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D-ITET
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Signal and System Theory II

This sheet is provided to you for ease of reference only.
***Do not* write your solutions here.**

Exercise 1

1	2	3	4	5(a)	5(b)	Exercise
5	4	7	3	4	2	25 Points

Consider the model of an averaged Buck converter depicted in Figure 1:

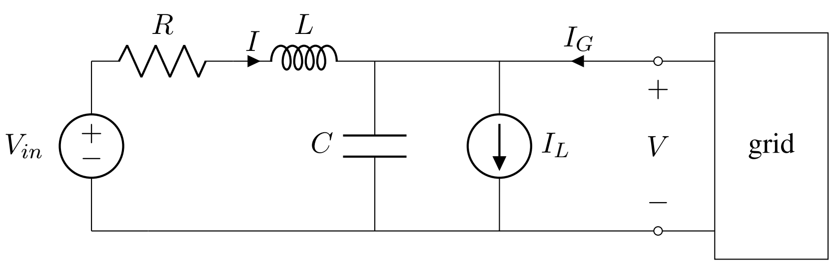

Figure 1: Buck converter connected to the main grid.

The input voltage V_{in} is designed to sustain an output voltage V over a local load I_L, via an RLC circuit with $R, L, C > 0$. The internal current is denoted with I, while the external current flowing from the main grid is I_G.

1. Consider the case where the load and grid currents are constant, $I_L(t) = \overline{I}_L$ and $I_G(t) = \overline{I}_G$ for all $t \geq 0$. Verify that the Buck converter dynamics can be represented by

$$\dot{x}(t) = Ax(t) + Bu(t) + d\,, \tag{1}$$

where $x(t) = \begin{bmatrix} V(t) \\ I(t) \end{bmatrix}$ is the state, $u(t) = V_{in}(t)$ is the input, d is a constant vector, and the matrices $A = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1/L \end{bmatrix}$ depend on the electrical parameters of the converter. Moreover, derive a formula for the vector d in terms of $\overline{I}_L$, $\overline{I}_G$, and the parameters R, L, C.

2. Consider the system (1) under a constant input voltage $V_{in}(t) = \overline{V}_{in}$ for all $t \geq 0$. Compute the equilibrium point $\bar{x}$ of system (1) and show that it is unique.

3. Investigate the stability of the equilibrium point $\bar{x}$ in part 2.
   Hint: Consider the change of coordinates $\tilde{x} = x - \bar{x}$.

4. Given a desired reference value V_r, select a constant $V_{in}(t) = \overline{V}_{in}$ for all $t \geq 0$ such that for system (1) it holds

$$\lim_{t \to \infty} V(t) = V_r \, .$$

5. Consider now a constant impedence load $I_L(t) = YV(t)$, where $Y > 0$ represents the load impedence, and a time-varying grid current $I_G(t)$.

   (a) Verify that, under the state-feedback controller $V_{in}(t) = -KI(t)$, with $K > 0$,

$$\frac{d}{dt} E(V(t), I(t)) \leq V(t) I_G(t) \, , \qquad (2)$$

   where $E(V(t), I(t)) = \frac{1}{2} C V(t)^2 + \frac{1}{2} L I(t)^2$.

   (b) Engineers refer to systems that satisfy inequality (2) as *dissipative systems*. Can you guess why? Make explicit reference to the electrical meaning of the terms in (2).

Exercise 2

1	2	3	4	5	6	7	Exercise
4	4	4	3	3	4	3	25 Points

Consider the following linear time invariant system with parameters $\alpha, \beta \in \mathbb{R}$:

$$\dot{x}(t) = \overbrace{\begin{bmatrix} -5 & 0 \\ \alpha & \beta \end{bmatrix}}^{A} x(t) + \overbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}^{B} u(t)$$
$$y(t) = \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{C} x(t).$$

1. Find all the values of (α, β) for which there exists a unique solution $Q^T = Q > 0$ to the equation $A^T Q + QA = -R$, where $R = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. You are not required to compute the solution Q. How does the stability of the system under $u(t) = 0$ depend on α?

2. For which values of (α, β) is the system observable?

3. For which values of (α, β) is the system controllable?

4. Is it possible to find an input $u(\cdot) : [0, 1] \to \mathbb{R}$ driving the system from $x(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ to $x(1) = \begin{bmatrix} 10 & 10 \end{bmatrix}^T$ for any values of (α, β)? You are not required to compute the input $u(t)$.

5. Consider a state feedback controller of the form
$$u(t) = K_1 x(t) = \begin{bmatrix} -\alpha & -\alpha \end{bmatrix} x(t).$$
For which values of (α, β) is the closed loop system $(A + BK_1)$ asymptotically stable?

6. Consider a feedback controller of the form
$$u(t) = K_2 x(t) = \begin{bmatrix} -\alpha & 0 \end{bmatrix} x(t).$$
Find the values of (α, β) for which the transfer function of the closed loop system $(A + BK_2)$ has a pole located at -2 and the system is stable, but not asymptotically stable.

7. Consider the case where $\alpha < \beta$. Is it possible to find a matrix $L \in \mathbb{R}^2$ such that the matrix $A - LC$ has eigenvalues with negative real part?

Exercise 3

1	2(a)	2(b)	2(c)	3(a)	3(b)	Exercise
6	3	3	3	5	5	25 Points

1. Consider the transfer function $L(s) = G_1(s)K(s)$, where

$$G_1(s) = \frac{s+3}{(s+1)(s+2)}, \quad K(s) = \frac{k}{s(s+0.1)},$$

for some $k \in \mathbb{R}$.

The right-hand side of Figure 2 shows the Nyquist plot of $L(s)$ for $k = 1$ with the closures at infinity (shown as a dotted line). The left-hand side of Figure 2 shows the contour used to draw the Nyquist plot, with the poles and the zeros marked with crosses and circles, respectively. Figure 3 shows the Nyquist plot of $L(s)$ for $k = 1$ in the vicinity of the origin.

Using the Nyquist criterion, provide the values of $k \in \mathbb{R}$ that ensure that the closed-loop transfer function

$$T(s) = \frac{L(s)}{1 + L(s)}$$

is asymptotically stable.

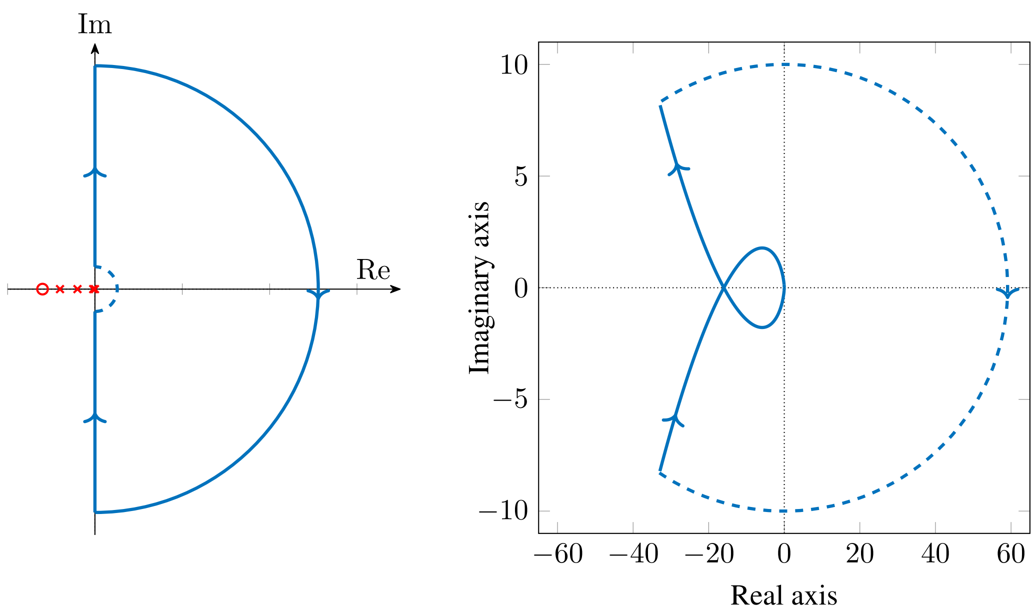

Figure 2: Nyquist plot of $L(s)$ with closures (right), and contour used to draw it (left).

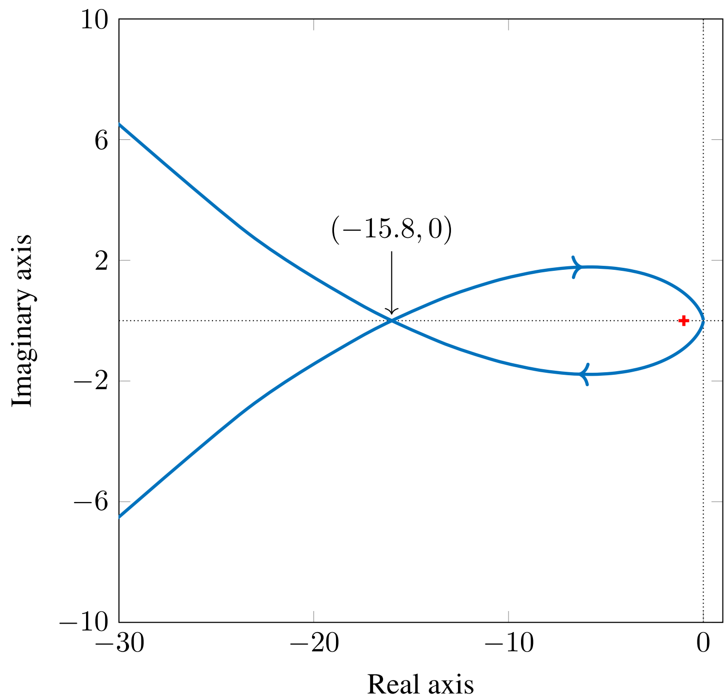

Figure 3: Nyquist plot of $L(s)$ near the origin. The cross shows the point $(-1, 0)$.

2. Consider the transfer function

$$G_2(s) = \frac{5}{s(s+5)(s+3)}.$$

The Bode plot of $G_2(s)$ is provided in Figure 4.

(a) Compute the gain margin of $G_2(s)$ analytically.

Hint: You can use the fact that $\mathrm{Im}(G_2(j\omega)) = 0$ for $\omega = \sqrt{15}$ rad/s without verifying it.

(b) Using the final value theorem, compute $\lim_{t\to\infty} y_1(t)$, where $y_1(t)$ is the impulse response of $G_2(s)$.

(c) Using the final value theorem, study $\lim_{t\to\infty} y_2(t)$, where $y_2(t)$ is the step response of $G_2(s)$.

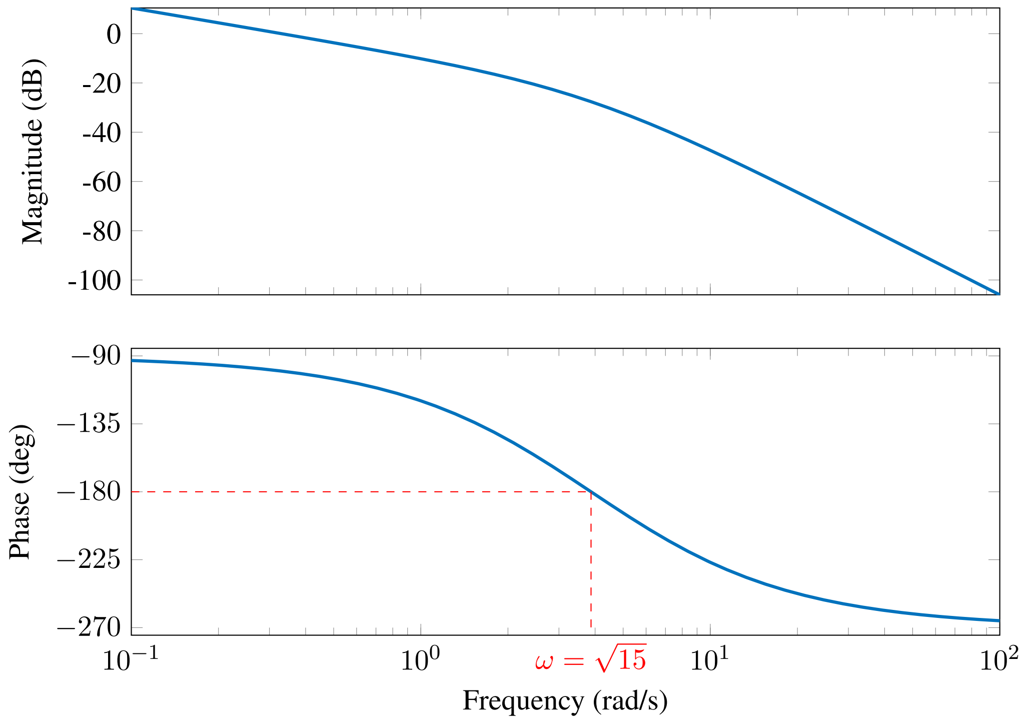

Figure 4: Bode diagram of $G_2(s)$.

3. Figure 5 shows the Bode plot of a transfer function $G_3(s)$.

   (a) Sketch the Nyquist plot of $G_3(s)$.

   (b) We know that the transfer function $G_3(s)$ has the form

$$G_3(s) = \frac{(s-1)^{n_z}}{(s+p)^{n_p}},$$

   with $p \in \{-1, +1\}$, and $n_z, n_p \in \mathbb{N}$.

   Using the Bode plot in Figure 5, provide the values of n_z, n_p, and determine whether $p = +1$ or $p = -1$.

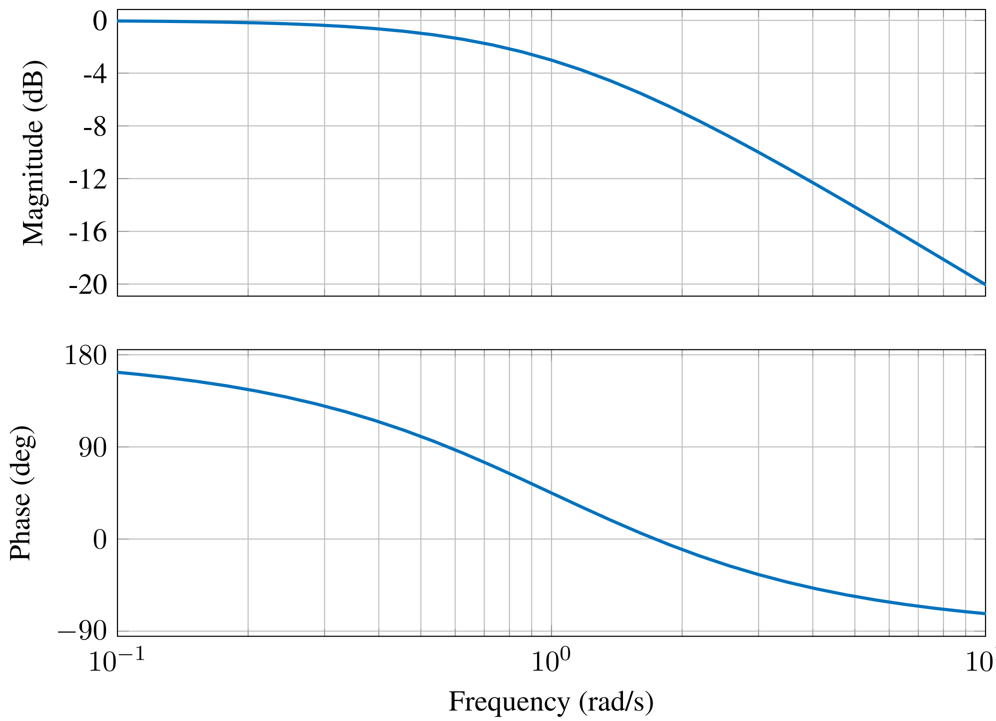

Figure 5: Bode diagram of $G_3(s)$.

Exercise 4

1	2	3	4	5	Exercise
3	6	7	7	2	25 Points

For $x_1(t),\ x_2(t),\ a \in \mathbb{R}$ and $t \geq 0$ consider the system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t)\,, \\ \dot{x}_2(t) &= -2x_1(t) - 2ax_2(t) - 4x_1^3(t)\,. \end{aligned} \tag{3}$$

1. Prove that for all $a \in \mathbb{R}$ the system (3) has a unique equilibrium point $\hat{x} = (0, 0)$.

2. Consider the case $a \in [-\sqrt{2}, \sqrt{2}]$. Study the stability of the equilibrium point $\hat{x} = (0, 0)$ using linearization. What conclusions can be drawn when $a = 0$, when $a \in (0, \sqrt{2}]$ and when $a \in [-\sqrt{2}, 0)$?

3. Consider now the case $a \geq 0$. Using the Lyapunov function

   $$V(x(t)) = 4x_1^2(t) + 2x_2^2(t) + 4x_1^4(t)\,,$$

   show that the equilibrium $\hat{x} = (0, 0)$ is stable. Can you also determine whether $\hat{x}$ is asymptotically stable using the same Lyapunov function?

4. Consider now the case $a > 0$, the same Lyapunov function as in Task 3, and the set

   $$S = \left\{ x(t) \in \mathbb{R}^2 \,|\, V(x) \leq \epsilon \right\}\,,$$

   for an arbitrary $\epsilon > 0$. Prove that $M = \{(0, 0)\}$ is the largest invariant set contained in the set

   $$\bar{S} = \left\{ x(t) \in S \,\middle|\, \dot{V}(x(t)) = 0 \right\}\,.$$

   Can we conclude that $\hat{x} = (0, 0)$ is globally asymptotically stable?

   Hint: You may assume that S is compact and invariant. Note that ϵ can be chosen arbitrarily large and $V(x) \to \infty$ whenever $\|x\| \to \infty$.

5. Figure 6 shows a phase plane plot of the trajectories for system (3) when $a = 0$. What can you conclude about the stability of the equilibrium $\hat{x} = (0, 0)$?

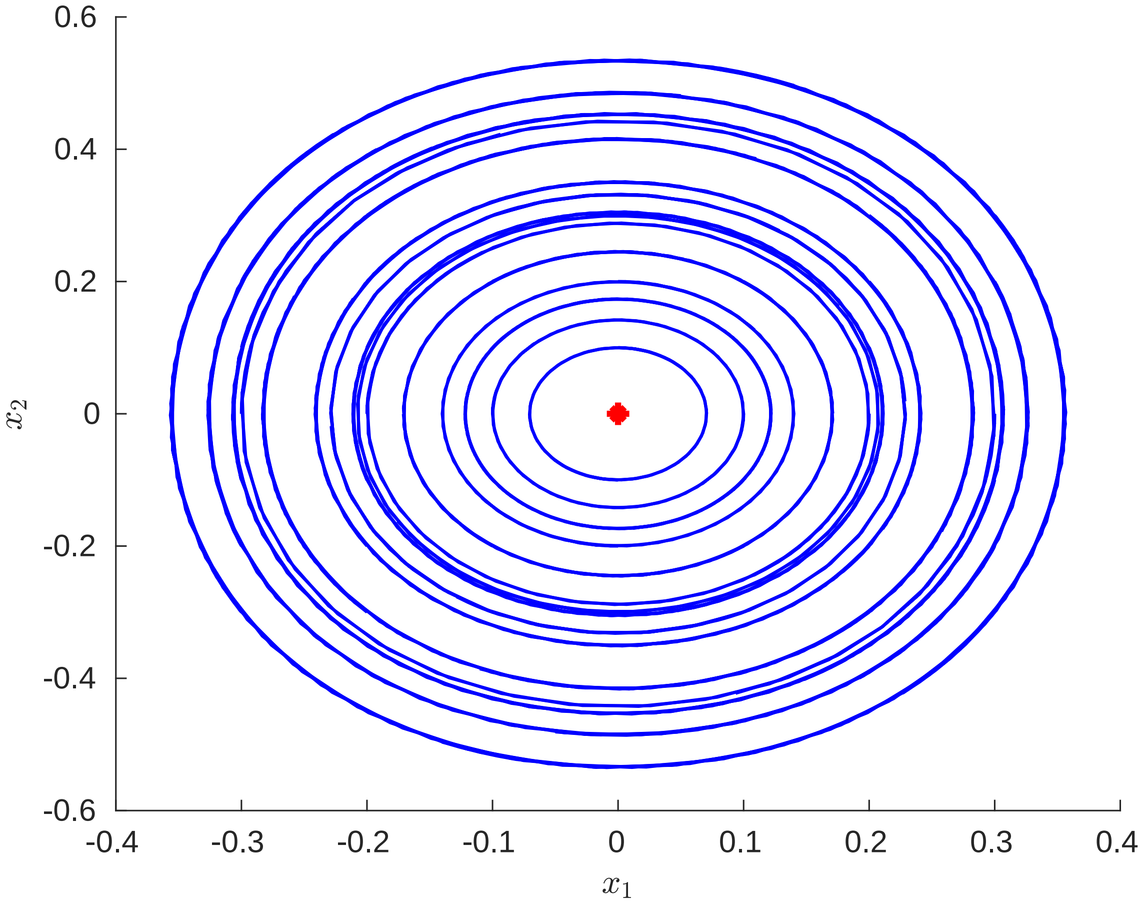

Figure 6: Trajectories portrait of system (3) when $a = 0$.