__Beispiel:__ Finde $\hat{f}(\omega)$ in Abhängigkeit von $\hat{g}(\omega)$

$$f(t) := \int_0^\infty g(t-s)\,ds$$

$$\sqrt{2\pi}\,\hat{f}(\omega) = \cdots = \hat{g}(\omega) \underbrace{\int_0^\infty e^{-i\omega s}\,ds}$$

$$\frac{1}{-i\omega} e^{-i\omega s}\Big|_0^\infty = \frac{1}{-i\omega}\left(\lim_{s\to\infty} e^{-i\omega s} - 1\right)$$

$F(t)$	$\widehat{F}(\omega)$	Notes	(0)
$f(t)$	$\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$	Definition.	(1)
$\frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\omega)e^{i\omega t}\,d\omega$	$\widehat{f}(\omega)$	Inversion formula.	(2)
$\widehat{f}(-t)$	$2\pi f(\omega)$	Duality property.	(3)
$e^{-at}u(t)$	$\frac{1}{a+i\omega}$	a constant, $\Re(a) > 0$	(4)
$e^{-a\|t\|}$	$\frac{2a}{a^2+\omega^2}$	a constant, $\Re(a) > 0$	(5)
$\beta(t) = \begin{cases} 1, & \text{if } \|t\| < 1, \\ 0, & \text{if } \|t\| > 1 \end{cases}$	$2\,\mathrm{sinc}(\omega) = 2\frac{\sin(\omega)}{\omega}$	Boxcar in time.	(6)
$\frac{1}{\pi}\,\mathrm{sinc}(t)$	$\beta(\omega)$	Boxcar in frequency.	(7)
$f'(t)$	$i\omega\widehat{f}(\omega)$	Derivative in time.	(8)
$f''(t)$	$(i\omega)^2\widehat{f}(\omega)$	Higher derivatives similar.	(9)
$tf(t)$	$i\frac{d}{d\omega}\widehat{f}(\omega)$	Derivative in frequency.	(10)
$t^2f(t)$	$i^2\frac{d^2}{d\omega^2}\widehat{f}(\omega)$	Higher derivatives similar.	(11)
$e^{i\omega_0 t}f(t)$	$\widehat{f}(\omega-\omega_0)$	Modulation property.	(12)
$f\left(\frac{t-t_0}{k}\right)$	$ke^{-i\omega t_0}\widehat{f}(k\omega)$	Time shift and squeeze.	(13)
$(f*g)(t)$	$\widehat{f}(\omega)\widehat{g}(\omega)$	Convolution in time.	(14)

$\frac{1}{\sqrt{2\pi}}$!

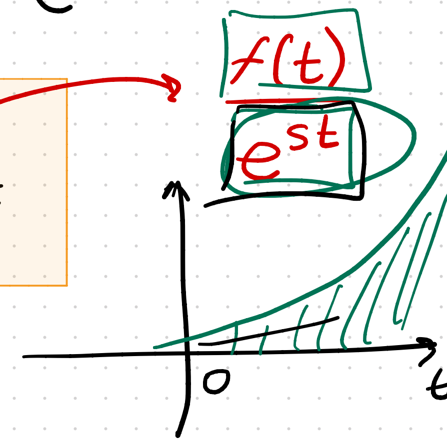
$$\widehat{\left(\frac{d}{dt}f(t)\right)}(\omega) = i\omega\hat{f}(\omega)$$

$$\widehat{(f*g)}(\omega) = \hat{f}(\omega)\cdot\hat{g}(\omega)$$

Laplacetransformation

$\cancel{f(t) = e^{t^2}} \rightarrow X$

$$\boxed{\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty \underbrace{f(t)\,e^{-st}}_{\mathrm{Re}}\,dt}$$



→ Existenz

1. f ist von exponentieller Ordnung

   $\exists\, c, s_0 > 0$ s.d. $|f(t)| \leq C e^{s_0 t}, \; t > 0$ $\Rightarrow$ $\mathcal{L}$ existiert für $\mathrm{Re}\{s\} \geq s_0$

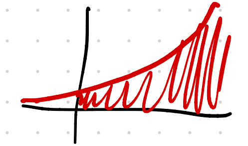
2. Integrierbarkeit

   $\int_0^T |f(t)|\,dt < \infty, \; T > 0$ $\Rightarrow$ f stetig zwischen 0 und ∞

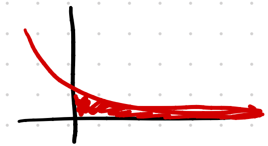
Beispiel:

$\rightarrow f(t) = e^{3t} \rightarrow \int_0^\infty e^{3t} e^{-st}\,dt = \int_0^\infty e^{\underbrace{t(3-s)}}\,dt$

$3-s \geq 0$



$3-s < 0$



$\mathcal{L}\{f\}\ \exists$ für $\operatorname{Re}\{s\} > 3 = s_0$ $\quad s_0 = 3$

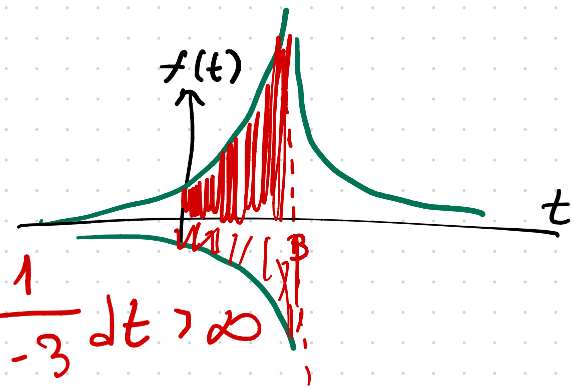
$\rightarrow f(t) = e^{t^2}$

$\mathcal{L}\{f\}\ \nexists$

$\rightarrow f(t) = \frac{1}{t-3}$ $\quad$ 1. exp Ord. ✓ $\quad$ 2. int.

$\int_0^T \left|\frac{1}{t-3}\right| dt \not< \infty$

$\int_0^3 \frac{1}{t-3}\,dt \rightarrow \infty$

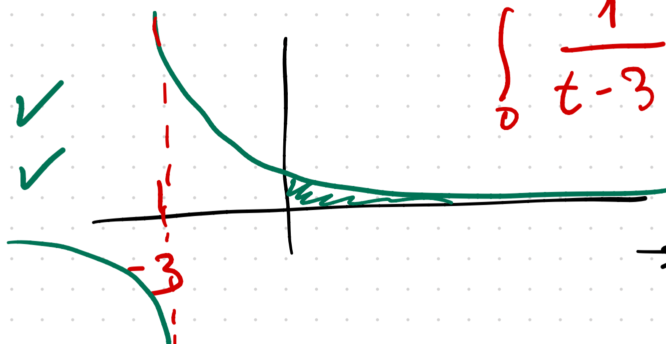


$\mathcal{L}\{f\}\ \nexists$

$\rightarrow f(t) = \frac{1}{t+3}$ $\quad$ 1. ✓ $\quad$ 2. ✓

$\boxed{t = -3}$

$[0, \infty[$



$\rightarrow \int_0^\infty \frac{1}{t+3} e^{-st}\,dt$

Laplacetransformation

$\mathcal{L}^{-1}\{\cdot\}$

$$\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$\mathcal{F}\{f\}(\omega) = \hat{f}(\omega)$$

$$\mathcal{L}\{f\}(s) = F(s)$$

→ Eigenschaften

1. $f(t) \circ\!\!-\!\!\bullet\ F(s)$
2. $f(t-a) \circ\!\!-\!\!\bullet\ e^{-as}F(s)$
3. $\frac{d}{dt} f(t) \circ\!\!-\!\!\bullet\ sF(s) - f(0)$
4. $t^n e^{-at} \circ\!\!-\!\!\bullet\ \frac{n!}{(s+a)^{n+1}}$

⋮

	$f(t) = \mathfrak{L}^{-1}\{F(s)\}$	$F(s) = \mathfrak{L}\{f(t)\}$
1.	1	$\frac{1}{s}$
3.	$t^n,\quad n = 1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{\left(s^2+a^2\right)^2}$
11.	$\sin(at) - at\cos(at)$	$\frac{2a^3}{\left(s^2+a^2\right)^2}$
13.	$\cos(at) - at\sin(at)$	$\frac{s\left(s^2-a^2\right)}{\left(s^2+a^2\right)^2}$
15.	$\sin(at+b)$	$\frac{s\sin(b) + a\cos(b)}{s^2+a^2}$

cos

Beispiel: Gegeben ist $F(s) = \frac{1}{s(s-1)}$. Finde $f(t)$ $\mathcal{L}\{f\}(s) = F(s)$

$G(s) = \frac{1}{s}$. Finde $g(t)$

$$G(s) = \frac{1}{s} = \frac{0!}{(s+0)^{0+1}} = \frac{1}{s}$$

$a = 0$
$n = 0$

$$\mathcal{L}\{1\}(s) = \frac{1}{s}$$

$$g(t) = t^0 e^{-0 \cdot t} = 1$$

$$F(s) = \frac{1}{s(s-1)} = \frac{A}{s} + \frac{B}{s-1} = -\frac{1}{s} + \frac{1}{s-1}$$

$$f(t) = -\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = -1 + t^0 e^{1 \cdot t} = -1 + e^t$$

$n = 0$
$a = 0$

$n = 0$
$a = -1$

$$\left[\, t^n e^{-at} \circ\!\!-\!\!\bullet \frac{n!}{(s+a)^{n+1}} \,\right]$$

Beispiel: Löse $\dot{f}(t) = e^t$, $f(0) = 0$

$\hookrightarrow \frac{d}{dt} f(t)$

1. $\mathcal{L}\{DGL\}$
2. $F(s) = \cdots$
3. $\mathcal{L}^{-1}\{F(s)\}(t)$

1. $\mathcal{L}\{\dot{f}(t)\}(s) = \mathcal{L}\{e^t\}(s)$ $\quad n=0$, $a=-1$

$$\mathcal{L}\left\{\frac{d}{dt} f(t)\right\}(s) = \frac{1}{s-1}$$

$$s \cdot F(s) - \underbrace{f(0)}_{0} = \frac{1}{s-1} \Rightarrow sF(s) = \frac{1}{s-1} \Rightarrow \boxed{F(s) = \frac{1}{s(s-1)}}$$

PBZ

$$\mathcal{L}^{-1} F(s) = \frac{1}{s(s-1)} = \frac{A}{s} + \frac{B}{s-1} = \mathcal{L}^{-1}\ \underline{-\frac{1}{s} + \frac{1}{s-1}}$$

$$= -1 + e^t$$

$$\left[\begin{array}{lcl} \frac{d}{dt} f(t) & \circ\!\!-\!\!\bullet & sF(s) - f(0) \\ t^n e^{-at} & \circ\!\!-\!\!\bullet & \frac{n!}{(s+a)^{n+1}} \end{array}\right]$$

Beispiel: Was ist $\mathcal{L}\{\cdot\}$ von $\ddot{y}(t)$ in Abhängigkeit von $Y(s) = \mathcal{L}\{y\}$?

$\hookrightarrow \frac{d^2}{dt^2} y(t)$

$$\mathcal{L}\Big\{\frac{d}{dt}\underbrace{\frac{d}{dt} y(t)}_{f(t)}\Big\}(s) = \mathcal{L}\Big\{\frac{d}{dt} f(t)\Big\}(s) = s\,F(s) - f(0)$$

$F(s)$: $\mathcal{L}\{\frac{d}{dt} y(t)\}$

$f(0)$: $\hookrightarrow \frac{d}{dt} y(t)\Big|_{t=0}$ $\dot{y}(0)$

$$= s\,\mathcal{L}\Big\{\frac{d}{dt} y(t)\Big\}(s) - \dot{y}(0)$$

$$= s\big(s\,Y(s) - y(0)\big) - \dot{y}(0) = \boxed{s^2 Y(s) - s\,y(0) - \dot{y}(0)}$$

$$t^n e^{-at} \circ\!\!-\!\!\bullet \frac{n!}{(s+a)^{n+1}}$$

$$\Big[\frac{d}{dt} f(t) \;-\!\!\bullet\; sF(s) - f(0)\Big]$$