Satz von Parseval

komplex

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi i n t}{T}}$$

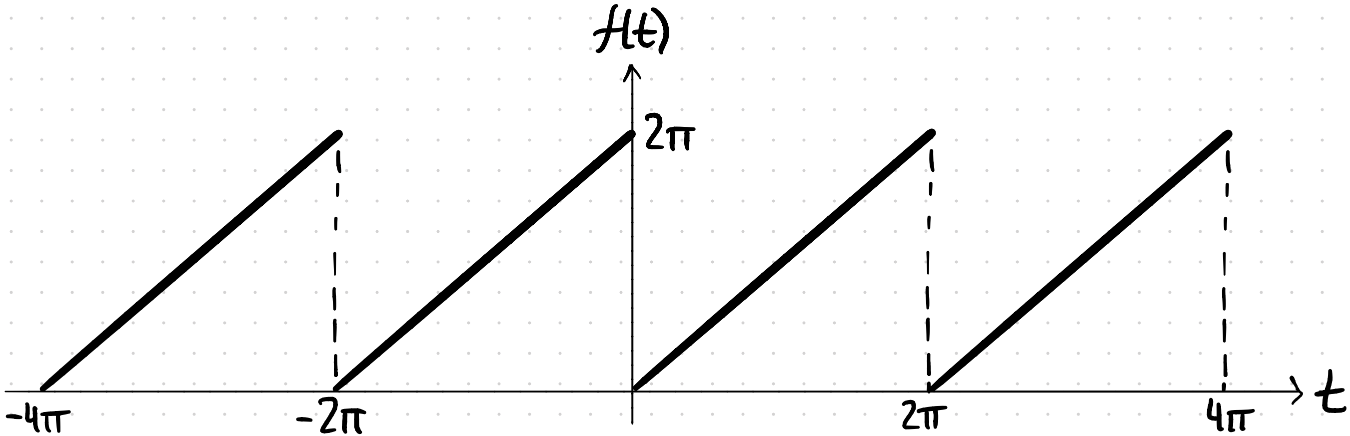
reell

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{T}\right) + b_n \sin\left(\frac{2\pi n t}{T}\right)$$

$$\boxed{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{k=-\infty}^{\infty} |c_k|^2 = \frac{a_0^2}{4} + \frac{1}{2}\sum_{k=1}^{\infty}\left(|a_k|^2 + |b_k|^2\right)}$$

[komplex] [reell]

Beispiel: Berechne $\sum_{k=1}^{\infty} \frac{1}{k^2}$ mit Hilfe der Fourierreihe von $\tilde{f}(t) := t$, $t \in [0, 2\pi]$ und Satz von Parseval



Satz von Parseval: $\frac{1}{2\pi} \int_0^{2\pi} |f(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |c_k|^2$

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$$c_k = \frac{1}{2\pi}\int_0^{2\pi} t e^{-ikt}\,dt = \dots \textit{Partielle Integration} \dots = \frac{i}{k}$$

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$$c_0 = \frac{1}{2\pi}\int_0^{2\pi} t\,dt = \pi$$

$$\Rightarrow c_k = \begin{cases} \pi, & k = 0 \\ \frac{i}{k}, & \text{sonst} \end{cases}$$

1. $$\frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2\,dt$$

2. $$\sum_{k=-\infty}^{\infty} |c_k|^2 =$$

$$1 \quad \frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |c_k|^2 \quad 2$$

$$\downarrow \qquad\qquad \downarrow$$

$$\frac{4}{3}\pi^2 \qquad\qquad \pi^2 + 2\sum_{k=1}^{\infty}\frac{1}{k^2}$$

$$\Rightarrow$$

Fouriertransformation

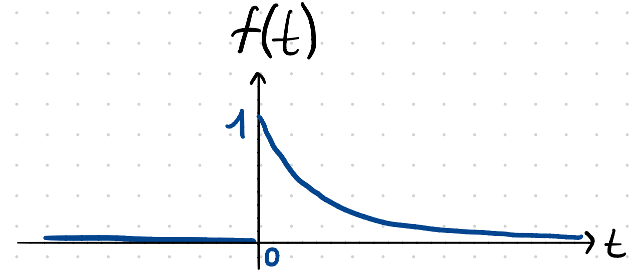
$$\boxed{\mathcal{F}\{f\}(\omega) = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt}$$

→ Inverse FT: $\mathcal{F}^{-1}\{\hat{f}\}(t) = f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega t}\, d\omega$

$$\Rightarrow \underbrace{\mathcal{F}^{-1}\{\underbrace{\mathcal{F}\{f\}(\omega)}_{\text{FT}}\}(t)}_{\text{Inv FT}} = f(t)$$

Beispiel

$$f(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & \text{sonst} \end{cases}$$



$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt$$

$$=$$

Satz von Plancherel

→ $\hat{f}(\omega)$ ist die FT von $f(t)$ $\rightsquigarrow$ $\hat{f}(\omega) = F\{f\}(\omega)$

$$\boxed{\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 \, d\omega}$$

1. Fourierreihe → Periodische Funktionen [T=2π]

→ Satz von Parseval

$$\boxed{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{k=-\infty}^{\infty} |c_k|^2 = \frac{a_0^2}{4} + \frac{1}{2}\sum_{k=1}^{\infty}\left(|a_k|^2 + |b_k|^2\right)}$$

2. Fouriertransformation → Alle [integrierbare] Funktionen

→ Satz von Plancherel

$$\boxed{\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega}$$

Beispiel: Berechne $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ mit dem Satz von Plancherel

$$f(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & \text{sonst} \end{cases} \implies \hat{f}(\omega) = -\frac{1}{\sqrt{2\pi}}\left(\frac{1+i\omega}{\omega^2+1}\right)$$

$\rightarrow$ Plancherel: $\underbrace{\int_{-\infty}^{\infty} |f(t)|^2\,dt}_{1} = \underbrace{\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\,d\omega}_{2}$

1. $\int_{-\infty}^{\infty} |f(t)|^2\,dt =$

2. $\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\,d\omega =$

$$① \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 \, d\omega \; ②$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\frac{1}{2} \qquad\qquad \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1+\omega^2} \, d\omega$$

$$\Longrightarrow$$