1. Fourierreihe → Periodische Funktionen [T=2π]

→ Satz von Parseval

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx = \sum_{k=-\infty}^{\infty}|c_k|^2 = \frac{a_0^2}{4}+\frac{1}{2}\sum_{k=1}^{\infty}\left(|a_k|^2+|b_k|^2\right)$$

2. Fouriertransformation → Alle [integrierbare] Funktionen

→ Satz von Plancherel

$$\int_{-\infty}^{\infty}|f(t)|^2\,dt = \int_{-\infty}^{\infty}|\hat{f}(\omega)|^2\,d\omega$$

Fouriertransformation

$$\boxed{\mathcal{F}\{f\}(\omega) = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt}$$

$\rightarrow$ Inverse FT: $\mathcal{F}^{-1}\{\hat{f}\}(t) = f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega t}\, d\omega$

$$\Rightarrow \underbrace{\mathcal{F}^{-1}\{\underbrace{\mathcal{F}\{f\}(\omega)}_{\text{FT}}\}(t)}_{\text{Inv FT}} = f(t)$$

$$f(t) \xrightarrow{FT} \hat{f}(\omega)$$

ω, ν, f

$$\widehat{\boxed{t f(t)}}(\omega) = \underline{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty}} \boxed{t f(t)}\, \underline{e^{-i\omega \boxed{t}}\, dt}$$

$$\underbrace{\widehat{e^{-at} f(t)}(\omega)} \xrightarrow{FT} \underbrace{\hat{f}(\omega - a)}$$

1. e^{-at} mit $f(t)$
2. FT

1. Finde FT f
2. Verschiebung um a

$F(t)$	$\widehat{F}(\omega)$	Notes	(0)
$f(t)$	$\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$	Definition.	(1)
$\frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\omega)e^{i\omega t}\,d\omega$	$\widehat{f}(\omega)$	Inversion formula.	(2)
$\widehat{f}(-t)$	$2\pi f(\omega)$	Duality property.	(3)
$e^{-at}u(t)$	$\frac{1}{a+i\omega}$	a constant, $\Re e(a) > 0$	(4)
$e^{-a\|t\|}$	$\frac{2a}{a^2+\omega^2}$	a constant, $\Re e(a) > 0$	(5)
$\beta(t) = \begin{cases} 1, & \text{if } \|t\| < 1, \\ 0, & \text{if } \|t\| > 1 \end{cases}$	$2\,\mathrm{sinc}(\omega) = 2\frac{\sin(\omega)}{\omega}$	Boxcar in time.	(6)
$\frac{1}{\pi}\,\mathrm{sinc}(t)$	$\beta(\omega)$	Boxcar in frequency.	(7)
$f'(t)$	$i\omega\widehat{f}(\omega)$	Derivative in time.	(8)
$f''(t)$	$(i\omega)^2\widehat{f}(\omega)$	Higher derivatives similar.	(9)
$tf(t)$	$i\frac{d}{d\omega}\widehat{f}(\omega)$	Derivative in frequency.	(10)
$t^2 f(t)$	$i^2\frac{d^2}{d\omega^2}\widehat{f}(\omega)$	Higher derivatives similar.	(11)
$e^{i\omega_0 t}f(t)$	$\widehat{f}(\omega-\omega_0)$	Modulation property.	(12)
$f\left(\frac{t-t_0}{k}\right)$	$ke^{-i\omega t_0}\widehat{f}(k\omega)$	Time shift and squeeze.	(13)
$(f*g)(t)$	$\sqrt{2\pi}\,\widehat{f}(\omega)\widehat{g}(\omega)$	Convolution in time.	(14)
$f\cdot g$	$(\hat{f}*\hat{g})(\omega)$		

$\frac{1}{\sqrt{2\pi}}$

$$\widehat{\left(\frac{d}{dt}f(t)\right)}(\omega) = i\omega\hat{f}(\omega)$$

$$\widehat{(f*g)}(\omega) = \hat{f}(\omega)\cdot\hat{g}(\omega)$$

Beispiel: Finde $\hat{f}(\omega)$ in Abhängigkeit von $\hat{g}(\omega)$ $\rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t) e^{-i\omega t} dt$

$$f(t) := \int_0^\infty g(t-s)\, ds$$

Methode 1:

$$\sqrt{2\pi}\, \hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} dt = \int_{-\infty}^{\infty} \int_0^\infty g(t-s)\, \boxed{ds}\, e^{-i\omega t} \boxed{dt}$$

$$= \boxed{\int_{-\infty}^{\infty}} \boxed{\int_0^\infty} g(t-s) e^{-i\omega t} \boxed{ds}\boxed{dt} = \int_0^\infty \int_{\boxed{-\infty}}^{\boxed{\infty}} g(\underbrace{t-s}_{\ell}) e^{-i\omega t} dt\, ds$$

$+\infty$ $\quad$ $+\infty$

$-\infty$ $\quad$ $-\infty$

$$\ell = t - s \qquad \frac{d\ell}{dt} = 1 \qquad \boxed{dt = d\ell}$$

$$t = \ell + s$$

$$= \int_0^\infty \int_{-\infty}^\infty g(\ell)\, \underbrace{e^{-i\omega(\ell+s)}}_{e^{-i\omega\ell} \cdot \boxed{e^{-i\omega s}}} \, d\ell\, ds$$

$$= \int_0^\infty \underbrace{\int_{-\infty}^\infty g(\ell)\, e^{-i\omega\ell}\, d\ell}_{\hat{g}(\omega)} \cdot e^{-i\omega s}\, ds = \int_0^\infty \hat{g}(\omega)\, e^{-i\omega s}\, ds$$

$$= \hat{g}(\omega) \underbrace{\int_0^\infty \boxed{e^{-i\omega s}}\, ds}_{\hat{a}(\omega)}$$

$$= \hat{g}(\omega) \frac{1}{-i\omega} e^{-i\omega s} \Big|_0^\infty = \hat{g}(\omega) \frac{1}{i\omega}$$

$$\boxed{\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{i\omega}\, \hat{g}(\omega)}$$

$\boxed{A2}$

Beispiel: Finde $\hat{f}(\omega)$ in Abhängigkeit von $\hat{g}(\omega)$

$$f(t) := \int_0^\infty 1\, g(t-s)\,ds$$

Methode 2: A3

$$a(t),\ b(t) \longrightarrow (a*b)(t) = \int_{-\infty}^{\infty} a(s)\cdot b(t-s)\,ds$$

$$\boxed{b(t)} = g(t)$$

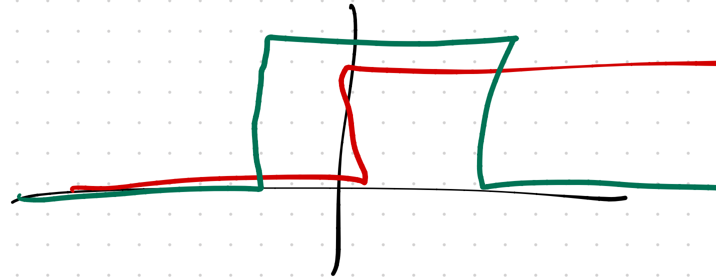
$$\boxed{a(t)} = \begin{cases} 1, & t \geq 0 \\ 0, & \text{sonst} \end{cases}$$

$$(a*b)(t) = \int_0^\infty 1\cdot b(t-s)\,ds$$

$$f(t) = (a*g)(t), \quad \text{wobei} \quad a(t) = \begin{cases} 1, & t \geq 0 \\ 0, & \text{sonst} \end{cases}$$

$$\hat{f}(\omega) = \widehat{(a*g)}(\omega)$$

$$= \hat{a}(\omega) \cdot \boxed{\hat{g}(\omega)}$$



$$\hat{a}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a(t) e^{-i\omega t} \, dt = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-i\omega t} \, dt = \frac{1}{\sqrt{2\pi}} \frac{1}{i\omega}$$

$$\Rightarrow \quad \hat{f}(\omega) = \hat{a}(\omega)\hat{g}(\omega) = \frac{1}{\sqrt{2\pi}} \frac{1}{i\omega} \cdot \hat{g}(\omega)$$

Reelle Integrale mit exp, sin, cos

$$\int_{|z|=1} f(z)\,dz \xrightarrow[t\in[0,1]]{\gamma(t):=e^{2\pi i t}} \int_0^1 f(\underbrace{e^{2\pi i t}}_{\gamma(t)})\cdot \underbrace{2\pi i\, e^{2\pi i t}}_{\gamma'(t)}\,dt$$

$$\int_{|z|=1} f(z) = \frac{z}{1+2z}\,dz$$

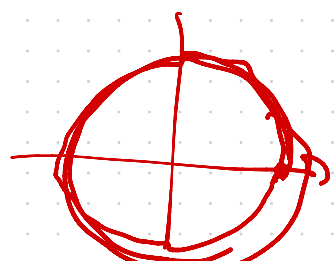
Beispiel

$$f(e^{it}) = \int_0^{4\pi} \frac{e^{it}}{1+2e^{it}}\,dt = \int_{2\times |z|=1} \frac{z}{1+2z}$$

z

$z = e^{2\pi i t}$

$t \in [0,1]$



$e^{it} = z$

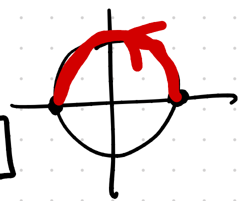
$1\times$

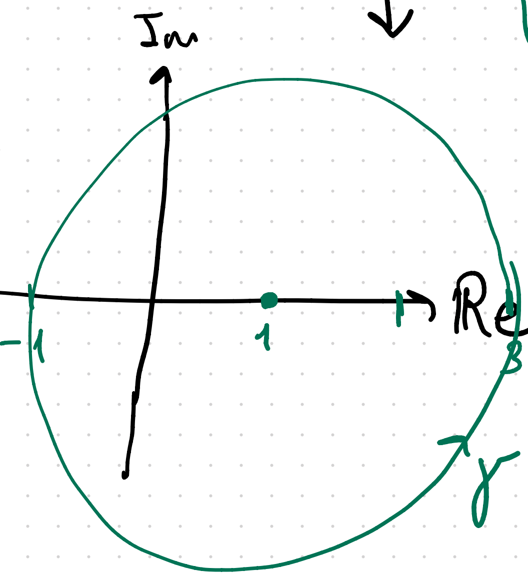
$t \in [0, 4\pi] \to 2\times$ EK

$$\int_{0}^{\pi} \frac{e^{it}}{1+2e^{it}}\,dt \qquad z = e^{it}$$

$$z = \boxed{1+2e^{it}} \qquad t \in [0, 4\pi]$$

$$z = e^{it}$$
$$t \in [0,\pi]$$





$$\int_{\gamma} \frac{\frac{z-1}{2}}{z}\,dt$$

$$|z-1| = 2 \qquad \frac{dz}{dt}$$

$$z = 1+2e^{it}$$
$$e^{it} = \frac{z-1}{2}$$

Reelle Integrale mit exp, sin, cos

$$\int_{|z|=1} f(z)\,dz \xrightarrow[t\in[0,1]]{\gamma(t):=e^{2\pi i t}} \int_0^1 f(\underbrace{e^{2\pi i t}}_{\gamma(t)})\cdot \underbrace{2\pi i e^{2\pi i t}}_{\gamma'(t)}\,dt$$

$$\boxed{\int_0^T f\left(e^{\frac{2\pi}{T} i t}\right) dt \xrightarrow{z:=e^{\frac{2\pi}{T} i t}} \int_{|z|=1} f(z)\cdot\frac{1}{2\pi i z}\,dz}$$

$$\left[\frac{dz}{dt} = \frac{2\pi}{T} i\, e^{\frac{2\pi}{T} i t} = \frac{2\pi}{T} i z \quad \Rightarrow dt = \frac{1}{2\pi i z}\right]$$

1. Alle period. Fkt. haben gleiche Periode
2. Grenzen $\Rightarrow$ <u>komplette</u> Umdrehungen (Vielfaches der Periode)

$\underline{\text{Beispiel}}$: Finde $\displaystyle\int_0^{4\pi} \frac{e^{it}}{1+2e^{it}}\, dt$

1. ✓

2. $(4\pi - 0) = 4\pi = 2 \cdot \underbrace{2\pi}_{T}$

$z = e^{it} \longrightarrow [0, 4\pi] \Rightarrow \underline{\underline{2}}$ Umdrehungen bei $|z| = 1$

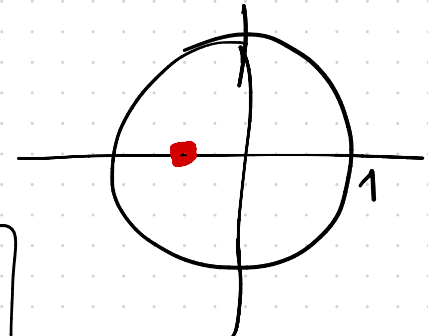
$$\frac{dz}{dt} = \underbrace{e^{it}}_{z} \cdot i \Rightarrow dt = \frac{1}{iz}$$

$$\int_0^{4\pi} \frac{e^{it}}{1+2e^{it}}\, dt = 2 \oint_{|z|=1} \frac{\cancel{z}}{(1+2z)} \cdot \frac{1}{(i\cancel{z})}\, dz = \frac{2}{i} \oint_{|z|=1} \frac{1}{\boxed{1+2z}}\, dz$$

Sing. bei $z_0 = -\frac{1}{2}$

$$= \frac{2}{i}\int_{|z|=1} \frac{1}{\boxed{1+2z}}\,dz$$

Sing. bei $\boxed{z_0 = -\frac{1}{2}}$



$$\boxed{\int_0^{4\pi} \frac{e^{it}}{1+2e^{it}}\,dt = 2\pi}$$

$$\mathrm{Res}\left(\frac{1}{1+2z}\right) = \lim_{z\to -\frac{1}{2}} \left(z+\frac{1}{2}\right)\frac{1}{1+2z} = \frac{1}{2}$$

$$\int_0^{4\pi} \frac{e^{it}}{1+2\sin(t)}\,dt$$

$$\Rightarrow \frac{2}{i}\int_{|z|=1} \frac{1}{1+2z}\,dz = \frac{\cancel{2}}{\cancel{i}}\cdot\frac{1}{\cancel{2}}\cdot 2\pi \cancel{i} = 2\pi$$

$$\sin(t) = \frac{\boxed{e^{it}} - e^{-it}}{2i}$$

$$e^{-it} = \frac{1}{e^{it}} = \boxed{\frac{1}{z}}$$

$\gamma : |z| = 1$

$z = e^{it}$