$$f(z) = \frac{\sinh(z) - z}{z^3} \qquad \left[\sinh(z) = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} \right]$$

$z = 0$

$$= \frac{1}{z^3} \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} - \frac{1}{z^2} = \sum_{k=0}^{\infty} \frac{z^{2k-2}}{(2k+1)!} - \underbrace{\cancel{\frac{1}{z^2}}}_{2.\ \text{Ordn.}}$$

$k=0$ $\frac{1}{z^2} \Rightarrow$ 2. Ordn.

$\Rightarrow$ Sing. Ordn. 2 X

$k=0 \Rightarrow \cancel{\frac{1}{z^2}}$

$$f(z) = z - \underbrace{\cancel{\frac{1}{z}}}_{1\ \text{Ord}} + \underbrace{\cancel{\frac{1}{z}}}_{1\ \text{Ord}}$$

$f(z=0)$ ist nicht definiert $\Rightarrow$ 1. Ordnung X

$= z \Rightarrow z = 0 \Rightarrow$ hebbar

$$m := \boxed{2}k$$

$$\sum_{k=0}^{\infty} \frac{z^{\boxed{2k}-2}}{(2k)!} = \sum_{m=0}^{\infty} \frac{z^{\boxed{2m}}}{m!}$$

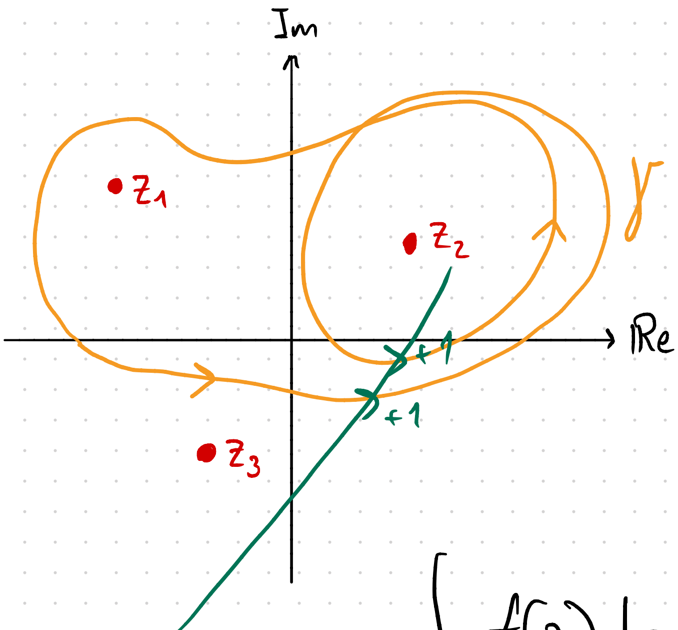
$k, m = 0, 1, 2, 3$

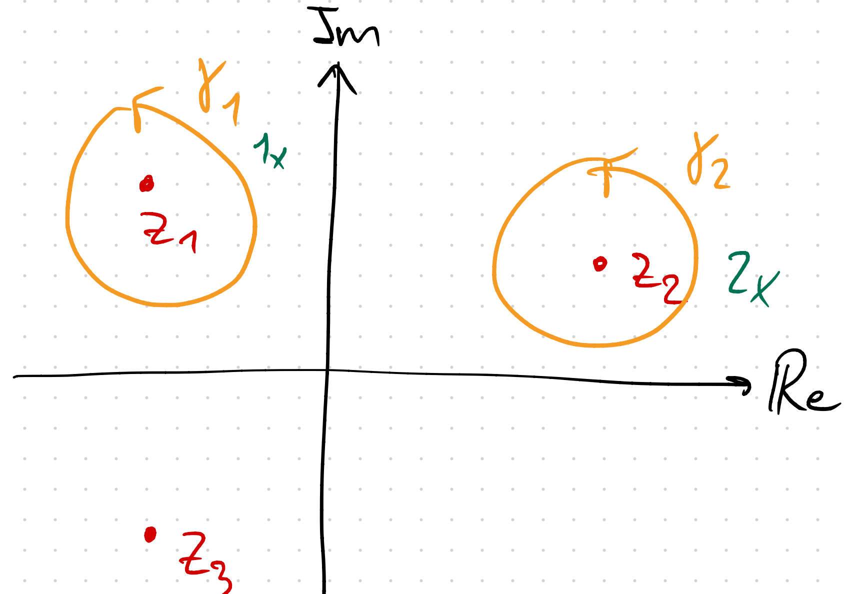
Residuensatz

$f: U \subset \mathbb{C} \to \mathbb{C}$, f holomorph

• Singularität von f

→ Was wir schon wissen





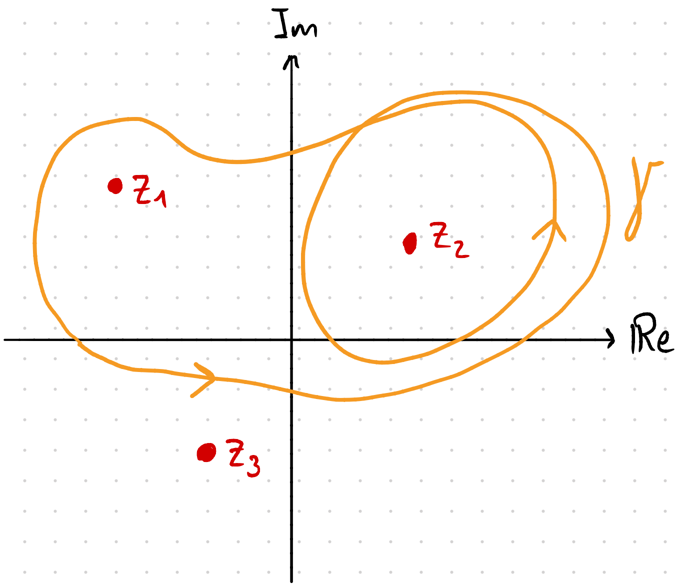
$$\int_{\gamma} f(z)\,dz = 1\int_{\gamma_1} f(z)\,dz + 2\int_{\gamma_2} f(z)\,dz$$

Residuensatz

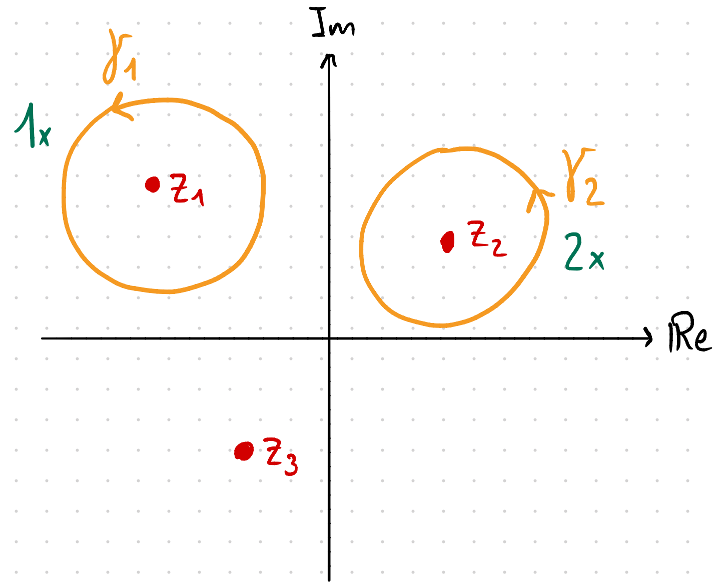
$f: U \subset \mathbb{C} \to \mathbb{C}$, f holomorph

• Singularität von f

→ Was wir schon wissen



$\xrightarrow[\text{Invarianz}]{\text{Homotopie}}$



$$\int_\gamma f(z)\,dz = \sum_k \mathrm{Ind}_{\gamma_k}(z_k) \int_{\gamma_k} f(z)\,dz$$

$\gamma_k \longrightarrow 1\times$ in Math. pos. Richtung

Residuensatz

$f: U \subset \mathbb{C} \to \mathbb{C}$, f holomorph

- Singularität von f

0, sonst

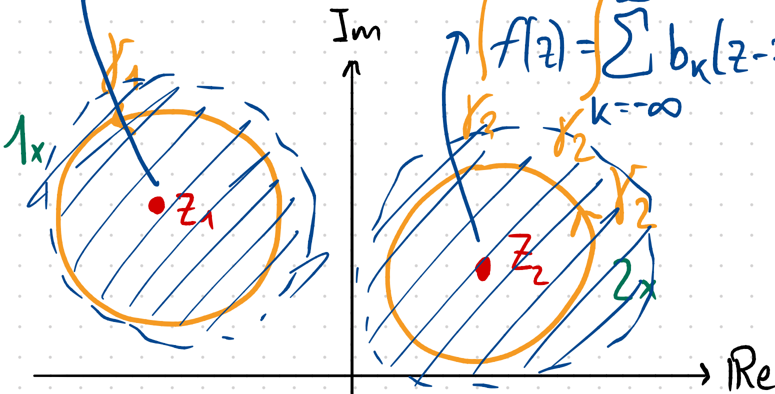
1. Laurententwicklung

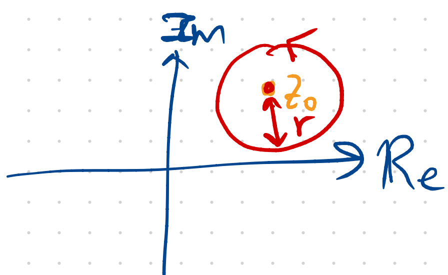
$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z-z_0)^k, \quad |z-z_0| < \sigma$$

2. Integral von $(z-z_0)^k$

$$\int_{\gamma_1} = 2\pi i a_{-1}$$

$$\int_{\gamma_2} f(z) = \int_{\gamma_2} \sum_{k=-\infty}^{\infty} b_k (z-z_2)^k = 2\pi i b_{-1}$$





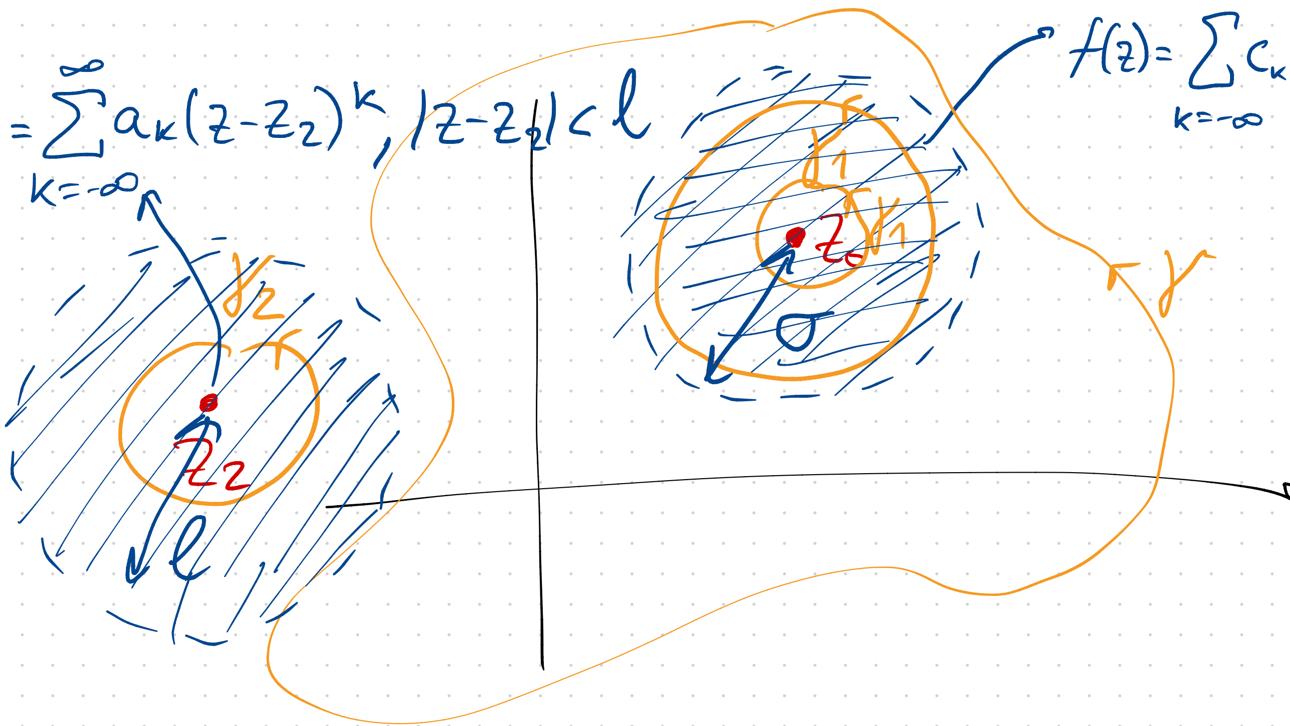
Serie 4 $\boxed{A4}$

$$\int_{|z-z_0|=r} (z-z_0)^k \, dz = \begin{cases} 2\pi i, & k=-1 \\ 0, & \text{sonst} \end{cases}$$

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_2)^k, \quad |z-z_2| < \ell$$

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z-z_0)^k$$

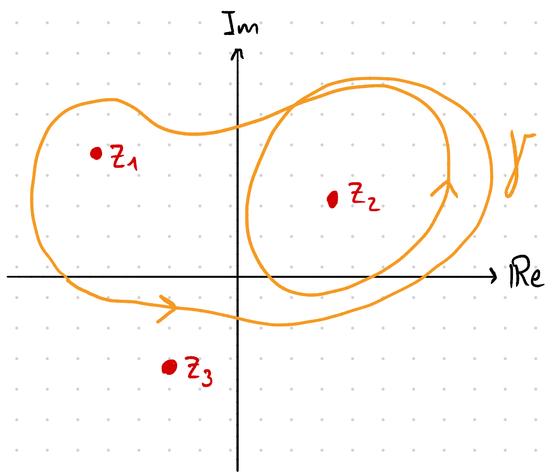
$$|z-z_0| < \sigma$$



Residuensatz

$f: U \subset \mathbb{C} \to \mathbb{C}$, f holomorph

- Singularität von f

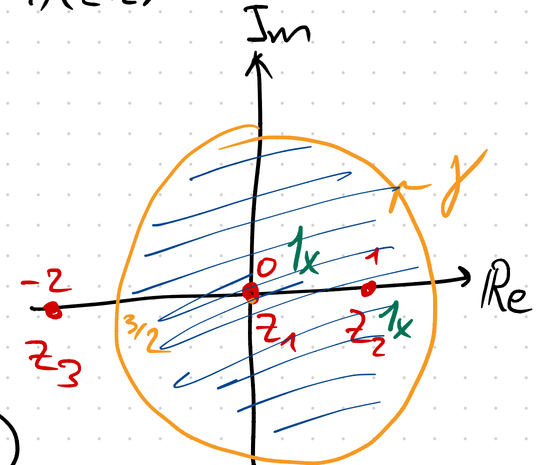


$$\boxed{\int_\gamma f(z)\,dz = 2\pi i \sum_k \mathrm{Ind}_\gamma(z_k) \cdot \mathrm{Res}(f \mid z_k)}$$

$\forall$ Sing $\in A(\gamma)$

C_{-1} von der LE mit EP $z = z_k$

Beispiel: $\displaystyle\int_{|z|=\frac{3}{2}} f(z)\,dz$ für $f(z) = \dfrac{e^z}{z(z-1)(z+2)}$



1. Singularitäten:

→ $z_1 = 0$ 1. Ord

→ $z_2 = 1$ 1. Ord

→ $z_3 = -2$ 1. Ord

$$\left[\lim_{z\to 0} z\, \frac{e^z}{z(z-1)(z+2)} = -\tfrac{1}{2}\right]$$

2. Residuensatz

$$\int_\gamma f(z)\,dz = 2\pi i \left(1\,\mathrm{Res}(f|z_1) + 1\,\mathrm{Res}(f|z_2) + \cancel{0\,\mathrm{Res}(f|z_3)}\right) = 2\pi i\left(-\tfrac{1}{2}\right) + 2\pi i\,\frac{e}{3}$$

$$= -\pi i + \frac{2}{3}\pi i e$$

→ $\mathrm{Res}(f|z_1) = \lim\limits_{z\to 0} z\, \dfrac{e^z}{z(z-1)(z+2)} = -\tfrac{1}{2}$

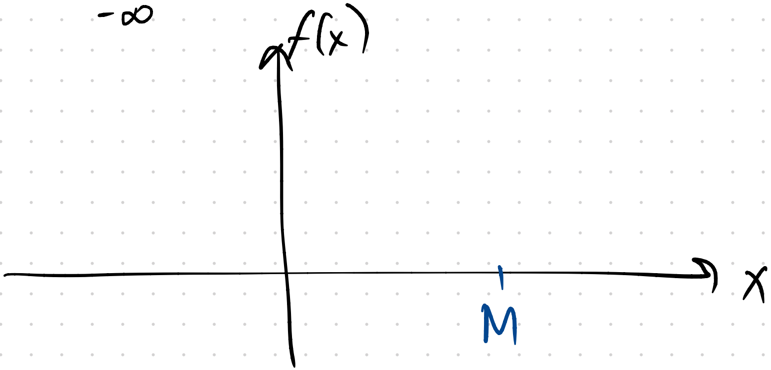
→ $\mathrm{Res}(f|z_2) = \lim\limits_{z\to 1} \cancel{(z-1)} \cdot \dfrac{\underbrace{e^z}_{\to e}}{\underbrace{z}_{1}\cancel{(z-1)}\underbrace{(z+2)}_{3}} = \dfrac{e}{3}$

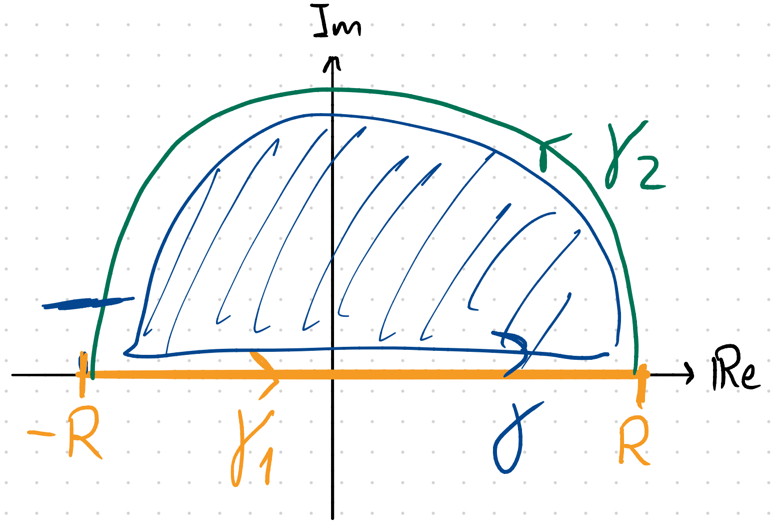
Uneigentliche Integrale [in $\mathbb{R}$]

$\rightarrow f: \mathbb{R} \rightarrow \mathbb{R}, \quad |f(x)| < cx^{-2} \; \forall \; |x| > M \qquad [c, M \in \mathbb{R}]$

↳ $f(x)$ wächst nicht schneller als cx^{-2}

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\, dx = ?$$





$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R \to \infty} \int_{\gamma_1} f(z)\, dz$$

$$\gamma := \gamma_1 * \gamma_2$$

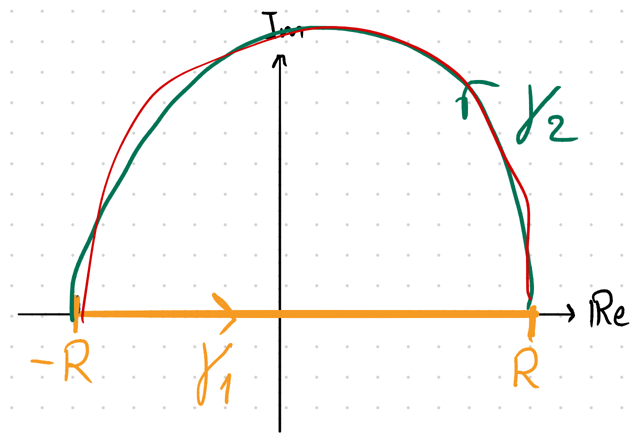
Uneigentliche Integrale [in $\mathbb{R}$]

$\rightarrow f: \mathbb{R} \rightarrow \mathbb{R}$, $|f(x)| < cx^{-2}$ $\forall\ |x| > M$ $\quad [c, M \in \mathbb{R}]$

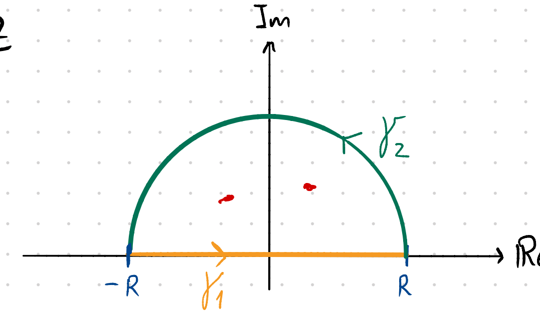
$$\gamma = \gamma_1 * \gamma_2$$

$$\int_{\gamma} f(z)\,dz = \lim_{R\to\infty} \int_{\gamma_1} f(z)\,dz + \lim_{R\to\infty} \int_{\gamma_2} f(z)\,dz$$

1 $\quad$ 2 $\quad$ 3



$$\int_{\gamma} f(x)\,dx = \int_{\gamma_1} f(x)\,dx + \int_{\gamma_2} f(z)\,dz$$



(1) $$\lim_{R\to\infty} \int_{\gamma} f(z)\,dz = 2\pi i \sum_{\substack{k \\ \mathrm{Im} \geq 0}} \operatorname{Res}(f|z_k)$$

$$\gamma_2(t) = Re^{\pi i t}, \quad t \in [0,1]$$

(2) $$\lim_{R\to\infty} \int_{\gamma_1} f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx$$

$$z_0 = 0$$

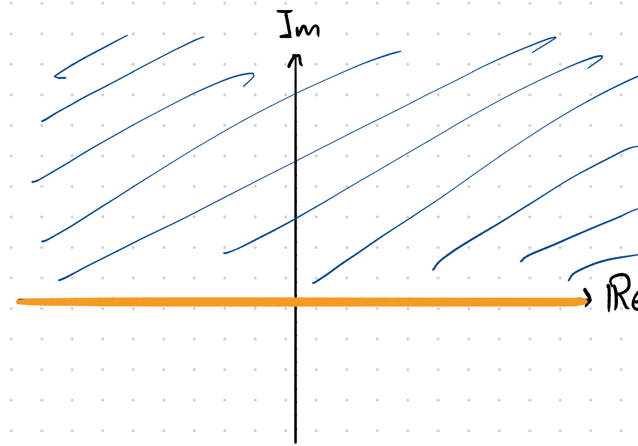
(3) $$\lim_{R\to\infty} \int_{\gamma_2} f(z)\,dz = \lim_{R\to\infty} \int_0^1 f(z_0 + Re^{\pi i t})\,\pi i Re^{\pi i t}\,dt$$

$$= \lim_{R\to\infty} \int_0^1 \underbrace{f(|R|e^{\pi i t}|)}_{R^{-2}}\, \pi i \underbrace{R}_{R} e^{\pi i t}\,dt \overset{\lim_{R\to\infty}}{\leq} \int_0^1 \frac{R}{R^2} = 0$$

Uneigentliche Integrale [in $\mathbb{R}$]

$\rightarrow f: \mathbb{R} \rightarrow \mathbb{R}, \ |f(x)| < cx^{-2} \ \forall \ |x| > M \qquad [c, M \in \mathbb{R}]$

$$\boxed{\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum_{\substack{k \\ \mathrm{Im} \geq 0}} \mathrm{Res}(f \mid z_k)}$$

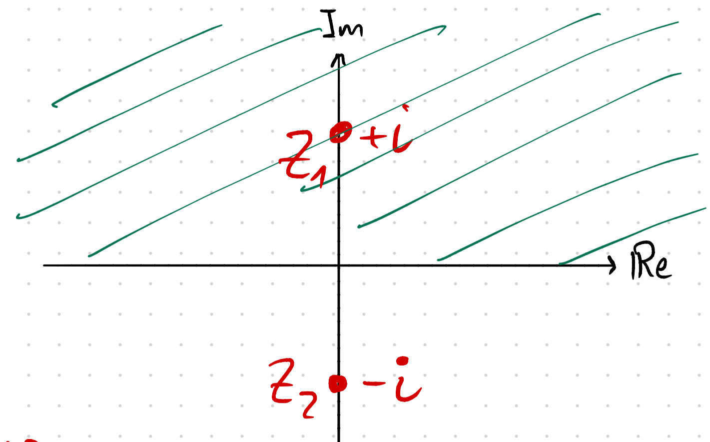


Beispiel: $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$

i. uneigentlich ⇝ $-\infty, \infty$ ✓

ii. $f(x)$ „schwächer" als x^{-2} ✓

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = 2\pi i \sum_{\substack{k \\ \boxed{\mathrm{Im}>0}}} \mathrm{Res}(f \mid z_k)$$



$$f(z) = \frac{1}{z^2+1} \longrightarrow z^2+1=0 \Rightarrow z = \pm i$$

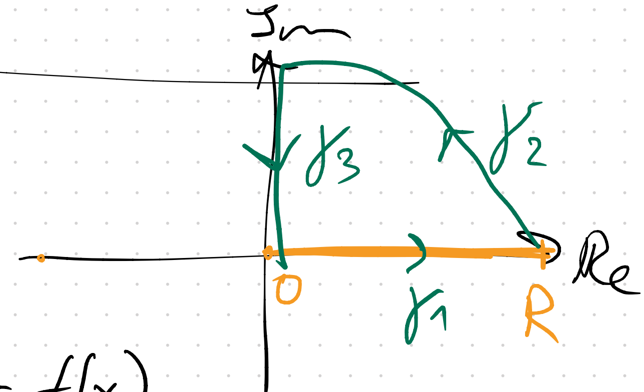
$$= \frac{1}{\underbrace{(z+i)}_{z_2}\underbrace{(z-i)}_{z_1}} \rightarrow \text{Pol 1. Ordnung}$$

$$\mathrm{Res}(f \mid z_1 = i) = \lim_{z \to i} \cancel{(z-i)} \frac{1}{(z+i)\cancel{(z-i)}} = \frac{1}{2i}$$

A 3b

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = 2\pi i \sum_{\substack{k \\ \mathrm{Im}>0}} \mathrm{Res}(f/z_k)$$

$$= \cancel{2}\pi \cancel{i} \frac{1}{\cancel{2i}} = \pi$$



$$\int_0^{\infty} \underbrace{\frac{1}{x^2+1}}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$$

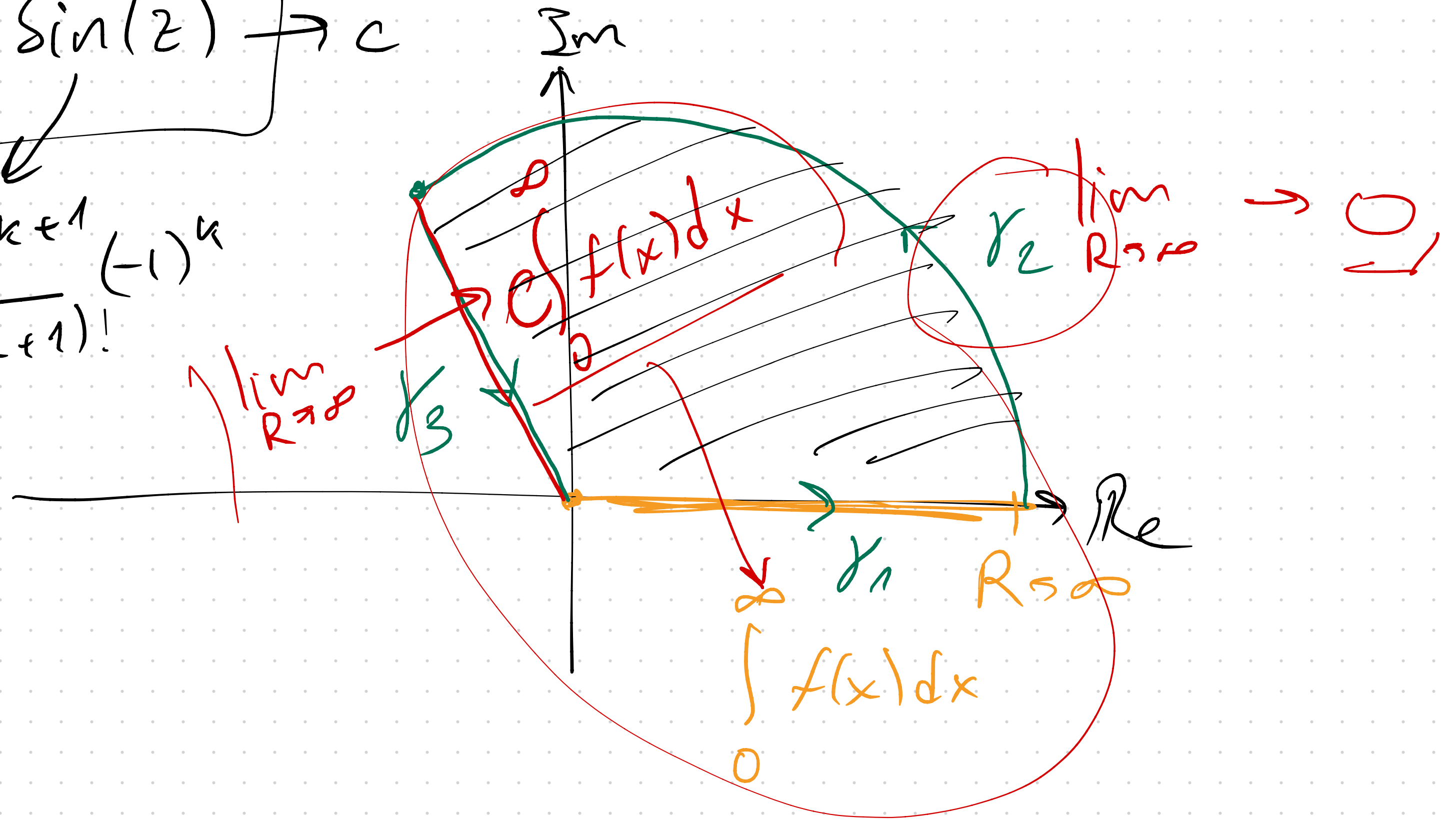
↳ $f(x)$ ist gerade $\Rightarrow$ $f(-x) = f(x)$

$$\Rightarrow \boxed{\int_{-a}^{a} f(x)\,dx} = 2\int_{-a}^{0} f(x)\,dx = \boxed{2\int_0^{a} f(x)\,dx}$$

A 3a

$$\boxed{\lim_{z\to\infty} \sin(z)} \to c$$

$$\sum \frac{z^{2k+1}}{(2k+1)!}(-1)^k$$



2.– Konvergenzradius

→ Zum wissen wann/wo man eine gegebene Reihendarstellung verwe
kann ist es nützlich den Konvergenzradius zu berechnen. Er ist gege
durch:

$$\boxed{\rho = \lim_{k\to\infty} \left|\frac{a_k}{a_{k+1}}\right| = \lim_{k\to\infty} \frac{1}{\sqrt[k]{a_k}}}$$

Serie 6

Beispiel 2.2: $f(z) = \sum_{n=0}^{\infty} (-1)^n \cdot \left(\frac{3}{2}\right)^n \cdot z^n$. Für welche z ist die Potenzreihe definiert?

⇝ Wir suchen den Konvergenzradius von $f(z)$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n = \sum_{n=0}^{\infty} (-1)^n \left(\frac{3}{2}\right)^n z^n$$

$$\lim_{z \to 0} \cancel{z^3} \frac{\sinh(z) - z}{\cancel{z^3}} = \lim_{z \to 0} \sinh(z) - z = 0$$

$z^2 \cdot \sin = 0 \quad z = 0$

hebbar

$$\frac{1}{z^2}$$

$$\lim_{z \to z_0} (z - z_0)^{m} f(z) \neq 0$$

$$\lim_{z \to z_0} (z - z_0)^{m-1} f(z) \to \pm\infty$$

$$\lim_{z \to z_0} (z - z_0)^{m+1} f(z) = 0$$

Ord 1? $\lim_{z \to 0} z \cdot \frac{1}{z^2} \to \infty$

Ord 2? $\lim_{z \to 0} z^2 \cdot \frac{1}{z^2} = 1 \neq 0$

Ord 3? $\lim_{z \to 0} z^3 \cdot \frac{1}{z^2} = z = 0$

1. trig. Fkt. Arg $\frac{1}{z}$ $\cos(1/z)$, $e^{1/z}$

$$\sum_{k=0}^{\infty} \cdot \frac{1}{z^k}$$

$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!} \frac{1}{z^{2k}}$$

$c_{10000} \neq 0$

$\Rightarrow$ Laurententwicklung

$$f(z) = \boxed{\sum_{k=-\infty}^{\infty} c_k (z-z_0)^k}$$

Pol 2. Ordnung

$c_k = 0$ für $k < -2$ $\quad$ $c_k \neq 0$ für $\forall k < 0$

$c_{-3} = 0$

$c_{-4} = 0$

$$f(z) = \frac{1}{z^2} + \frac{1}{z} + 2 + 3z + 1z^2 + \dots z^{k>0}$$

c_{-2}, c_1, c_0, c_1

$$e^{1/z} = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \frac{1}{z^k} = \sum_{k=-\infty}^{0} \frac{1}{k!} \cdot z^k$$

$$\downarrow$$

$$1/z = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot z^{-k} \rightsquigarrow \sum_{m=-\infty}^{0} \frac{1}{(-m)!} z^m$$

$m = -k$

$k = 0 \Rightarrow m = 0$

$k = \infty \Rightarrow m = -\infty$

$$\lim_{z \to z_0} (z - z_0)^m f(z) \neq 0$$

$m \to \infty$

$$\sum_{k=0}^{\infty} (-1)^k \frac{z^{2k-2}}{(2k-2)!}$$

$k \longrightarrow m$ Faktor

$k = 0 \quad 1 \quad 2 \quad 3 \longleftrightarrow 0 \quad 1 \quad 2 \quad 3$

$2k \to 0 \quad 2 \quad 4 \quad 6$

$m = 2k$

$0 \quad 1 \quad 2 \quad 3$

$$\frac{1}{z^2} \sum_{k=0}^{\infty} \frac{z^k}{k!} = \sum_{k=0}^{\infty} \frac{z^{k-2}}{k!}$$