

1. Fourierreihe → Periodische Funktionen [T=2π]

→ Satz von Parseval

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{k=-\infty}^{\infty} |c_k|^2 = \frac{a_0^2}{4} + \frac{1}{2} \sum_{k=1}^{\infty} (|a_k|^2 + |b_k|^2)$$

2. Fouriertransformation → Alle [integrierbare] Funktionen

→ Satz von Plancherel

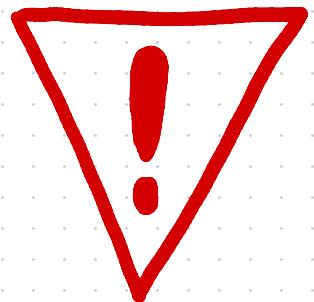
$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega$$

Fouriertransformation

$$\mathcal{F}\{f\}(\omega) = \hat{f}(\omega) = \frac{1}{T2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$\rightarrow \text{Inverse FT: } \mathcal{F}^{-1}\{\hat{f}\}(t) = f(t) = \frac{1}{T2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega$$

$$\Rightarrow \mathcal{F}^{-1}\left\{ \underbrace{\mathcal{F}\{f\}(\omega)}_{\substack{\text{FT} \\ \text{Inv FT}}}(t)\right\} = f(t)$$



$$f(t) \xrightarrow{\text{FT}} \hat{f}(\omega)$$

ω, \mathcal{T}, f

$$\boxed{tf(t)}(\omega) = \frac{1}{12\pi} \int_{-\infty}^{\infty} \boxed{tf(t)} e^{-i\omega t} dt$$

$$\underbrace{e^{-at} f(t)}(\omega) \xrightarrow{\text{FT}} \hat{f}(\omega - a)$$

1. e^{-at} mit $f(t)$

2. FT

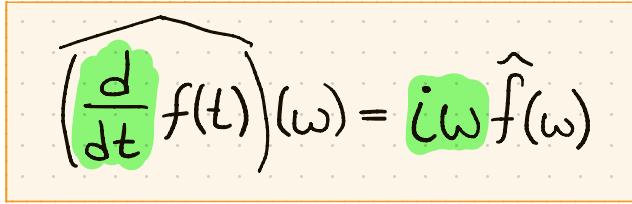
1. Finde FT f

2. Verschiebung um a

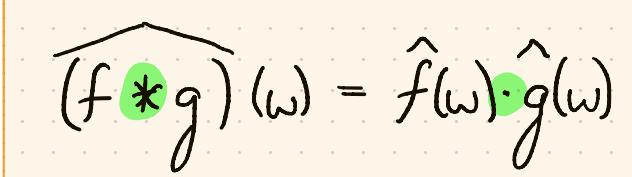
$F(t)$	$\widehat{F}(\omega)$	Notes	(0)
$f(t)$	$\int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt$	Definition.	(1)
$\frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{f}(\omega) e^{i\omega t} d\omega$	$\widehat{f}(\omega)$	Inversion formula.	(2)
$\widehat{f}(-t)$	$2\pi f(\omega)$	Duality property.	(3)
$e^{-at}u(t)$	$\frac{1}{a+i\omega}$	a constant, $\Re(a) > 0$	(4)
$e^{-a t }$	$\frac{2a}{a^2 + \omega^2}$	a constant, $\Re(a) > 0$	(5)
$\beta(t) = \begin{cases} 1, & \text{if } t < 1, \\ 0, & \text{if } t > 1 \end{cases}$	$2 \operatorname{sinc}(\omega) = 2 \frac{\sin(\omega)}{\omega}$	Boxcar in time.	(6)
$\frac{1}{\pi} \operatorname{sinc}(t)$	$\beta(\omega)$	Boxcar in frequency.	(7)
$f'(t)$	$i\omega \widehat{f}(\omega)$	Derivative in time.	(8)
$f''(t)$	$(i\omega)^2 \widehat{f}(\omega)$	Higher derivatives similar.	(9)
$tf(t)$	$i \frac{d}{d\omega} \widehat{f}(\omega)$	Derivative in frequency.	(10)
$t^2 f(t)$	$i^2 \frac{d^2}{d\omega^2} \widehat{f}(\omega)$	Higher derivatives similar.	(11)
$e^{i\omega_0 t} f(t)$	$\widehat{f}(\omega - \omega_0)$	Modulation property.	(12)
$f\left(\frac{t-t_0}{k}\right)$	$ke^{-i\omega_0 t} \widehat{f}(k\omega)$	Time shift and squeeze.	(13)
$(f * g)(t)$	$\sqrt{2\pi} \widehat{f}(\omega) \widehat{g}(\omega)$	Convolution in time.	(14)
$f \cdot g$	$(\widehat{f} * \widehat{g})(\omega)$		



 $\frac{1}{\sqrt{2\pi}}$



$$\left(\frac{d}{dt} f(t) \right)(\omega) = i\omega \widehat{f}(\omega)$$



$$(f * g)(\omega) = \widehat{f}(\omega) \cdot \widehat{g}(\omega)$$

Beispiel: Finde $\hat{f}(\omega)$ in Abhängigkeit von $\hat{g}(\omega)$

$$f(t) := \int_0^{\infty} g(t-s) ds$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t) e^{-i\omega t} dt$$

Methode 1:

$$\begin{aligned}\sqrt{2\pi} \hat{f}(\omega) &= \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt = \int_{-\infty}^{\infty} \int_0^{\infty} g(t-s) ds e^{-i\omega t} dt \\ &= \left[\int_{-\infty}^{\infty} \int_0^{\infty} g(t-s) e^{-i\omega t} ds dt \right]_{t=s}^{t=\infty} = \int_0^{\infty} \int_{-\infty}^{\infty} g(t-s) e^{-i\omega t} dt ds\end{aligned}$$

$$l = t - s$$

$$t = l + s$$

$$\frac{dl}{dt} = 1$$

$$dt = dl$$

$$= \int_0^\infty \int_{-\infty}^\infty g(l) e^{-i\omega(l+s)} dl ds$$

↓
 $\underbrace{e^{-i\omega l}} \cdot \boxed{\underbrace{e^{-i\omega s}}}$

$$= \int_0^\infty \int_{-\infty}^\infty g(l) e^{-i\omega l} dl \cdot e^{-i\omega s} ds = \int_0^\infty \hat{g}(\omega) \underbrace{e^{-i\omega s}} ds$$

$\underbrace{\hat{g}(\omega)}$
 $\boxed{e^{-i\omega s}}$

$$= \hat{g}(\omega) \int_0^\infty \boxed{e^{-i\omega s}} ds$$

$$= \hat{g}(\omega) \frac{1}{-i\omega} e^{-i\omega s} \Big|_0^\infty = \hat{g}(\omega) \frac{1}{i\omega} \underbrace{\Big[e^{-i\omega s} \Big]}_{\hat{a}(\omega)}$$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{i\omega} \hat{g}(\omega)$$

A2

Beispiel: Finde $\hat{f}(\omega)$ in Abhängigkeit von $\hat{g}(\omega)$

$$f(t) := \int_0^\infty g(t-s) ds$$

Methode 2: A3

$$a(t), b(t) \rightarrow (a * b)(t) = \int_{-\infty}^{\infty} a(s) \cdot b(t-s) ds$$

$$b(t) = g(t)$$

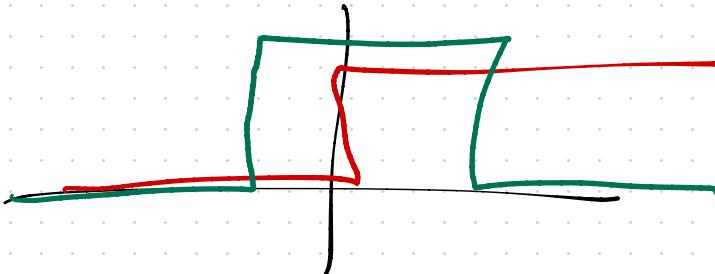
$$a(t) = \begin{cases} 1, & t > 0 \\ 0, & \text{sonst} \end{cases}$$

$$(a * b)(t) = \int_0^\infty 1 \cdot b(t-s) ds$$

$$f(t) = (a * g)(t), \text{ wobei } a(t) = \begin{cases} 1, & t \geq 0 \\ 0, & \text{sonst} \end{cases}$$

$$\hat{f}(\omega) = \widehat{(a * g)}(\omega)$$

$$= \hat{a}(\omega) \cdot \boxed{\hat{g}(\omega)}$$



$$\hat{a}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a(t) e^{-i\omega t} dt = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-i\omega t} dt = \frac{1}{\sqrt{2\pi}} \frac{1}{i\omega}$$

$$\Rightarrow \hat{f}(\omega) = \hat{a}(\omega) \hat{g}(\omega) = \frac{1}{\sqrt{2\pi}} \frac{1}{i\omega} \cdot \boxed{\hat{g}(\omega)}$$

Reelle Integrale mit exp, sin, cos

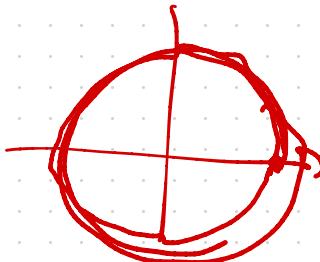
$$\int f(z) dz \xrightarrow{\substack{z = e^{2\pi i t} \\ t \in [0,1]}} \int_0^1 f(e^{2\pi i t}) \cdot 2\pi i e^{2\pi i t} dt$$

$$\int_{|z|=1} f(z) = \frac{z}{1+2z} dz$$

Beispiel

$$z = e^{2\pi i t}$$

$$t \in [0,1]$$



$$f(e^{it}) = \frac{e^{it}}{1+2e^{it}} dt = \frac{z}{1+2z} dt$$

$2 \times |z| = 1$

$e^{it} = z$

$t \in [0, 4\pi] \rightarrow 2 \times EK$

Diagram illustrating the substitution $z = e^{2\pi i t}$ for $t \in [0, 1]$. The right side shows the integral $\int_0^1 f(e^{2\pi i t}) \cdot 2\pi i e^{2\pi i t} dt$ being transformed into $\int_0^{4\pi} f(z) \cdot 2\pi i z dz$. The interval $t \in [0, 4\pi]$ is divided into two segments: $[0, 2\pi]$ and $[2\pi, 4\pi]$, each corresponding to a full revolution of the unit circle. The factor $2\pi i z$ is shown as a vector from the origin to a point z on the circle, and the differential dt is shown as a small arc length element.

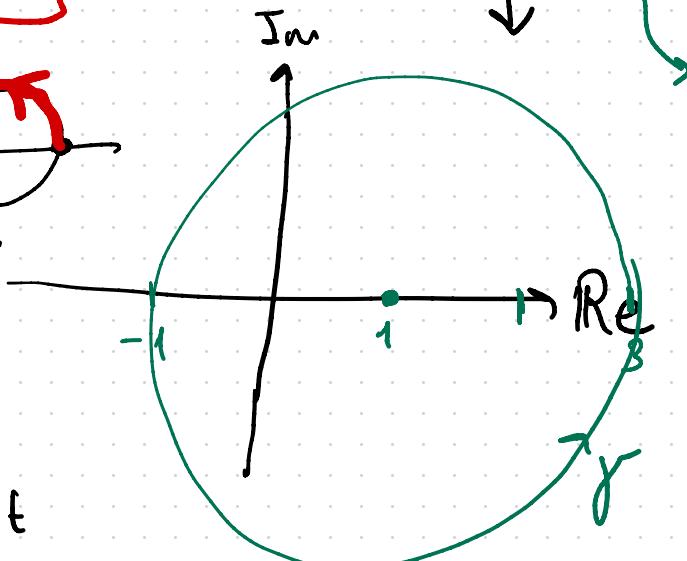
$$z = e^{it}$$

$$z = e^{it}$$

$$t \in [0, \pi]$$

$$z = \frac{1+2e^{it}}{1+2e^{-it}}$$

$$t \in [0, 4\pi]$$



$$\int \frac{\frac{z-1}{z}}{z} dt$$

$$|z-1|=2 \quad \frac{dz}{dt}$$

$$z = 1+2e^{it}$$

$$e^{it} = \frac{z-1}{2}$$

Reelle Integrale mit exp, sin, cos

$$\int f(z) dz \xrightarrow[\substack{|z|=1 \\ t \in [0,1] \\ \gamma(t) := e^{2\pi i t}}]{} \int_0^1 f(e^{2\pi i t}) \cdot 2\pi i e^{2\pi i t} dt$$

$$f'(t)$$

$$\int_0^T f(e^{\frac{2\pi}{T}it}) dt \xrightarrow[z := e^{\frac{2\pi}{T}it}]{} \int_{|z|=1} f(z) \cdot \frac{1}{2\pi i z} dz$$

$$\left[\begin{aligned} \frac{dz}{dt} &= \frac{2\pi}{T} i e^{\frac{2\pi}{T}it} = \frac{2\pi}{T} i z \\ \Rightarrow dt &= \frac{1}{2\pi i z} \end{aligned} \right]$$

- Alle period. Fkt. haben gleiche Periode
- Grenzen \Rightarrow komplette Umdrehungen
(Vielfaches der Periode)

Beispiel: Finde $\int_0^{4\pi} \frac{e^{it}}{1+2e^{it}} dt$

1. ✓
 2. $(4\pi - 0) = 4\pi = 2 \cdot \frac{2\pi}{T}$

$z = e^{it} \rightarrow [0, 4\pi] \Rightarrow \underbrace{2}_{z} \text{ Umdrehungen bei } |z|=1$

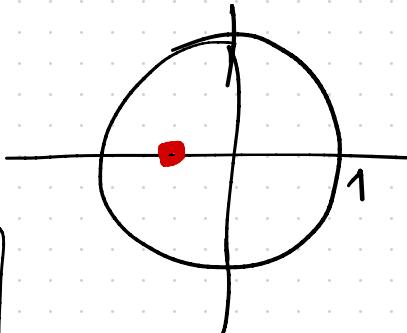
$$\frac{dz}{dt} = \underbrace{e^{it}}_z \cdot i \Rightarrow dt = \frac{1}{iz} dz$$

$$\int_0^{4\pi} \frac{e^{it}}{1+2e^{it}} dt = 2 \int_{|z|=1} \frac{z}{(1+2z)} \cdot \frac{1}{iz} dz = \frac{2}{i} \int_{|z|=1} \frac{1}{1+2z} dz$$

Sing. bei $z_0 = -\frac{1}{2}$

$$= \frac{2}{i} \int_{|z|=1} \frac{1}{1+2z} dz$$

Sing. bei $z_0 = -\frac{1}{2}$



$$\text{Res}\left(\frac{1}{1+2z}\right) = \lim_{z \rightarrow -\frac{1}{2}} \left(z + \frac{1}{2}\right) \frac{1}{1+2z} = \frac{1}{2}$$

$$\Rightarrow \frac{2}{i} \int_{|z|=1} \frac{1}{1+2z} dz = \frac{2}{i} \cdot \frac{1}{2} \cdot 2\pi i = 2\pi$$

$$e^{-it} = \frac{1}{e^{it}} = \boxed{\frac{1}{z}}$$

$$\gamma: |z|=1$$

$$z = e^{it}$$

$$\int_0^{4\pi} \frac{e^{it}}{1+2e^{it}} dt = 2\pi$$

$$\int_0^{4\pi} \frac{e^{it}}{1+2\sin(t)} dt$$

$$\sin(t) = \frac{e^{it} - e^{-it}}{2i}$$