Satz von Parseval

komplex

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi i n t}{T}}$$

reell

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{T}\right) + b_n \sin\left(\frac{2\pi n t}{T}\right)$$

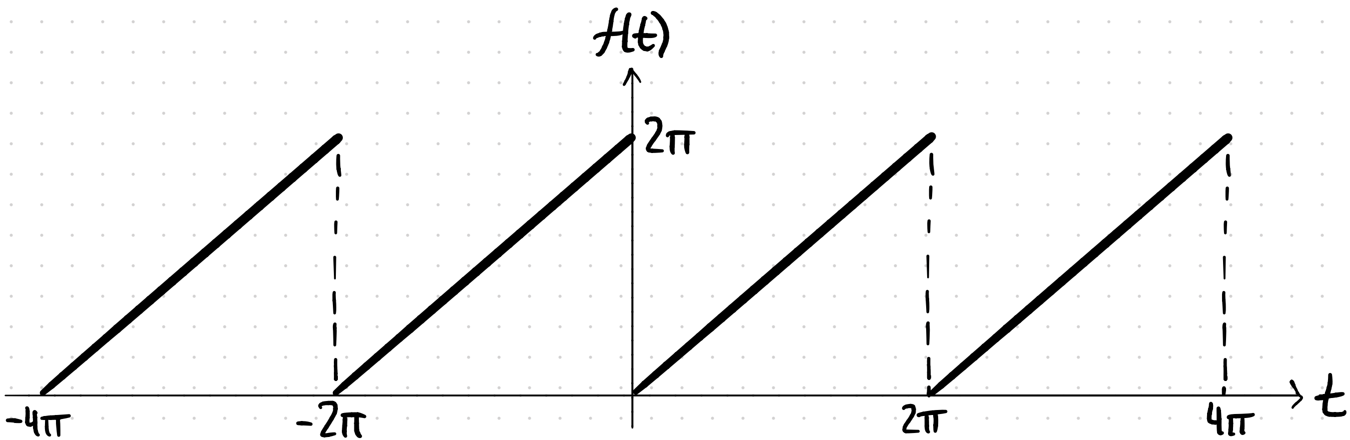
$$\boxed{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 \, dx = \sum_{k=-\infty}^{\infty} |c_k|^2 = \frac{a_0^2}{4} + \frac{1}{2}\sum_{k=1}^{\infty}\left(|a_k|^2 + |b_k|^2\right)}$$

$T = 2\pi$

[komplex]

[reell]

Beispiel: Berechne $\sum_{k=1}^{\infty} \frac{1}{k^2}$ mit Hilfe der Fourierreihe von $\tilde{f}(t) := t, \ t \in [0, 2\pi]$ und Satz von Parseval



Satz von Parseval: $\frac{1}{2\pi} \int_0^{2\pi} |f(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |c_k|^2$

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$\boxed{S9A4}$

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} t e^{-ikt}\,dt = \dots \textit{Partielle Integration} \dots = \frac{i}{k}$$

$$c_0 = \frac{1}{2\pi}\int_0^{2\pi} t\,dt = \pi \qquad \Longrightarrow \qquad c_k = \begin{cases} \pi, & k=0 \\ \frac{i}{k}, & \text{sonst} \end{cases}$$

① $$\frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2 dt = \frac{1}{2\pi}\int_0^{2\pi} t^2 dt = \frac{1}{2\pi}\frac{1}{3}t^3\Big|_0^{2\pi} = \frac{4}{3}\pi^2$$

② $$\sum_{k=-\infty}^{\infty} |c_k|^2 = \sum_{k=-\infty}^{-1}\left|\frac{i}{k}\right|^2 + c_0^2 + \sum_{k=1}^{\infty}\left|\frac{i}{k}\right|^2 = \pi^2 + 2\sum_{k=1}^{\infty}\frac{1}{k^2}$$

$|i| = 1$

$$\frac{1}{k^2} \longleftrightarrow \frac{1}{k^2}$$

$$|c_{-k}|^2 = |c_k|^2$$

$$① \quad \frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |c_k|^2 \quad ②$$

$$\underline{\frac{4}{3}\pi^2} = \underline{\pi^2} + \underline{2}\sum_{k=1}^{\infty}\frac{1}{k^2}$$

$$\Rightarrow \sum_{k=1}^{\infty}\frac{1}{k^2} = \left(\frac{4}{3}\pi^2 - \pi^2\right)\frac{1}{2} = \frac{\pi^2}{6}$$

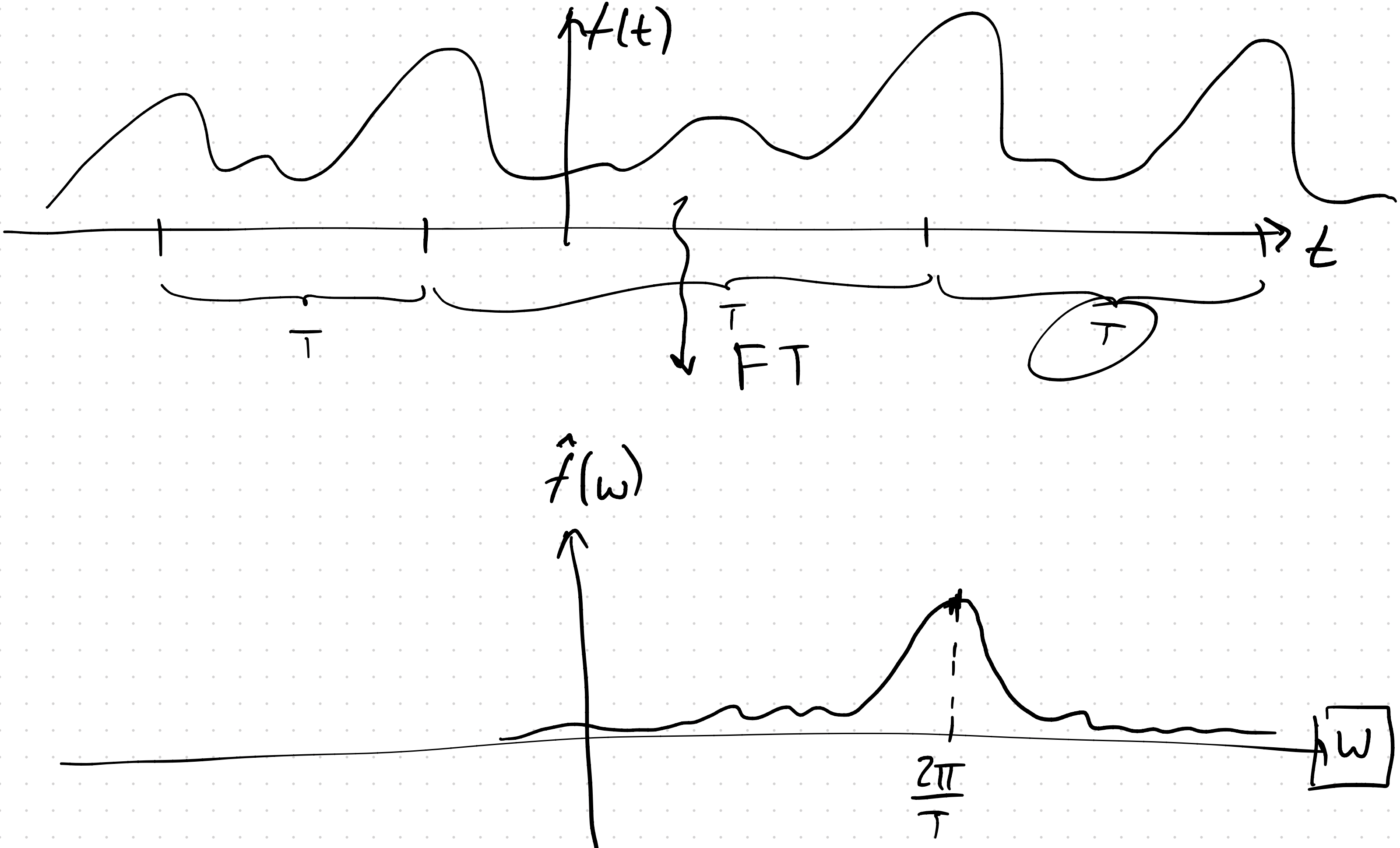
$$\boxed{\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}}$$

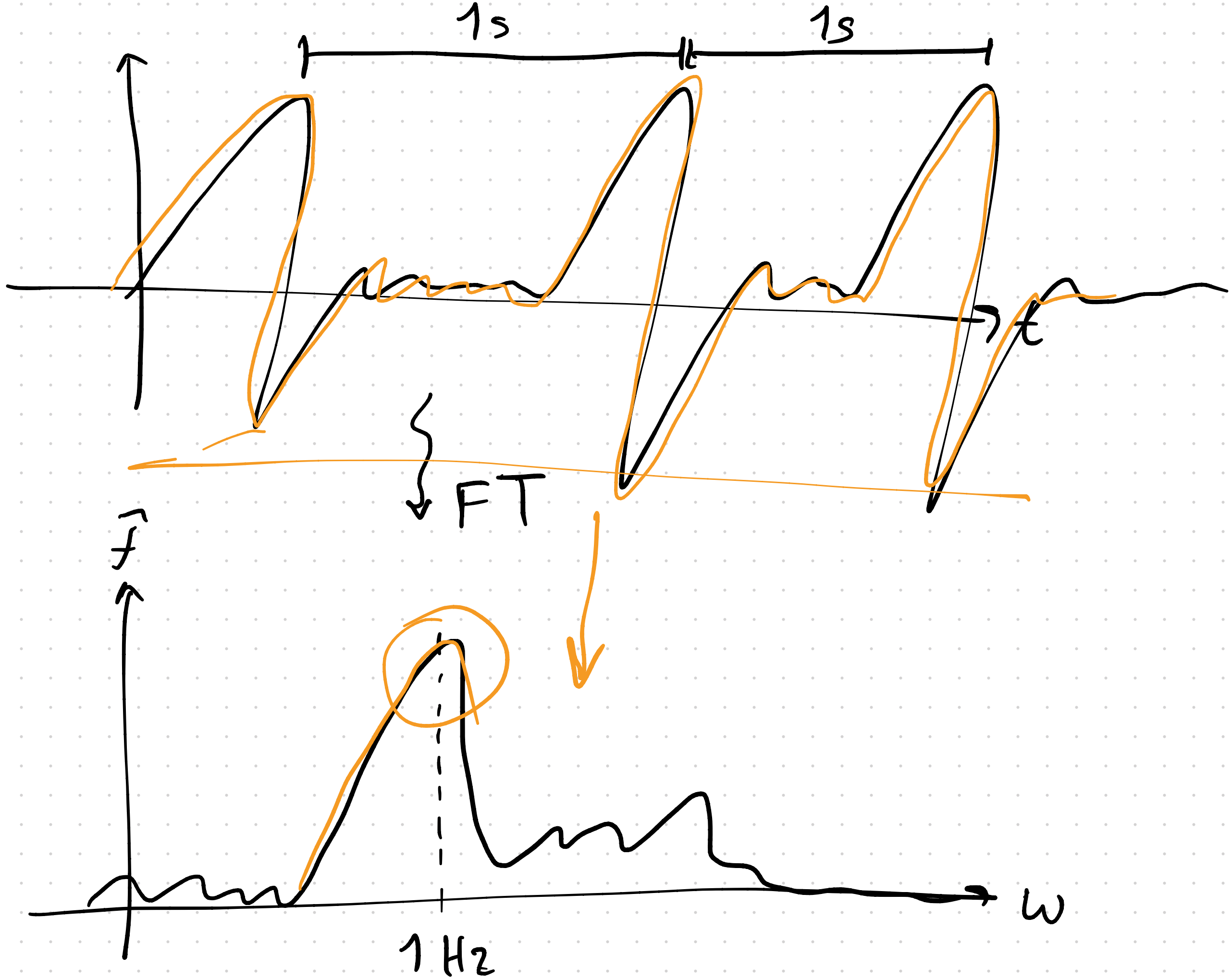
Fouriertransformation

$$\mathcal{F}\{f\}(\omega) = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

→ Inverse FT: $\mathcal{F}^{-1}\{\hat{f}\}(t) = f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega t}\, d\omega$

$$\Rightarrow \underbrace{\mathcal{F}^{-1}\{\underbrace{\mathcal{F}\{f\}(\omega)}_{\text{FT}}\}(t)}_{\text{Inv FT}} = f(t)$$


f(t)
t
T
T
T
FT
f̂(ω)
2π/T
ω

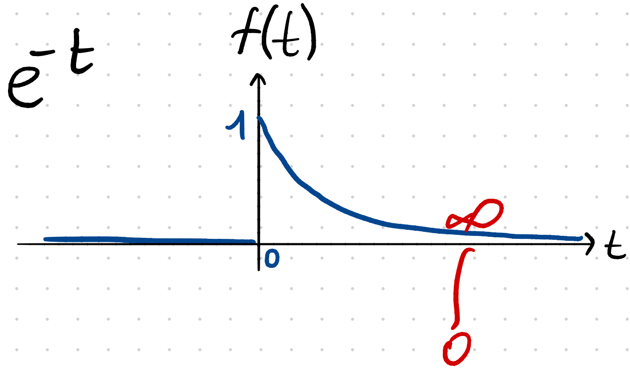


1s
1s
t
FT
$\hat{f}$
1 Hz
ω

Beispiel

$$\boxed{f(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & \text{sonst} \end{cases}}$$

$|e^{-t}| = e^{-t}$



$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-t} e^{-i\omega t}\, dt = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \underline{e^{-t(1+i\omega)}}\, dt$$

a) FT

b) $\int_{-\infty}^{\infty} \frac{1}{1+\omega^2}\, d\omega$

$$= \frac{1}{\sqrt{2\pi}} \left(-\frac{1}{1+i\omega}\right) e^{-t(1+i\omega)} \Big|_0^{\infty}$$

$\lim_{t \to \infty} e^{-t} = 0$

$$= \frac{1}{\sqrt{2\pi}} \frac{1\,(1-i\omega)}{(1+i\omega)\,(1-i\omega)} = \frac{1 - i\omega}{\sqrt{2\pi}\,(1+\omega^2)}$$

Satz von Plancherel

→ $\hat{f}(\omega)$ ist die FT von $f(t)$ ⟿ $\hat{f}(\omega) = F\{f\}(\omega)$

$$\boxed{\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 \, d\omega}$$

$$|f(t)|^2 = f(t) \cdot \overline{f(t)}$$

1. Fourierreihe → Periodische Funktionen [T=2π]

→ Satz von Parseval

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx = \sum_{k=-\infty}^{\infty}|c_k|^2 = \frac{a_0^2}{4}+\frac{1}{2}\sum_{k=1}^{\infty}\left(|a_k|^2+|b_k|^2\right)$$

2. Fouriertransformation → Alle [integrierbare] Funktionen

→ Satz von Plancherel

$$\int_{-\infty}^{\infty}|f(t)|^2\,dt = \int_{-\infty}^{\infty}|\hat{f}(\omega)|^2\,d\omega$$

Beispiel: Berechne $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ mit dem Satz von Plancherel

$$f(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & \text{sonst} \end{cases} \Rightarrow \hat{f}(\omega) = -\frac{1}{\sqrt{2\pi}}\left(\frac{1+i\omega}{\omega^2+1}\right)$$

→ Plancherel: $\boxed{\underbrace{\int_{-\infty}^{\infty} |f(t)|^2\,dt}_{1} = \underbrace{\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\,d\omega}_{2}}$

$z = a + bi$

$|z| = \sqrt{a^2+b^2}$

$|z|^2 = a^2+b^2$

1 $\int_{-\infty}^{\infty} |f(t)|^2\,dt = \int_{-\infty}^{\infty} e^{-2t} = \frac{1}{2}$

2 $\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\,d\omega = \int_{-\infty}^{\infty} \left|\frac{1}{\sqrt{2\pi}}\left(\frac{1+i\omega}{\omega^2+1}\right)\right|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{1}{\omega^2+1} + i\frac{\omega}{\omega^2+1}\right|^2 d\omega$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{(\omega^2+1)^2} + \frac{\omega^2}{(\omega^2+1)^2}\,d\omega = \frac{1}{2\pi}\boxed{\int_{-\infty}^{\infty} \frac{1}{\omega^2+1}\,d\omega}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(\omega^2+1)^2} + \frac{\omega^2}{(\omega^2+1)^2} \, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\boxed{1+\omega^2}}{(\boxed{1+\omega^2})^2} \rightarrow 1 \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\cancel{1+\omega^2}}{\cancel{(1+\omega^2)}(1+\omega^2)} \, d\omega$$

① $\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega$ ②

$$\frac{1}{2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1+\omega^2} d\omega$$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} \frac{1}{1+\omega^2} d\omega} = \frac{1}{2} \cdot 2\pi = \underline{\pi}$$

$$\int_{-\infty}^{\infty} \underbrace{\frac{1}{1+x^2}}_{f(x)\,\leq\, a x^{-2}} dx = 2\pi i \sum_{\mathrm{Im}>0} \mathrm{Res}(f)$$

$$\text{Sing. } 1+x^2=0 \Rightarrow x=\pm i \quad \begin{array}{l} \nearrow \boxed{x_1 = i} \\ \searrow x_2 = -i \end{array}$$

$$\mathrm{Res}\left(\frac{1}{1+x^2}, x=i\right) = \lim_{x\to i} \cancel{(x-i)} \frac{1}{(x+i)\cancel{(x-i)}} = \frac{1}{2i}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \cancel{2}\pi \cancel{i} \cdot \frac{1}{\cancel{2i}} = \pi$$

Eigenschaften

$$F\{f(t)\cdot g(t)\}(\omega) = \widehat{f(t)\cdot g(t)}(\omega) = \hat{f}(\omega) * \hat{g}(\omega)$$

$$F\{f(t-a)\}(\omega) = \widehat{f(t-a)} = e^{-ia\omega}\,\hat{f}(\omega)$$

$$e^{-(t+2)} \quad \downarrow$$

$$f(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0, & \text{sonst} \end{cases} \qquad g(t) := \begin{cases} e^{-t-2}, & t \geq -2 \\ 0, & \text{sonst} \end{cases} \qquad a = 2$$

$$\hat{g}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cdots\cdots, \quad e^{+i2\omega}\,\hat{f}(\omega)$$