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D-ITET
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Signal and System Theory II

This sheet is provided to you for ease of reference only.
***Do not* write your solutions here.**

Exercise 1

1	2	3	4(a)	4(b)	5	Exercise
3	3	4	8	2	5	25 Points

In an effort to home-brew some beer, you read the well-known beer brewing manual "*the drunken brewchure*", and learn that the fermentation process can be modeled through two differential equations, one for the sugar concentration $S(t)$, and the other for the yeast population $Y(t)$. The dynamics obey the following principles:

i. the rate at which the yeast consumes the sugar is proportional to the product of the yeast population and the sugar concentration, with proportionality constant $\alpha > 0$;

ii. sugar naturally decays at a rate proportional to its concentration, with proportionality constant $\beta > 0$;

iii. yeast cells replicate at a rate proportional to the rate at which they consume sugar, with a different constant of proportionality $\gamma > 0$;

iv. yeast cells die at rate proportional to their population, with proportionality constant $\delta > 0$.

1. Which of the following models are consistent with the principles stated above? Explain your reasoning.

(a) $$\begin{cases} \dot{S}(t) = -\alpha S(t)Y(t) - \beta S(t), \\ \dot{Y}(t) = -\gamma S(t)Y(t) - \delta Y(t), \end{cases}$$

(b) $$\begin{cases} \dot{S}(t) = -\alpha S(t)Y(t) - \beta S(t), \\ \dot{Y}(t) = \gamma S(t)Y(t) - \delta Y(t), \end{cases}$$

(c) $$\begin{cases} \dot{S}(t) = \alpha S(t)Y(t) - \beta S(t), \\ \dot{Y}(t) = -\gamma S(t)Y(t) - \delta Y(t), \end{cases}$$

(d) $$\begin{cases} \dot{S}(t) = \alpha S(t)Y(t) - \beta S(t), \\ \dot{Y}(t) = \gamma S(t)Y(t) + \delta Y(t), \end{cases}$$

2. For your chosen model in Part 1, which values of S and Y ensure that the system is at an equilibrium? Do you think all the equilibria you found make physical sense?

3. Let $\bar{S}, \bar{Y}$ denote an equilibrium point of the system in Part 2. Construct a first-order approximation of the system of the form

$$\dot{x}(t) = Ax(t), \tag{1}$$

where

$$x(t) = \begin{bmatrix} S(t) - \bar{S} \\ Y(t) - \bar{Y} \end{bmatrix}.$$

4. You are not satisfied with the quality of your beer. You decide to equip the fermentation tank with a valve that can release sugar, thereby allowing you to control the sugar concentration. After substituting appropriate values for the constants, the dynamics in Part 3 now become

$$\dot{x}(t) = Ax(t) + Bu(t),$$

where $u(t)$ represents the amount of sugar released by the valve, and

$$A = \begin{bmatrix} -0.05 & 0 \\ 0 & -0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

You decide to design a state feedback controller $u(t) = Kx(t)$, with $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$.

(a) How would you choose $x(0) \in \mathbb{R}^2$, $k_1 \in \mathbb{R}$, and $k_2 \in \mathbb{R}$, with $x(0) \neq [0\ 0]^T$, so that the state response of the system is $x(t) = e^{-0.2t}x(0)$?

(b) Assuming $k_1 = 0.01$, can you choose $x(0)$ such that $\|x(t)\| \to +\infty$ as $t \to \infty$?

5. To implement the controller in Part 4 on a real computer, you need to discretize the continuous-time dynamics (1). You decide to use the forward Euler discretization scheme with sampling time $T_s > 0$. Assuming $k_1 = -0.15$ and $k_2 = 1$, what is the maximum value of T_s for which the resulting discrete time system is still asymptotically stable?

Exercise 2

1	2	3	4	5	Exercise
5	5	5	5	5	25 Points

Consider the following linear time invariant system with parameters $\beta_1, \beta_2, \gamma_1, \gamma_2 \in \mathbb{R}$:

$$\begin{aligned} \dot{x}(t) &= \overbrace{\begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}}^{A} x(t) + \overbrace{\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}}^{B} u(t) \\ y(t) &= \underbrace{\begin{bmatrix} \gamma_1 & \gamma_2 \end{bmatrix}}_{C} x(t). \end{aligned} \tag{2}$$

1. For which values of β_1, β_2 is system (2) controllable?

2. Suppose that either $\beta_1 = 0$ or $\beta_2 = 0$, so you can only actuate one state. Can you select the other, non-zero β value (respectively, β_2 or β_1) such that the system is stabilizable? If so, for what values of β_1 or β_2 is the system stabilizable? Analyze both cases.

3. Suppose $\beta_1, \beta_2, \gamma_1, \gamma_2$ are chosen such that (2) is controllable but not observable. What can you say about the controllability and observability of system (3)?

$$\begin{aligned} \dot{x}(t) &= A^T x(t) + C^T u(t) \\ y(t) &= B^T x(t) \end{aligned} \tag{3}$$

   Hint: compare the controllability and observability matrices of (2) *and* (3).

4. Let $\beta_1 = 1, \beta_2 = 0, \gamma_1 = 1, \gamma_2 = 1$, and $u(t) = 6e^{-2t}$. Suppose you measure the system output for a short time and find that $y(0) = 4$ and $\dot{y}(0) = 2$. Compute $x(0)$. Comment on (a) how one can use this estimate of $x(0)$, the known input, $u(t)$, and the dynamics in (2) to estimate the value of the state $x(t)$ for some $t \geq 0$ and (b) the disadvantages of using this state estimation procedure.

5. Another way to construct an estimate, $\tilde{x}(t)$, of the state, $x(t)$, is through an observer given by

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + Bu(t) + L[y(t) - C\tilde{x}(t)],$$

   Again let $\beta_1 = 1$, $\beta_2 = 0$, $\gamma_1 = 1$, and $\gamma_2 = 1$. For what values of α does the observer with $L = [\alpha \;\; 0]^T$ guarantee that the estimation error $e(t) = x(t) - \tilde{x}(t)$ goes to zero as $t \to \infty$?

Exercise 3

1	2	3	4	5	Exercise
6	4	5	6	4	25 Points

Consider the following transfer function of a controllable and observable system

$$G(s) = \frac{10}{(s+1)(s+10)}. \tag{4}$$

1. Compute the poles of the transfer function. Can we infer the stability of the system by inspecting the poles of the transfer function? If yes, explain why and then determine the stability properties of the system.

2. Determine which of the following Figures 1-4 is the Bode plot of transfer function (4). Justify your answer properly.

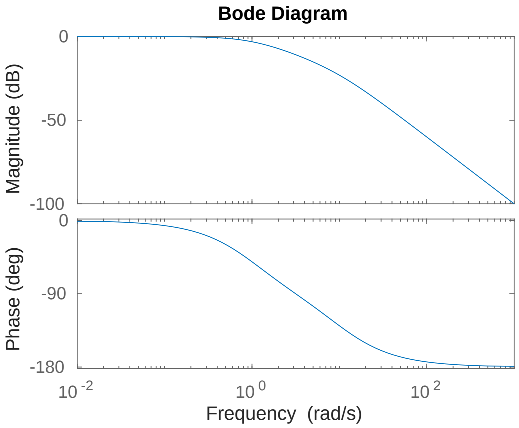

Figure 1: Bode plot 1.

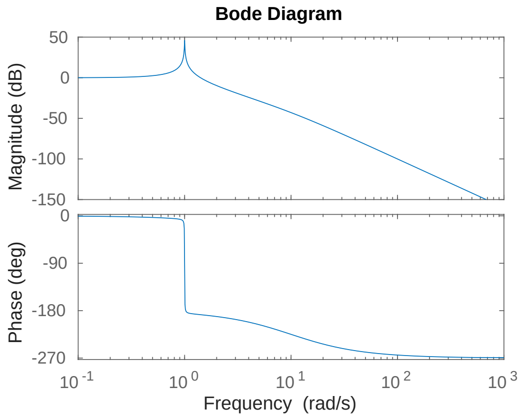

Figure 2: Bode plot 2.

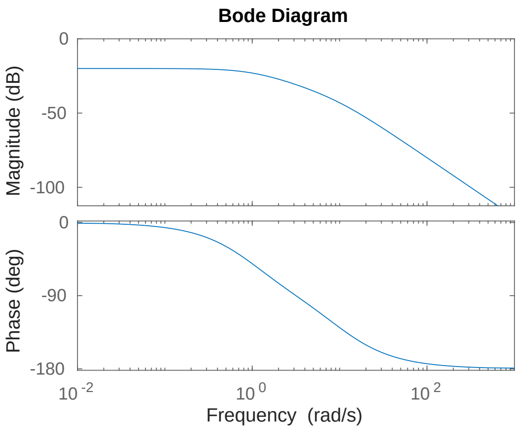

Figure 3: Bode plot 3.

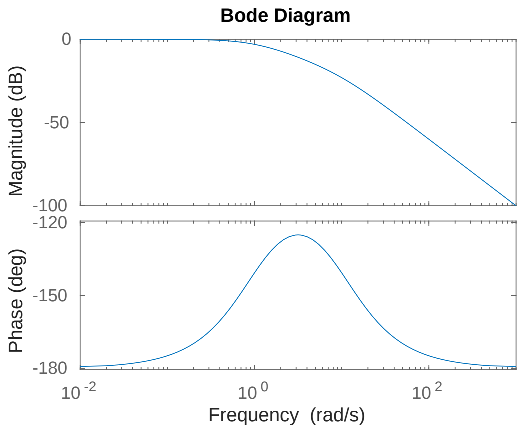

Figure 4: Bode plot 4.

3. Compute the steady state response of a system with transfer function $G(s)$ given by (4) to the sinusoidal input $u(t) = 10\sin(t)$. *Hint:* $\frac{10}{\sqrt{9^2+11^2}} \approx 0.7$ *and* $\tan^{-1}(11/9) \approx 0.885$ *rad.*

4. The Nyquist plot of the transfer function (4) is shown in Figure 5.

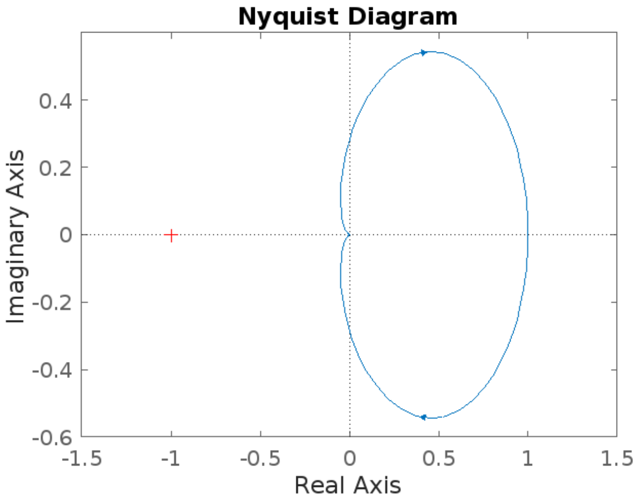

Figure 5: Nyquist plot of $G(s)$.

   $G(s)$ is inserted in a negative feedback loop with gain K, leading to the closed-loop transfer function $H(s) = \frac{KG(s)}{1+KG(s)}$. For which values of K is the closed-loop system asymptotically stable?

5. Independently of $G(s)$, consider the Bode magnitude plot in Figure 2. What causes the peak in magnitude at $\omega = 1$ rad/sec? Explain the phenomenon and its effect on magnitude and phase of a sinusoidal input around this frequency.

Exercise 4

1	2	3	4(a)	4(b)	4(c)	5	Exercise
2	4	6	3	5	2	3	25 Points

For $x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^2$ and $t \geq 0$, a non-linear system is described by the following dynamics

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -x_2(t) + x_2^3(t) - x_1^5(t) \,. \end{aligned} \tag{5}$$

1. Prove that $\hat{x}(t) = (0, 0)$ is the only equilibrium of system (5).
2. What can you conclude about the stability of the equilibrium using linearization?
3. Consider the system (5). Using the Lyapunov function

   $$V(x(t)) = x_1^6(t) + 3x_2^2(t)\,, \tag{6}$$

   show that the equilibrium $\hat{x} = (0, 0)$ is stable. Using the same Lyapunov function, can we also determine whether it is asymptotically stable? *Hint: Think about what happens for small values of* x_2.
4. For the same Lyapunov function, consider the set

   $$S = \left\{(x_1(t), x_2(t)) \in \mathbb{R}^2 \text{ such that } |x_2(t)| \leq 1\right\}\,.$$

   a) Derive the following set

   $$\bar{S} = \left\{ x \in S \text{ such that } \frac{d}{dt}V(x(t)) = 0 \right\}\,. \tag{7}$$

   b) Prove that $M = \{(0, 0)\}$ is the largest invariant set in (7).

   c) Based on (a) and (b), what can we conclude on the asymptotic stability of the equilibrium $\hat{x}$? *Hint: You may assume that S is compact and invariant.*
5. The phase plane plots in Figures 6-8 show the trajectories of some nonlinear system with two states for different values of a parameter a. In each of the three cases, comment on the stability of the equilibrium $\hat{x} = (0, 0)$ and the general behavior of the system for an arbitrary initial condition $x(0) \in \mathbb{R}^2$.

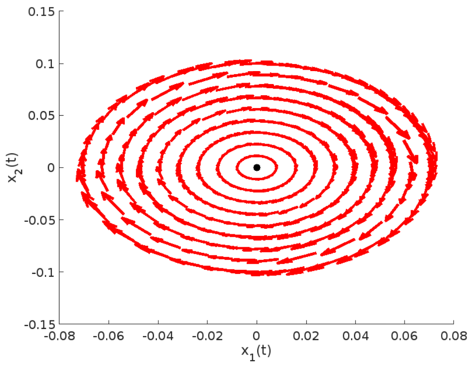

Figure 6: $a = 0$

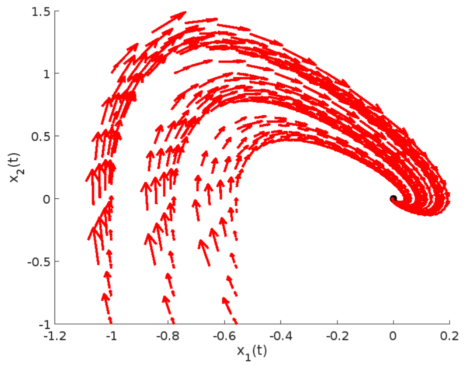

Figure 7: $a = -2$

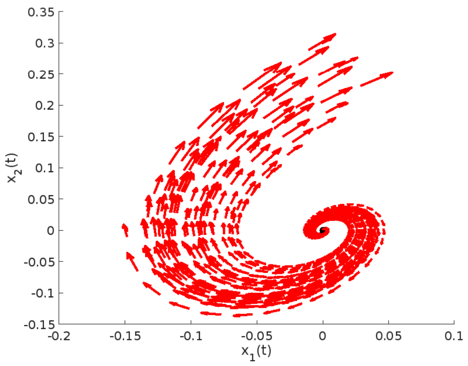

Figure 8: $a = 1$