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D-ITET
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Signal and System Theory II

This sheet is provided to you for ease of reference only.
***Do not* write your solutions here.**

Exercise 1

1	2	3	4	5	Exercise
5	3	5	6	6	25 Points

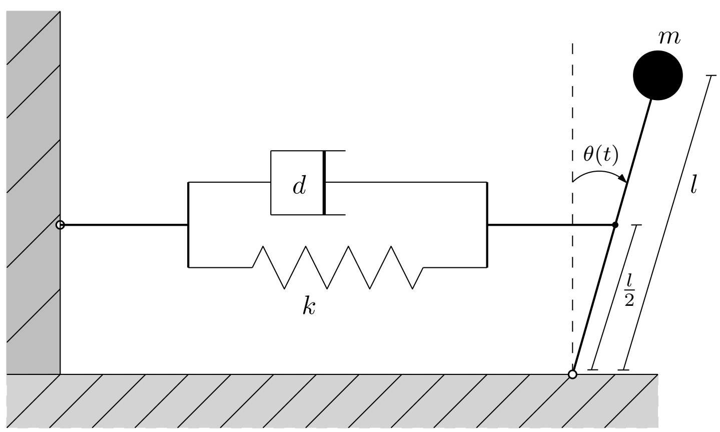

Figure 1: Inverted pendulum connected to a wall by a spring and a damper.

The inverted pendulum depicted in Figure 1 consists of a point mass m attached to the top of a weightless rod of length l. The middle of the rod is connected to a wall through a spring-damper system with spring constant k and damping coefficient d, and to the ground by a pivot. The spring is at a rest position when $\theta(t) = 0$, where $\theta(t)$ is the angular displacement of the rod. For simplicity, vertical movements are neglected.

The system dynamics are described by the following equation:

$$ml^2\ddot{\theta}(t) = mgl\sin(\theta(t)) - \frac{l}{2}\cos(\theta(t))F(t) \tag{1}$$

where $F(t)$ is the force resulting from the spring-damper system.

1. Verify that the force $F(t)$ exerted by the spring-damper system as a function of $\theta(t)$ and $\dot{\theta}(t)$ is given by:

$$F(t) = \frac{kl}{2}\sin(\theta(t)) + \frac{dl}{2}\cos(\theta(t))\dot{\theta}(t). \tag{2}$$

2. Is the system linear or nonlinear? Is it a first or a second order system? Is it autonomous? Justify your answers.

3. Assume that $\frac{4mg}{kl} < 1$ and calculate all equilibria of system (1).

4. Choose $x(t) = \begin{bmatrix}\theta(t) & \dot{\theta}(t)\end{bmatrix}^\top$ and show that the system can be described by the following linear time-invariant dynamics when linearizing it around the origin.

$$\dot{x}(t) = A\,x(t) = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} - \frac{k}{4m} & \frac{-d}{4m} \end{bmatrix} x(t) \tag{3}$$

5. Assume now that $m = 0.25$ and $l = g$. For which values of k and d is the linearised system in (3) asymptotically stable? What can you deduce about the stability of the origin for the nonlinear system (1)-(2)?

Exercise 2

1	2	3	4	5	6	7	Exercise
3	4	4	4	3	3	4	25 Points

Consider the following linear time invariant system with parameter $\alpha \in \mathbb{R}$:

$$\begin{aligned}\dot{x}(t) &= \overbrace{\begin{bmatrix}1 & 1\\ 0 & \alpha\end{bmatrix}}^{A_1} x(t) + \overbrace{\begin{bmatrix}1\\ 1\end{bmatrix}}^{B} u(t)\\ y(t) &= \underbrace{\begin{bmatrix}\alpha & -1\end{bmatrix}}_{C} x(t).\end{aligned} \tag{4}$$

1. Consider the input $u(t) = 0$. For which values of α is system (4) asymptotically stable, stable, and unstable?

2. For which values of α is system (4) observable?

3. For which values of α is system (4) controllable?

4. For which values of α is it possible to find an input $u(\cdot) : [0, 1] \to \mathbb{R}$ driving system (4) from $x(0) = \begin{bmatrix}3 & 3\end{bmatrix}^T$ to $x(1) = \begin{bmatrix}4 & 4\end{bmatrix}^T$?

5. Consider a state feedback controller of the form

$$u(t) = Kx(t) = \begin{bmatrix}-\alpha & -1\end{bmatrix} x(t), \tag{5}$$

and a second linear time invariant system

$$\begin{aligned}\dot{x}(t) &= \overbrace{\begin{bmatrix}1 & 1\\ 0 & \alpha+1\end{bmatrix}}^{A_2} x(t) + \overbrace{\begin{bmatrix}1\\ 1\end{bmatrix}}^{B} u(t)\\ y(t) &= \underbrace{\begin{bmatrix}\alpha & -1\end{bmatrix}}_{C} x(t).\end{aligned} \tag{6}$$

Are there values of α for which both closed loop systems $(A_1 + BK)$ and $(A_2 + BK)$ are asymptotically stable? If yes, which ones?

6. For the remainder of this exercise, consider the case where $\alpha = 1$. Find all points at which the closed loop system $(A_1 + BK)$ is at equilibrium. Find all points at which the closed loop system $(A_2 + BK)$ is at equilibrium. Are there points $\hat{x} = \begin{bmatrix}\hat{x}_1 & \hat{x}_2\end{bmatrix}^T$ at which both closed loop systems $(A_1 + BK)$ and $(A_2 + BK)$ are at equilibrium? If yes, state the values of $\hat{x}$.

7. For the $\hat{x}$ determined in Part 6, both closed loop systems $(A_1 + BK)$ and $(A_2 + BK)$ are initialized at $x_\epsilon(0) = \begin{bmatrix}\hat{x}_1 + \epsilon & \hat{x}_2\end{bmatrix}^T$, with $\epsilon \neq 0$. Will the two systems return to $\hat{x}$ or diverge? Based on this observation, comment on the stability of the two systems.

Exercise 3

1	2(a)	2(b)	2(c)	3(a)	3(b)	Exercise
4	4	3	3	5	6	25 Points

1. Consider the linear time-invariant system given by

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(t), \\ y(t) &= \begin{bmatrix} \frac{1}{2} & -1 \end{bmatrix} x(t). \end{aligned} \tag{7}$$

   Show that the transfer function $G(s)$ of system (7) from the input $u(t)$ to the output $y(t)$ is given by

$$G(s) = \frac{s-2}{(s+1)(s+2)}.$$

2. The system with transfer function $G(s)$ in Part 1 is controlled by a proportional feedback controller $K(s) = k$, with $k \in \mathbb{R}$. The closed-loop system is shown in Figure 2.

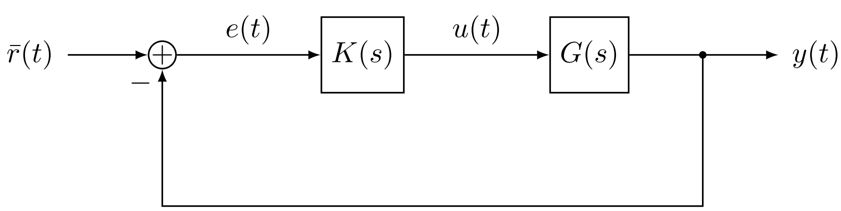

Figure 2: Proposed control architecture.

   (a) Show that the transfer function $T_1(s)$ from the input $\bar{r}(t)$ to the output $y(t)$ is given by

$$T_1(s) = \frac{k(s-2)}{s^2 + s(3+k) + 2 - 2k}.$$

   (b) For what values of $k \in \mathbb{R}$ is the closed loop system asymptotically stable?

   (c) Assuming the closed loop system is asymptotically stable, compute the steady-state value of its step response as a function of k.

3. Your colleague from EPFL proposes a modification of the control architecture. His idea is to use a second controller $K_{\text{ff}}(s)$ in combination with $K(s)$ as shown in Figure 3.

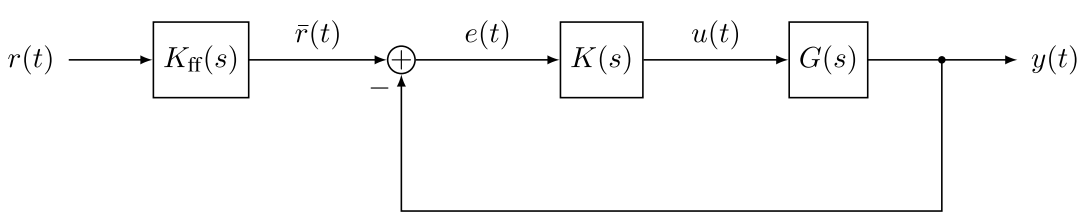

Figure 3: The architecture proposed by your colleague from EPFL.

You colleague suggests using $k = 1$ and

$$K_{\text{ff}}(s) = \frac{s}{s-2}.$$

However, you have doubts about his proposal and would like to verify it.

(a) Compute the transfer functions $T_2(s)$ from $r(t)$ to $y(t)$ and $T_3(s)$ from $r(t)$ to $u(t)$.

(b) Compute the step response of $T_3(s)$. Do you think the solution proposed by your colleague should be deployed on the real system?

Hint: remember that u is the physical input to a real system (e.g., voltage, torque) and therefore bounded in practice.

Exercise 4

1	2	3	4(a)	4(b)	Exercise
3	7	6	6	3	25 Points

Consider the system

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\,,\\ \dot{x}_2(t) &= -x_2(t)\,(x_1(t)-2)^2 - x_1(t)\left(x_1^2(t)-4\right)\,.\end{aligned} \tag{8}$$

1. Determine all equilibria of the system.

2. Consider the function $V(x(t)) = \frac{1}{4}x_1(t)^2\left(x_1^2(t)-8\right) + \frac{1}{2}x_2^2(t)$. Can you use it to show that all state trajectories of the system converge to an equilibrium point?

   Hint: You may assume that $S = \{x(t) \in \mathbb{R}^2 \mid V(x) \leq K\}$ with arbitrary $K \geq 0$ is a compact invariant set and that $V \to \infty$ as $|x| \to \infty$.

3. What can we conclude about the stability of $\hat{x} = (0, 0)$ by using linearization?

4. Consider the function $U(x(t)) = V(x(t)) + 4$ where $V(x(t))$ is as in Part 2. The function $U(x(t))$ is plotted in Figures 4 and 5.

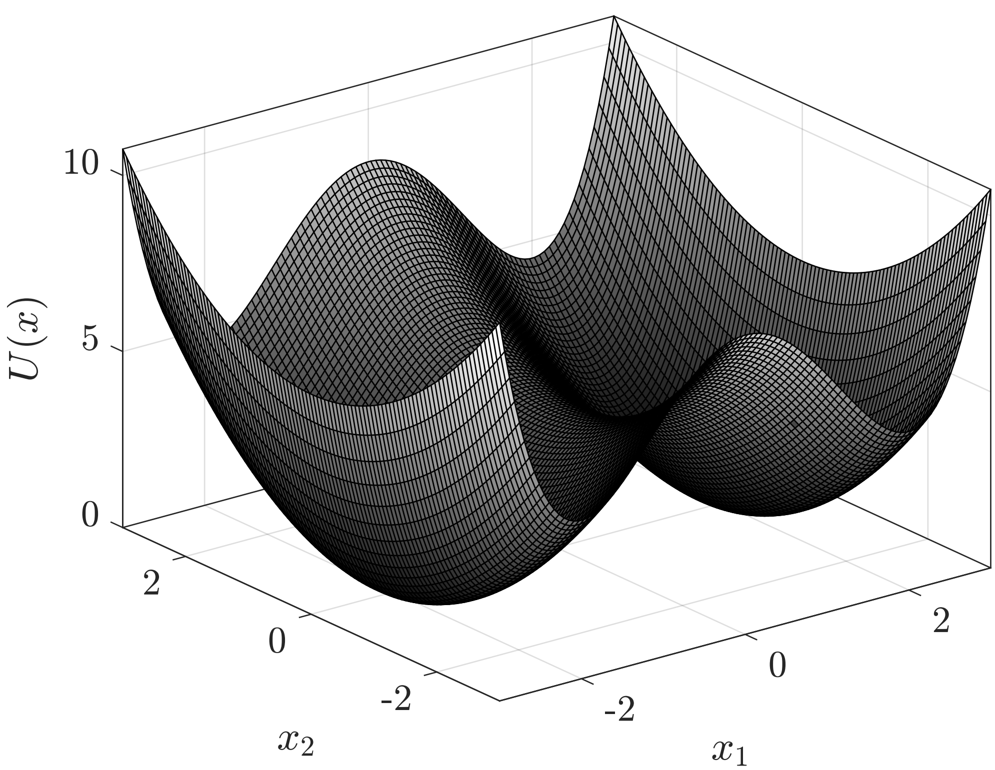

Figure 4: 3D representation of $U(x)$.

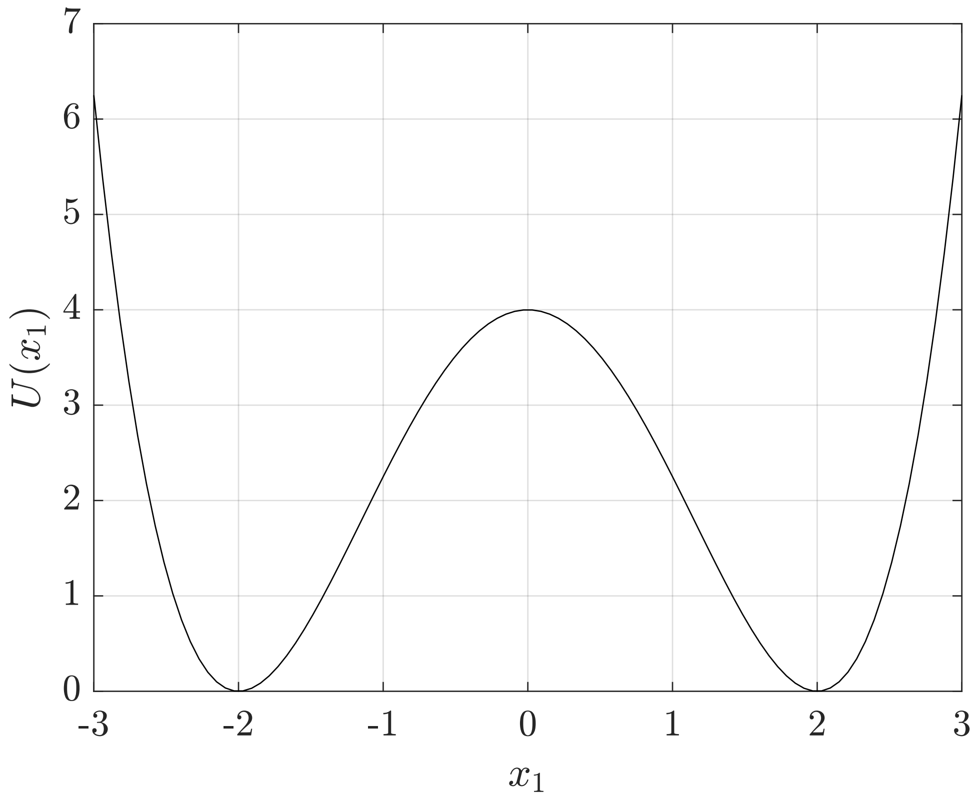

Figure 5: 2D representation of $U(x_1)$ for $x_2 = 0$.

(a) Using $U(x)$ or otherwise, show that the points $\hat{x} = (\pm 2, 0)$ are locally asymptotically stable equilibria.

Hint: Given the shape of $U(x)$ in Figure 4, you may assume that there exist open sets S_1 and S_2 each containing one of the equilibria in question and no other equilibrium.

(b) Estimate the domains of attraction of each of the two equilibria $\hat{x} = (\pm 2, 0)$.