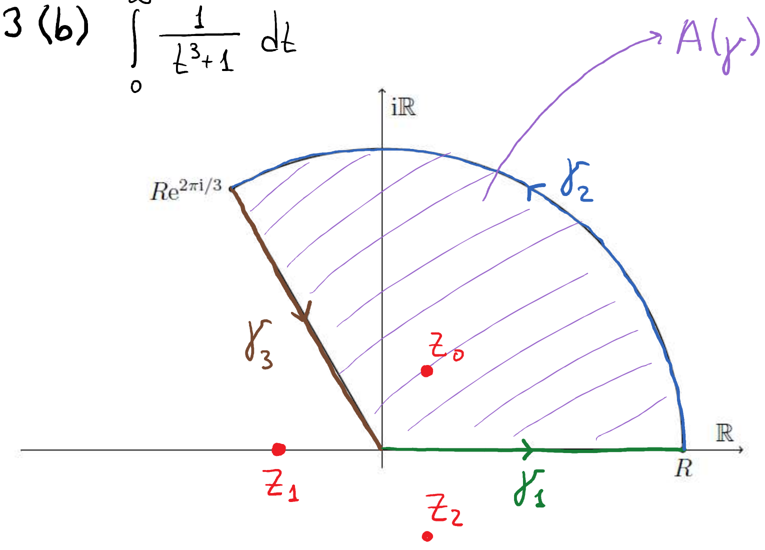
3 (b) $\int_0^\infty \frac{1}{t^3+1}\,dt$



$\gamma := \gamma_1 * \gamma_2 * \gamma_3$

$\gamma_1 : \ Rt, \quad t \in [0,1]$

$\gamma_2 : \ Re^{\frac{2\pi i}{3}t}, \quad t \in [0,1]$

$\gamma_3 : \ te^{\frac{2\pi i}{3}}, \quad t \in [R, 0]$

$\rightarrow z^3+1 = 0 \implies z^3 = -1 = e^{\pi i + 2\pi i k} \implies z = e^{\frac{\pi i + 2\pi i k}{3}}$ $\quad z_0 = e^{\frac{\pi i}{3}}, \quad z_1 = e^{\pi i}, \quad z_2 = e^{\frac{5\pi i}{3}}$

$\rightarrow$ We are effectively looking for $\lim\limits_{R\to\infty} \int_{\gamma_1} f(z)\,dz = \int_0^\infty \frac{1}{z^3+1}\,dz$

- $\lim\limits_{R\to\infty} \int_{\gamma_2} f(z)\,dz = \lim\limits_{R\to\infty} \int_0^1 \frac{1}{(Re^{\frac{2\pi i}{3}t})^3+1} \frac{2\pi i}{3} Re^{\frac{2\pi i t}{3}}\,dt \longrightarrow 0$

- $\lim\limits_{R\to\infty} \int_{\gamma_3} f(z)\,dz = \lim\limits_{R\to\infty} \int_R^0 \frac{1}{(te^{\frac{2\pi i}{3}})^3+1} e^{\frac{2\pi i}{3}}\,dt = -\lim\limits_{R\to\infty} e^{\frac{2\pi i}{3}} \int_0^R \frac{1}{t^3e^{2\pi i}+1}\,dt$

$$= -e^{\frac{2\pi i}{3}} \lim_{R\to\infty} \int_0^R \frac{1}{t^3+1}\,dt = -e^{\frac{2\pi i}{3}} \int_0^\infty \frac{1}{t^3+1}\,dt$$

$\rightarrow$ Only z_0 is inside $\lim\limits_{R\to\infty} A(\gamma)$

$$\implies \operatorname{Res}(f \mid z_0) = \lim_{z \to e^{\pi i/3}} \cancel{\left(z - \tfrac{\pi i}{3}\right)} \frac{1}{\cancel{\left(z - \tfrac{\pi i}{3}\right)}(z - e^{\pi i})(z - e^{\frac{5\pi i}{3}})} = \frac{1}{(e^{\frac{\pi i}{3}} - e^{\pi i})(e^{\frac{\pi i}{3}} - e^{\frac{5\pi i}{3}})}$$

$$= \frac{1}{(\frac{3}{2} + \frac{\sqrt{3}}{2}i)(\sqrt{3}i)} = \frac{1}{3e^{\frac{2\pi i}{3}}}$$

$$\rightarrow 2\pi i \operatorname{Res}(f \mid z_0) = \lim_{R\to\infty}\left[\int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz\right]$$

$$2\pi i \frac{1}{3e^{\frac{2\pi i}{3}}} = \int_0^\infty \frac{1}{t^3+1}\,dt + 0 - e^{\frac{2\pi i}{3}} \int_0^\infty \frac{1}{t^3+1}\,dt = \left(1 - e^{\frac{2\pi i}{3}}\right) \int_0^\infty \frac{1}{t^3+1}\,dt$$

$$\implies \int_0^\infty \frac{1}{t^3+1}\,dt = \frac{1}{1 - e^{\frac{2\pi i}{3}}} \cdot 2\pi i \cdot \frac{1}{3e^{\frac{2\pi i}{3}}} = \boxed{\frac{2\pi}{3\sqrt{3}}}$$