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D-ITET
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Signal and System Theory II

This sheet is provided to you for ease of reference only.
***Do not* write your solutions here.**

Exercise 1

1	2	3	4(a)	4(b)	4(c)	Exercise
3	4	6	4	4	4	25 Points

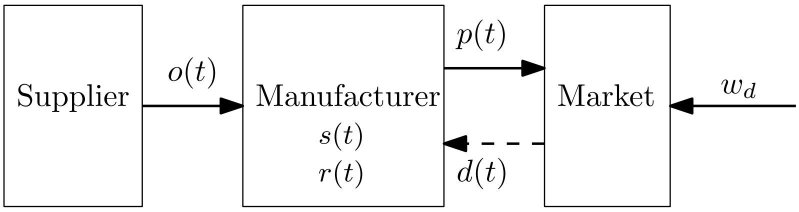

Figure 1: A schematic of the supply chain for a single manufacturer.

A manufacturer wants to optimize their operation and design a control law to stabilize their orders. Their stock $s(t)$ of manufactured goods and production rate $r(t)$ depends on their orders $o(t)$ from suppliers and the demand $d(t)$ and price $p(t)$ of the product they sell in the market, and is modeled by the following equations:

$$\dot{s}(t) = r(t) - d(t) \tag{1}$$

$$\dot{r}(t) = \frac{1}{\tau}(o(t) - r(t)) \tag{2}$$

$$d(t) = w_d - \beta p(t) \tag{3}$$

where w_d is the base demand for the product and $\tau > 0$.

1. For each equation (1)-(3), explain shortly why they make sense in the context of this supply chain problem.

2. Verify that the dynamics of the supply chain can be described by the following linear time invariant system with input $u(t) = \begin{bmatrix} o(t) \\ p(t) \end{bmatrix}$ and state $x(t) = \begin{bmatrix} s(t) \\ r(t) \end{bmatrix}$.

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{\tau} \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 0 & \beta \\ \frac{1}{\tau} & 0 \end{bmatrix}}_{B} u(t) + \underbrace{\begin{bmatrix} -1 \\ 0 \end{bmatrix}}_{D_d} w_d \tag{4}$$

3. In system (4) set $u(t) = 0$ and $w_d = 0$. What are the equilibrium points $\hat{x}$? Do they make intuitive sense in the context of the supply chain application? Are they stable or asymptotically stable?

4. The manufacturer wants to design a linear control law for automatically setting the orders $o(t)$ that is linear in the demand $d(t)$ and the difference from a safety stock level $\bar{s}$ such that $o(t) = -K(s(t) - \bar{s}) + d(t)$ with $K \neq 0$.
   a) Derive the new linear time-invariant dynamics as a function of the state $x(t) = \begin{bmatrix} s(t) \\ r(t) \end{bmatrix}$ and the single input $\tilde{u}(t) = p(t)$.
   b) Compute the equilibrium point of the autonomous system as a function of w_d, $\bar{s}$ and K when $\tilde{u}(t) = 0$. Give an explanation for why this equilibrium point makes sense for the supply chain application.
   c) For which values of K is the equilibrium point asymptotically stable?

Exercise 2

1	2	3	4	5	Exercise
4	5	5	5	6	25 Points

Consider the linear, time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t),$$

where

$$A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

1. Compute the eigenvalues and eigenvectors of A.
2. Compute the transition matrix $\Phi(t) = e^{At}$ by means of the eigenvalue decomposition of A.
3. Is the system stable? Is it stabilizable?
4. Assume that $u(t) = 0$ for all time. Sketch the phase plane plot of the system. In your sketch, particularly consider the eigenvectors and whether they correspond to stable or unstable modes.
5. Assume now that $u(t) = e^{-4t}$ and that you measure the output $y(t)$ for a short time and determine that $y(0) = 2$ and $\dot{y}(0) = 4$. Compute $x(0)$.

Exercise 3

1	2	3a	3b	3c	4	Exercise
5	3	5	3	3	6	25 Points

You are asked to design a controller $K(s)$ for the following plant

$$G(s) = \frac{1}{s^2 + 0.02s + 1},$$

where $K(s)$ and $G(s)$ are connected as depicted in Figure 2.

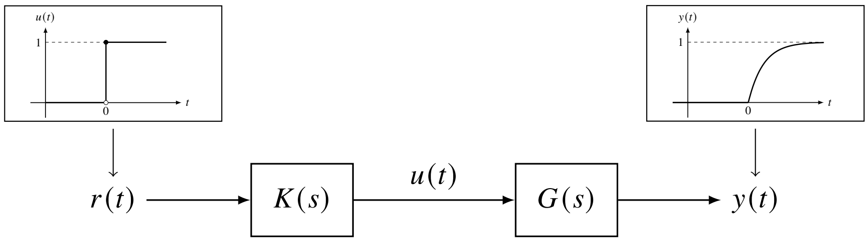

Figure 2: The system $G(s)$ and the controller $K(s)$.

Your task is to ensure that the system can track a *step reference* $r(t)$, that is, that the output $y(t)$ satisfies

$$\lim_{t\to\infty} y(t) = 1 \qquad \text{for} \qquad r(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0. \end{cases} \tag{5}$$

1. Consider first $K(s) = k_\mathrm{p}$. Can you fullfill your goal (5) with this controller? If so, what value of k_p should you choose?

   Hint: Use the final value theorem.

2. Suppose now that the system is affected by a disturbance $d(t)$ as in Figure 3. For the controller of Part 1, compute the value of $y(t)$ as $t \to \infty$ assuming $d(t) = d$, with $d \in \mathbb{R}$.

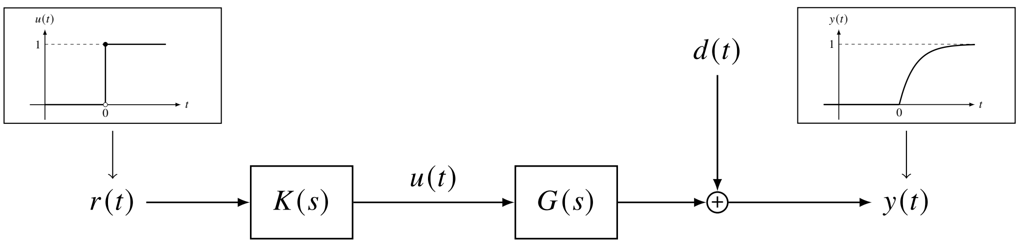

Figure 3: The system $G(s)$ and the controller $K(s)$ subject to a disturbance $d(t)$.

3. To mitigate the effect of the disturbance, you consider a new architecture as in Figure 4, where

$$K(s) = \frac{k_\mathrm{i}}{s}, \quad k_\mathrm{i} > 0.$$

   For now, assume that $d(t) = 0$ for all t.

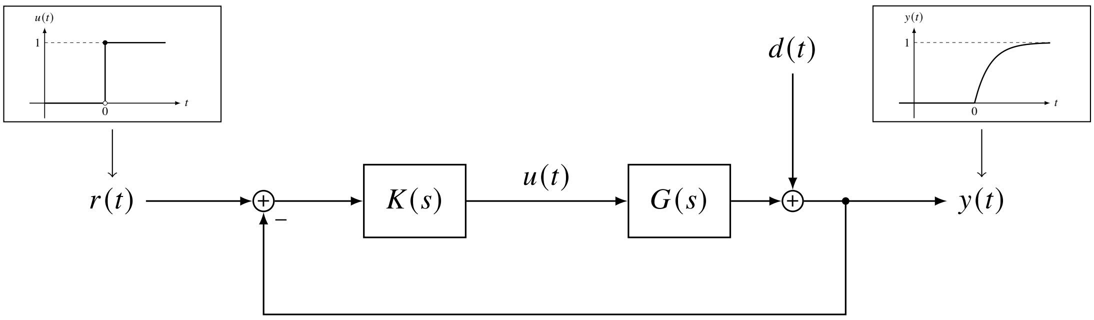

Figure 4: Feedback interconnection of system $G(s)$ and controller $K(s)$.

a) Verify that the phase cross-over frequency of the transfer function $L(s) = K(s)G(s)$ is $\omega_c = 1$ rad/s for any $k_i > 0$.

b) Compute the gain margin as a function of k_i.

Hint: Use the result in 3a. You can also use the Bode plot of the transfer function $L(s) = G(s)K(s)$, which is given in Figure 5 for $k_i = 0.01$.

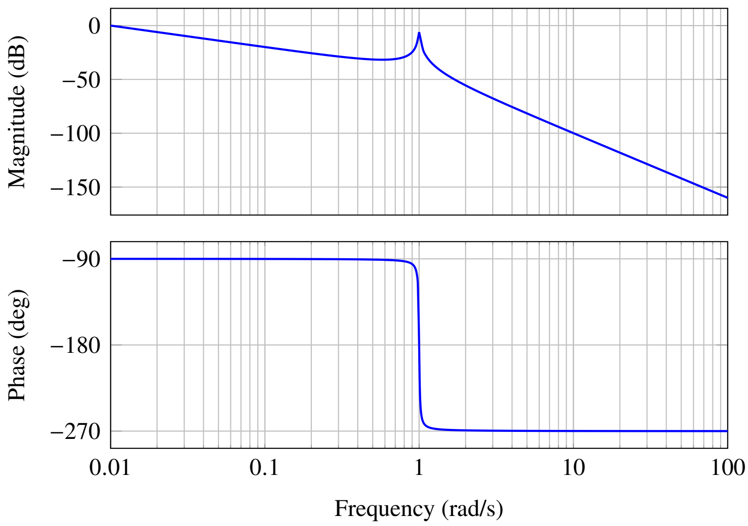

Figure 5: Bode plot of $L(s)$ for $k_i = 0.01$.

c) What values of k_i guarantee that the closed-loop system is stable?

4. Suppose now that $r(t) = 0$ and $d(t) = d$ for all t, with $d \in \mathbb{R}$. Assuming that k_i is chosen according to 3c, what is the value of $y(t)$ as $t \longrightarrow \infty$?

Exercise 4

1	2	3	4	5	Exercise
4	3	5	5	8	25 Points

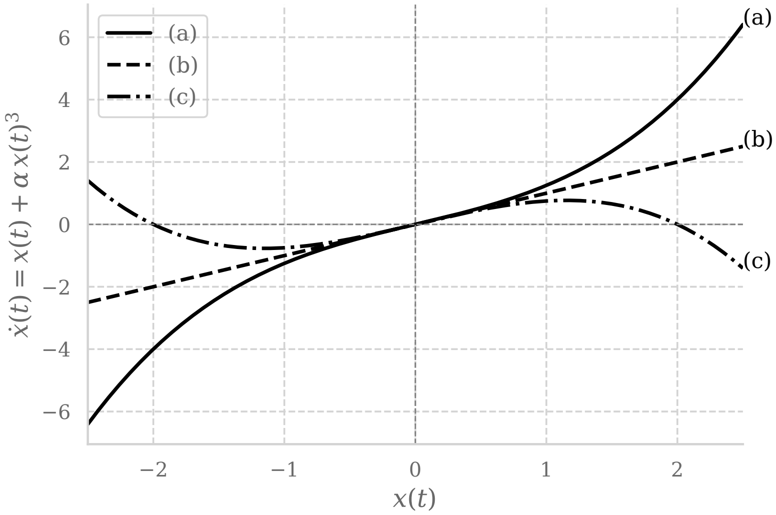

Figure 6: The dynamical system in (6) is plotted for three different values of α.

For $x(t) \in \mathbb{R}$, $\alpha \in \mathbb{R}$, and $t \geq 0$, a dynamical system is described by the following differential equation

$$\dot{x}(t) = x(t) + \alpha x(t)^3. \tag{6}$$

The dynamical system in (6) is plotted for different values of α in Figure 6.

1. What is the order of the system? Is it autonomous? For which values of α is the system linear and for which nonlinear?

2. Match the following values of α with the system dynamics (a) to (c) plotted in Figure 6.

α	Dynamics (a), (b), or (c)
-0.25	
0.25	
0	

3. Compute the equilibrium points $\hat{x}$ of system (6) as a function of α. Can you say something about the stability properties of the different equilibrium points through inspection of Figure 6?

4. For each of the equilibrium points $\hat{x}$ found in Part 3, analyze the stability properties using linearisation.

5. Consider the case $\alpha = -0.25$. If you found some asymptotically stable equilibria in Part 4., select one of them and verify your conclusion using a quadratic Lyapunov function of the form $V(x) = \frac{1}{2}(x - c)^2$ picking the constant c appropriately. Hence, or otherwise, provide an estimate of its domain of attraction.