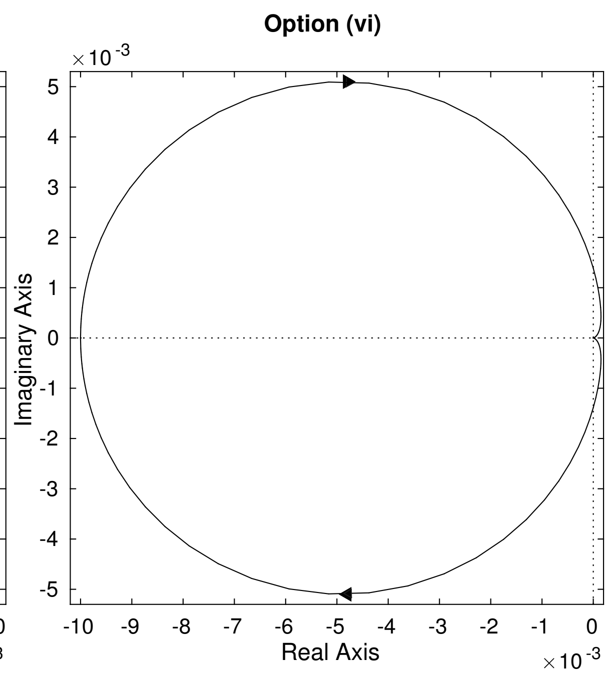

Figure 8: Nyquist plot - option (vi)

(a) $A = \begin{bmatrix} -100 & 1 \\ 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D = 0.$

(b) $A = \begin{bmatrix} -100 & 1 \\ 0 & -2 \end{bmatrix}$, $\quad B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $\quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $\quad D = 0.$

(c) $A = \begin{bmatrix} 100 & 1 \\ 0 & 2 \end{bmatrix}$, $\quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $\quad D = 0.$

(d) $A = \begin{bmatrix} -100 & 1 \\ 0 & -2 \end{bmatrix}$, $\quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\quad C = \begin{bmatrix} -1 & 0 \end{bmatrix}$, $\quad D = 0.$

2. You are given the system shown in Fig. 9. For any input signal $y(t)$ which satisfies:

$$y(t) \in \mathbb{R}, \qquad \forall t \geq 0 \tag{2}$$
$$y(t) = 0, \qquad \forall t < 0, \tag{3}$$

the system outputs a delayed version of the input signal.

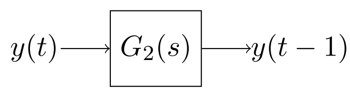

Figure 9: Block diagram for exercise 2

(a) Compute the transfer function $G_2(s)$.

(b) Plot the Bode plot of $G_2(j\omega)$ in Fig. 10.
Hint: Plot $\angle G_2(j\omega)$ *at* $\omega = 0.1, 0.2, \ldots, 1$ *while keeping in mind that* 1 *radian* $=$ $57.3°$.

(c) Find the phase margin of $G_3(s)$ rounded off to the closest integer,

$$G_3(s) = \frac{e^{-0.1s}}{s + 0.5}.$$

Hint:

ϕ	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin(\phi)$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos(\phi)$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0

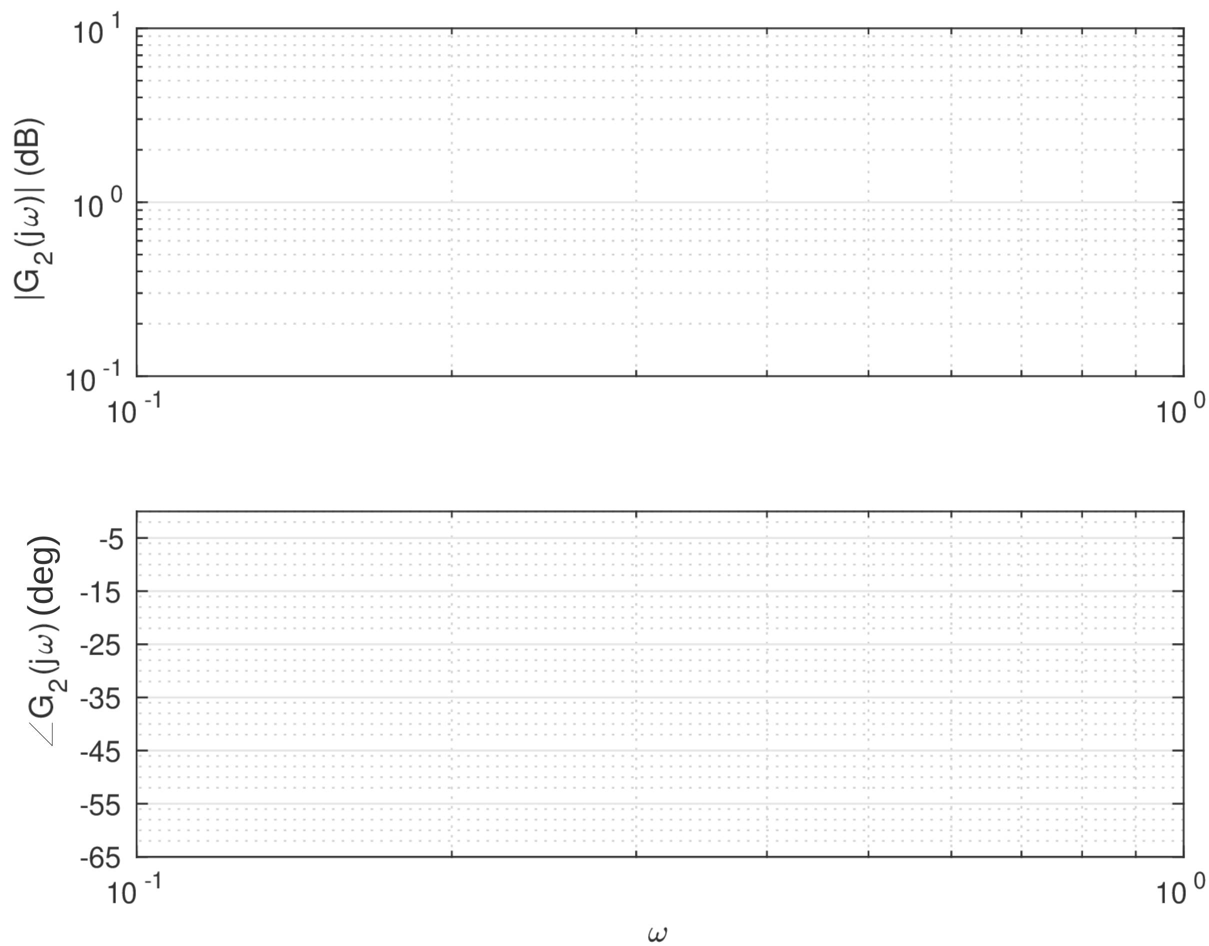

Figure 10: Bode plot of $G_2(s)$.

Exercise 4

1	2	3	4	5	Exercise
3	6	8	6	2	25 Points

For $y(t) \in \mathbb{R}$, $t \geq 0$, $d > 0$ and $k > 0$ a non-linear system is described by the following dynamics

$$\ddot{y}(t) + d\dot{y}(t)^3 + ky(t) = 0\,. \tag{4}$$

1. Write the system in the standard state space form using $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t)$ as the state variables. Then, show that $\hat{x} = (0, 0)$ is the only equilibrium of system (4).

   (alternative to previous point to avoid cascade errors) Show that the standard state space is

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -kx_1(t) - dx_2(t)^3\,.$$

To which temporal derivative of $y(t)$ do $x_1(t)$ and $x_2(t)$ correspond to? Show that $\hat{x} = (0, 0)$ is the only equilibrium of system (4).

2. Show that Lyapunov's linearization method fails to determine stability of the equilibrium.

3. Using Lyapunov's direct method with the Lyapunov function

$$V(x(t)) = \frac{1}{2}\left(kx_1(t)^2 + x_2(t)^2\right) ,$$

show that the equilibrium $\hat{x}$ is stable. Can you also determine whether it is asymptotically stable using the same method and Lyapunov function?

4. Let $\ell_1 > 0$ and $\ell_2 > 0$. Consider the following set

$$S = \left\{x(t) \in \mathbb{R}^2 \,|\, x_1(t) \in [-\ell_1, \ell_1],\, x_2(t) \in [-\ell_2, \ell_2]\right\} .$$

Using the same Lyapunov function as in 3, prove that $M = \{\hat{x}\}$ is the largest invariant set contained in the set

$$\bar{S} = \left\{x(t) \in S \,\middle|\, \dot{V}(x(t)) = 0\right\} .$$

Can we conclude that $\hat{x}$ is globally asymptotically stable?

Hint: If you can conclude that the origin is locally asymptotically stable by invoking LaSalle's theorem and, in addition, $V(x(t))$ is radially unbounded, then you can conclude that the origin is globally asymptotically stable.

5. Figure 11 shows a phase plane plot with its vector field for the non-linear system described by the following equations

$$\begin{aligned} \dot{x}_1(t) &= x_1(t) - \frac{x_1(t)^3}{3} + x_2(t) \\ \dot{x}_2(t) &= -x_1(t)\,. \end{aligned} \tag{5}$$

System (5) is known as the "Van Der Pol Oscillator" and has a unique equilibrium at $\hat{x} = (0, 0)$. Using the phase plane plot in Figure 11 and without doing further calculations, comment on the stability of the equilibrium and the general behavior of the system from an arbitrary initial condition $x(0) \in \mathbb{R}^2$.

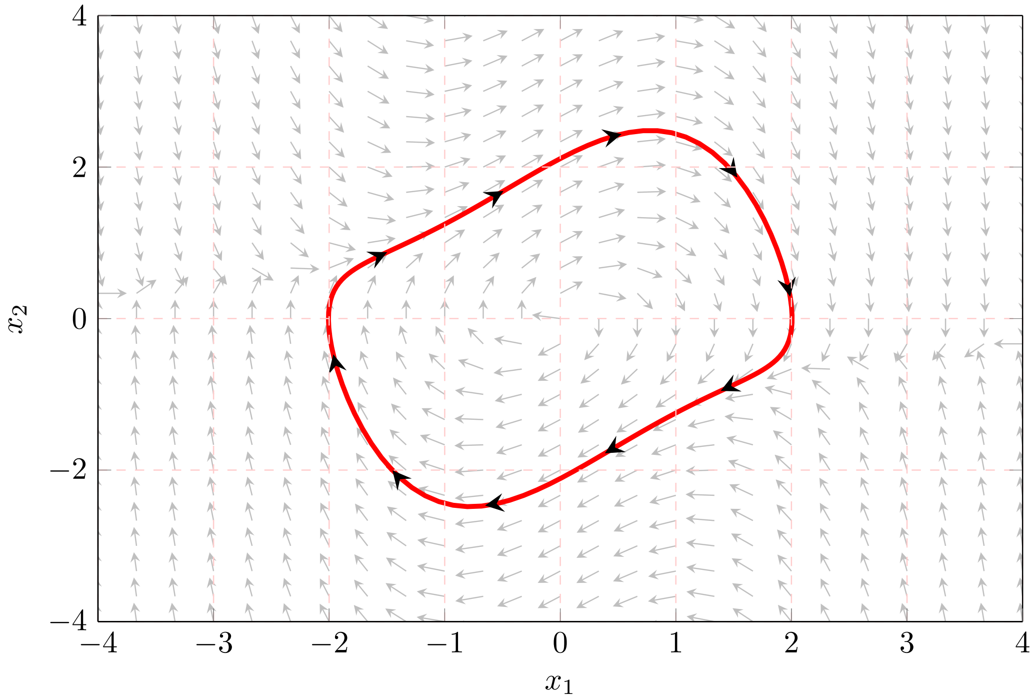

Figure 11: Phase portrait of the Van Der Pol Oscillator.