Theorie

1. Laplace - Transformation

$$\boxed{\mathcal{L}\{f\}(s) := \int_0^\infty f(t)\, e^{-st}\, dt}$$

→ Darstellung (Doetsch-Symbole $\circ\!\!-\!\!\bullet$)

Laplace-Transformation $\mathcal{L}\{f\}$

$\circ\!\!-\!\!\bullet$

Originalfunktion f

Beispiel:

$$f(t) \circ\!\!-\!\!\bullet F(s)$$

$$\frac{d}{dt} f(t) \circ\!\!-\!\!\bullet s F(s) - f(0)$$

$$t^n e^{-at} \circ\!\!-\!\!\bullet \frac{n!}{(s+a)^{n+1}}$$

→ Rücktransformation

Beispiel: Löse $\ddot{y}(t) - y(t) = e^t$, $\dot{y}(0) = y(0) = 0$

$$\left[t^n e^{-at} \} \circ\!\!-\!\!\bullet \frac{n!}{(S+a)^{n+1}} \right]$$

$$\to \mathcal{L}\left\{\frac{d^2}{dt^2} y(t)\right\}(s) = s\,\mathcal{L}\left\{\frac{d}{dt} y(t)\right\}(s) - \frac{d}{dt} y(t)\Big|_{t=0} = s\left(s\,\mathcal{L}\{y(t)\}(s) - y(0)\right) - \dot{y}(0)$$

$$= s^2 Y(s) - s\,y(0) - \dot{y}(0) = s^2 Y(s)$$

$$\to \mathcal{L}\{y(t)\}(s) = Y(s)$$

$$\to \mathcal{L}\{e^t\}(s) = \frac{1}{s-1} \quad \Longrightarrow \quad s^2 Y(s) - Y(s) = \frac{1}{s-1} \quad \Longrightarrow \quad Y(s) = \frac{1}{(s^2-1)(s-1)}$$

$$= \frac{1}{(S+1)(s-1)^2}$$

→ Rücktransformation: $y(t) = \mathcal{L}^{-1}\{Y(s)\}(t)$

$$Y(s) = \frac{1}{(s+1)(s-1)^2} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+1} = \frac{-1/4}{s-1} + \frac{1/2}{(s-1)^2} + \frac{1/4}{s+1}$$

$$y(t) = \underbrace{\mathcal{L}^{-1}\left\{-\frac{1}{4}\frac{1}{s-1}\right\}}_{n=0,\ a=-1} + \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{2}\frac{1}{(s-1)^2}\right\}}_{n=1,\ a=-1} + \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{4}\frac{1}{s+1}\right\}}_{n=0,\ a=1} = -\frac{1}{4}e^t + \frac{1}{2}te^t + \frac{1}{4}e^{-t}$$

$$= \frac{1}{2}te^t - \frac{1}{2}\sinh(t)$$

<u>Beispiel:</u> $Y(s) = \frac{3}{(s+1)^4}$

$$Y(s) = \frac{3}{(s+1)^4} = \frac{3}{3!}\,\frac{3!}{(s+1)^4} \Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{2}\,\frac{3!}{(s+1)^4}\right\} = \frac{1}{2}\,t^3 e^{-t}$$

$n = 3$

$a = 1$

$$\Rightarrow y(t) = \frac{1}{2}\,t^3 e^{-t}$$