PVK

Komplexe Analysis

FS 2020

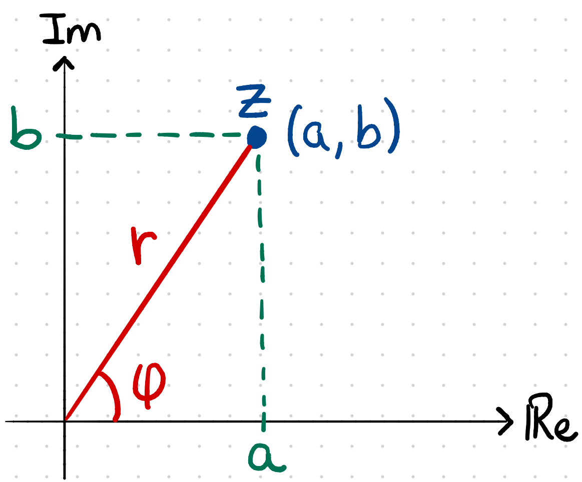
Eine kleine

Einführung

Eine komplexe Zahl

→ Alle komplexe Zahlen lassen sich mit *Realteil* und *Imaginärteil* beschreiben

$$z = a + bi \implies \left.\begin{array}{l} a = \mathrm{Re}\{z\} = \text{Realteil} \\ b = \mathrm{Im}\{z\} = \text{Imaginärteil} \end{array}\right\} a, b \in \mathbb{R}\,!$$



Kartesische Form

$$z = a + bi$$

Polarform

$$z = \underbrace{r}_{r = |z| = \text{Betrag}} e^{i\varphi} = r[\cos(\underbrace{\varphi}_{\varphi = \arg(z) = \text{Argument}}) + i\sin(\varphi)]$$

	Kart. Form	Polarform
Realteil Re	a	$r\cos(\varphi)$
Imaginärteil Im	b	$r\sin(\varphi)$
Betrag $\lvert z\rvert$	$\sqrt{a^2+b^2}$	r
Argument $\arg(z)$	$\tan^{-1}\left(\frac{b}{a}\right)$	φ

<u>Beispiel</u>: $z = -\sqrt{3} + i$

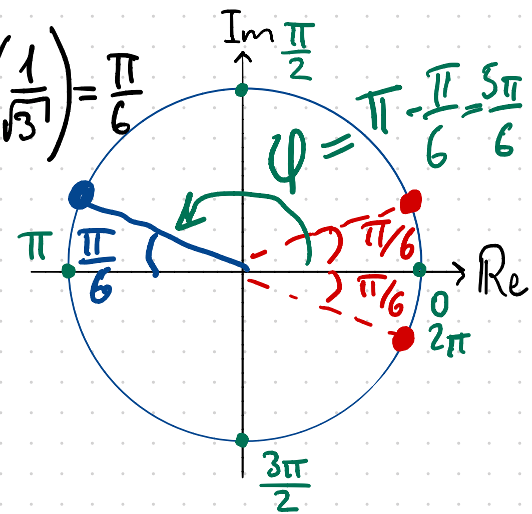
$$\Rightarrow r = \sqrt{3+1} = 2$$

$$\Rightarrow \varphi = \cancel{\tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) = -\frac{\pi}{6}}$$

$$\tan^{-1}\left(\left|\frac{1}{-\sqrt{3}}\right|\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

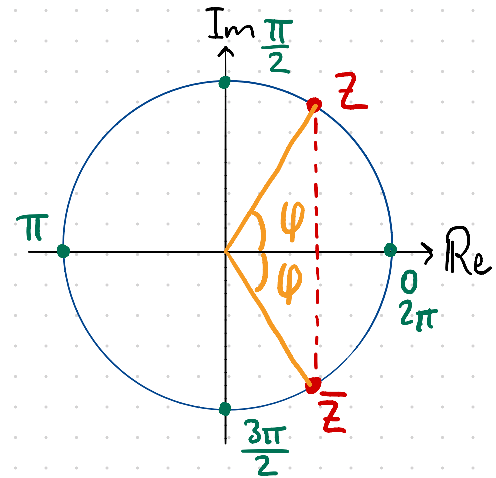
$$\varphi = \frac{5\pi}{6}$$

$$\boxed{z = 2\cdot e^{i\cdot\frac{5\pi}{6}}}$$



→ Eigenschaften

- $z + \overline{z} = 2a = 2\,\mathrm{Re}\{z\}$
- $z - \overline{z} = 2bi = 2i\,\mathrm{Im}\{z\}$
- $z \cdot \overline{z} = a^2 + b^2 = |z|^2$
- $\overline{z} = re^{-i\varphi}$



→ Moivrescher Satz

- 2π-Periodizität: $e^{i\varphi + 2\pi i k} = e^{i\varphi}, \quad k \in \mathbb{Z}$

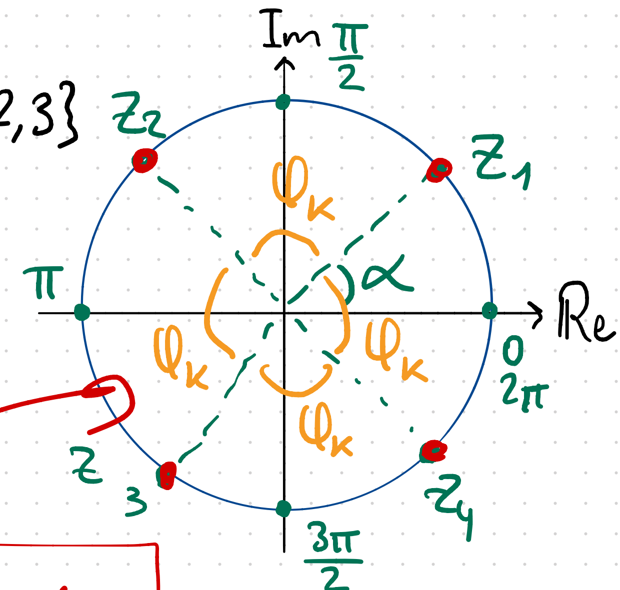
$$z^n = a = re^{i\varphi} \implies \boxed{z = r^{1/n} e^{i\left(\frac{\varphi + 2\pi k}{n}\right)}}, \quad k = \{0, 1, \dots, n-1\}$$

Beispiel: Finde alle Singularitäten von $f(z) = \dfrac{1}{z^4+1}$

FS 2016

$$(z^4)^{1/4} = -1 = 1e^{i\pi} = \left(e^{i\pi + 2\pi i k}\right)^{1/4}, \quad k \in \mathbb{Z}$$

$$= e^{\underbrace{\boxed{\frac{i\pi}{4}}}_{\alpha} + \underbrace{\boxed{\frac{\pi}{2} i k}}_{\varphi_k}}, \quad k \in \mathbb{Z} \qquad k = \{0, 1, 2, 3\}$$



$$z^n = a$$

$$\boxed{z^4 = \frac{1}{-64}}$$

$$\boxed{z^4 = \frac{1}{+64}}$$

$$\boxed{z^8 = 1 \ldots}$$

Funktionen

Funktionen in $\mathbb{C}$

$$\rightarrow \; f: \mathbb{C} \to \mathbb{C} \quad \approx \quad \mathbb{R}^2 \to \mathbb{R}^2$$

$\mathbb{R}e, \mathbb{I}m$ $\quad$ $\mathbb{R}e, \mathbb{I}m$

$$z \longmapsto f(z) = f(x+yi) = \tilde{f}(x,y)$$

$[z = x+yi]$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$$

$\to \mathbb{R}e\{f(z)\}$

$\to \mathbb{I}m\{f(z)\}$

Beispiel: $f(z) = z^2 = (x+yi)^2 = \underbrace{x^2-y^2}_{\boxed{\mathbb{R}e\{f\}}\ u} + \underbrace{2xy}_{\boxed{\mathbb{I}m\{f\}}\ v}i$

$z = x+yi$

$$u(x,y) = x^2 - y^2$$
$$v(x,y) = 2xy$$

$$u, v: \mathbb{R}^2 \to \mathbb{R}$$

→ Cauchy-Riemannsche Gleichungen

Variante 1

$$\boxed{\underline{i\frac{\partial}{\partial x} f(x+yi)} \overset{!}{=} \frac{\partial}{\partial y} f(x+yi)}$$

$$\left[\, i f_x = f_y \,\right]$$

$$\frac{\partial}{\partial x} f(x+yi) = \frac{\partial}{\partial x} u(x,y) + \frac{\partial}{\partial x} i\, v(x,y)$$

Variante 2

$$\boxed{\begin{aligned} \frac{\partial}{\partial x} u(x,y) &\overset{!}{=} \frac{\partial}{\partial y} v(x,y) \\ \frac{\partial}{\partial y} u(x,y) &\overset{!}{=} -\frac{\partial}{\partial x} v(x,y) \end{aligned}}$$

$$\left[\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}\right]$$

⚠ holomorph = komplex differenzierbar = analytisch

Beispiel: $f(z) = z + 3\bar{z}$ → Ist f holomorph?

$$f(x+yi) = (x+yi) + 3(x-yi)$$
$$= \boxed{4x - 2yi}$$

u, v

Variante 1

$$\underbrace{i\frac{\partial}{\partial x} f(x+yi)}_{①} = \underbrace{\frac{\partial}{\partial y} f(x+yi)}_{②}$$

① $i\frac{\partial}{\partial x} f(x+yi) = i\frac{\partial}{\partial x}(4x - 2yi) = \boxed{i \cdot 4}$

② $\frac{\partial}{\partial y} f(x+yi) = \boxed{-2i}$

=?

f ist nicht holomorph

Variante 2

$$\boxed{u_x = v_y} \;\&\; u_y = -v_x$$

$$u(x,y) = 4x, \quad v(x,y) = -2y$$

$\boxed{u_x = 4}$ $\overset{?}{=}$ $\boxed{v_x = 0}$

$\boxed{u_y = 0}$ $\overset{?}{=}$ $\boxed{v_y = -2}$

Nein ⇒ f ist nicht holomorph

$\left[i f_x = f_y\right]$ oder $\begin{bmatrix} u_x = v_y \\ u_y = -v_x \end{bmatrix}$

Prüfung

→ Ist f holomorph?

↳ CR-Gleichungen

→ $f(x+yi, \alpha)$. Für welche $\underline{\underline{\alpha \in \mathbb{R}}}$ ist f holomorph?

Bsp.: $f(z) = \dfrac{z + 3\alpha\bar{z}}{z\bar{z}}$ $\quad \boxed{\alpha \in \mathbb{R}}$

↳ In CR-Gleichungen einsetzen ✓

Bis jetzt:

1. Komplexe Zahlen
   - → Kartesische Form $\rightleftarrows$ Polarform
   - → De Moivre
2. Funktionen
   - → CR-Gleichungen

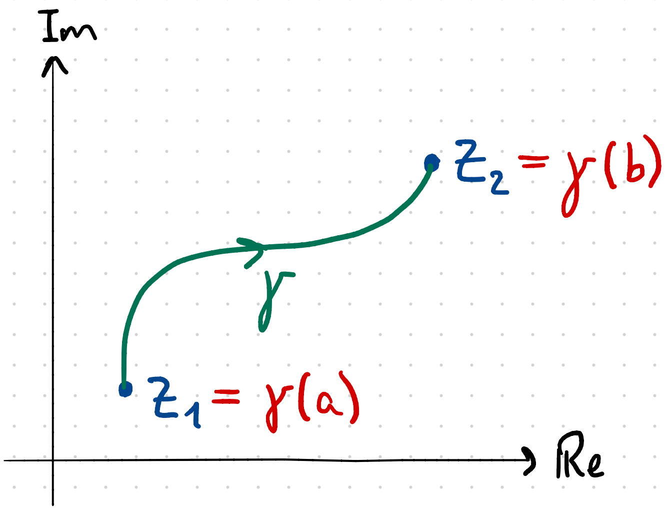
$$f(z) = \frac{\overline{z}}{|z|^2} = \frac{\overline{z}}{z \cdot \overline{z}} = \frac{1}{z}$$

Integrale

Kurvenintegrale

→ Interpretation ⟹ Extras

→ Parametrisierung

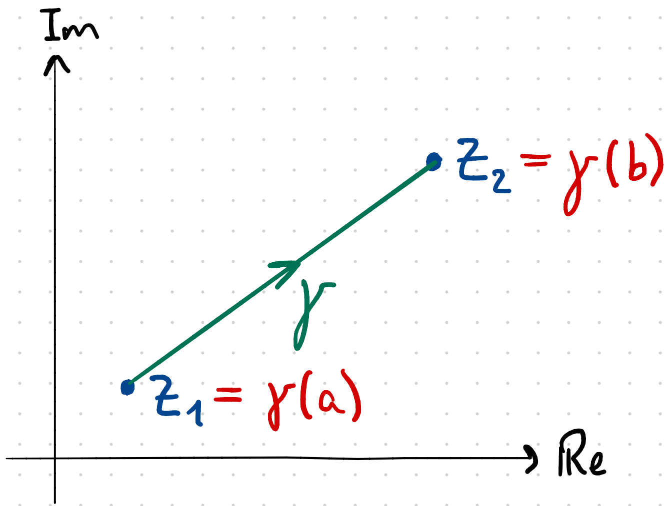


[alle z auf γ → $z = \gamma(t)$]

$\gamma : \mathbb{R} \to \mathbb{C},\ t \mapsto \gamma(t),\ t \in [a,b]$

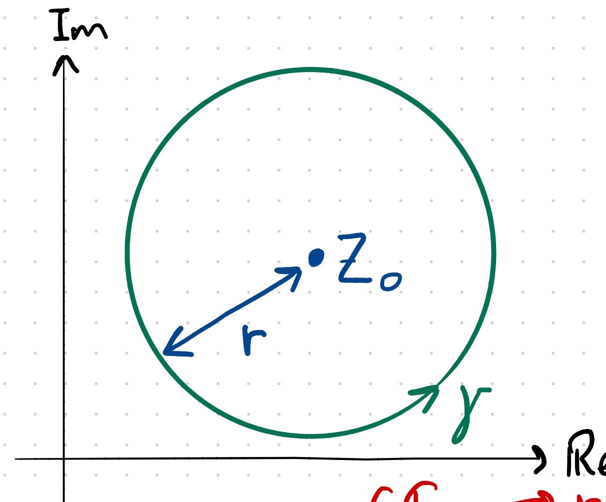
$$\boxed{\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t)) \cdot \dot{\gamma}(t)\, dt}$$

Gerade



$$\gamma(t) = z_2 t + (1-t) z_1$$
$$t \in [0, 1]$$

Kreis



$\in \mathbb{C}$ $\quad r \in \mathbb{R}$

$$\gamma(t) = z_0 + r e^{it}$$

$t \in \mathbb{R}$

$$t \in [0, 2\pi]$$

$$z_0 + r e^{2\pi i t}, \quad t \in [0, 1]$$

Beispiel: Berechne $\int\limits_{\boxed{|z|=1}} \frac{1}{z}\, dz$

↳ Kreis mit Mittelpunkt $(0,0)$ und Radius $r=1$

$$\gamma(t) = z_0 + re^{it}, \quad t \in [0, 2\pi]$$

$$z_0 = 0 + 0i = 0$$

$$\gamma(t) = e^{it}, \quad t \in [0, 2\pi] \qquad \dot{\gamma}(t) = \frac{\partial}{\partial t}\left(e^{it}\right) = ie^{it}$$

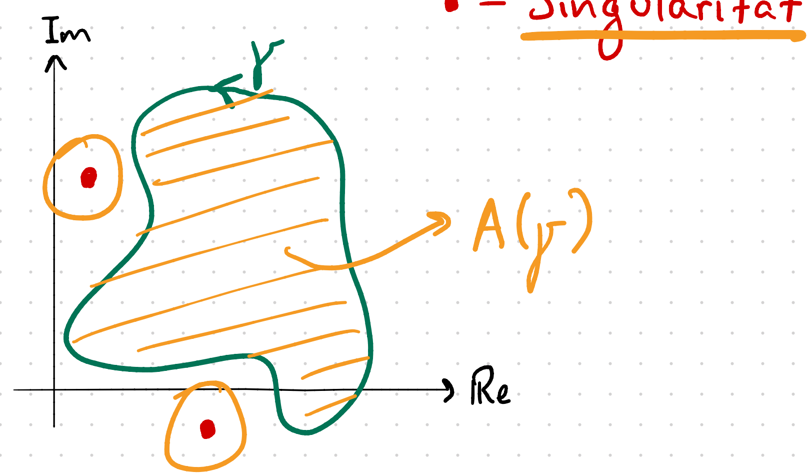
$$\int\limits_{|z|=1} \frac{1}{z}\, dz = \int_0^{2\pi} \frac{1}{\cancel{e^{it}}} \cdot i\cancel{e^{it}}\, dt = i\int_0^{2\pi} 1\, dt = 2\pi i$$

γ geschlossen, f holomorph

Satz von Cauchy

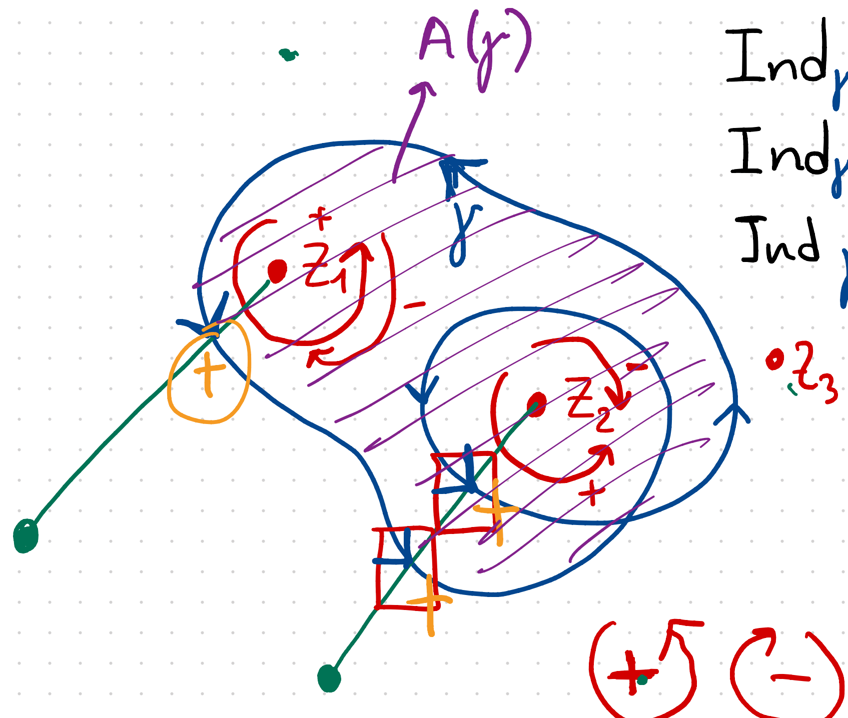
→ γ geschlossen, f holomorph. Falls es keine Singularität innerhalb der von γ eingeschlossene Fläche gibt, so ist $\int_\gamma f(z) = 0$

$$\boxed{\int_\gamma f(z)\,dz = 0}$$



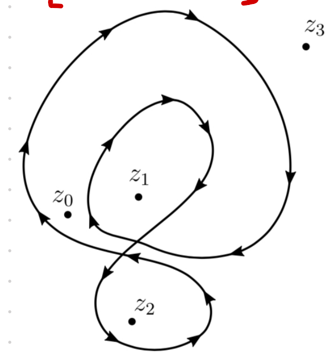
Umlaufzahl → $\mathrm{Ind}_\gamma(z_k)$

→ Sagt uns wie oft eine Singularität von einer Kurve γ umgelaufen wird

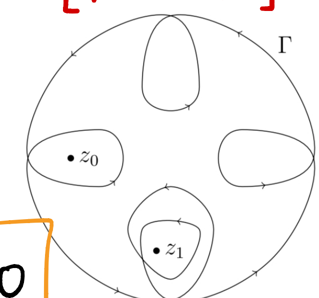


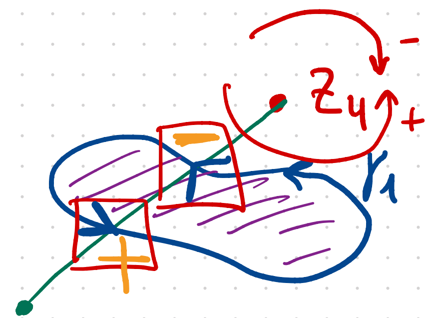
$$\mathrm{Ind}_\gamma(z_1) = 1$$
$$\mathrm{Ind}_\gamma(z_2) = 2$$
$$\mathrm{Ind}_\gamma(z_3) = 0$$

[FS2017]



[FS2016]

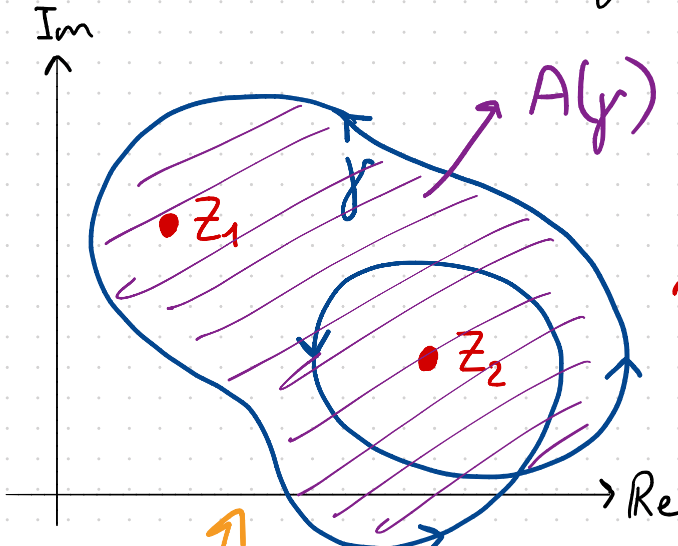


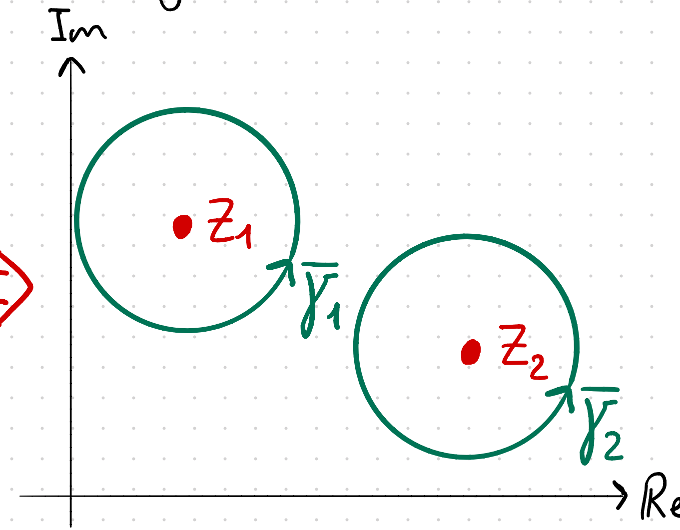


$$\boxed{\mathrm{Ind}_{\gamma_1}(z_4) = 0}$$

Homotopie Invarianz

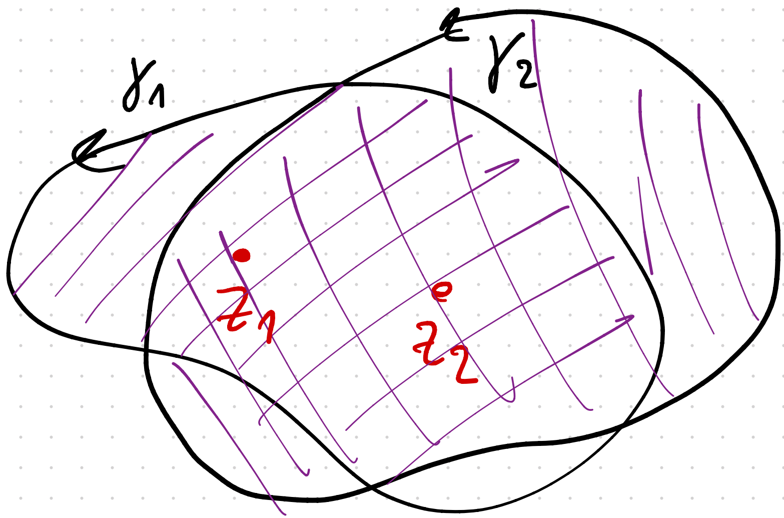
→ Homotopie Invarianz ermöglicht uns die Singularitäten zu isolieren





$$\boxed{\int_\gamma f(z)\,dz} = \sum_k^N \mathrm{Ind}_\gamma(z_k) \int_{\overline{\gamma}_k} f(z)\,dz$$

$\overline{\gamma}_k$ ↳ 1x in math pos. Richtung



$$\boxed{\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz}$$

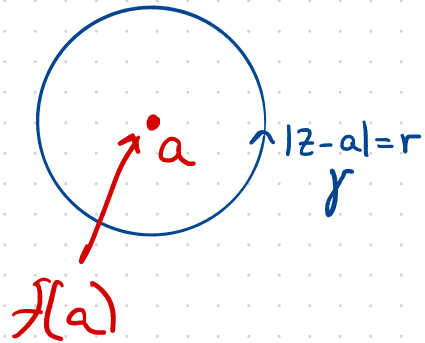
$$H(t) = t\gamma_1 + (1-t)\gamma_2$$

Mittelwertsatz

→ $f(a)$ = Mittelwert der Abbildungen am Rand (Kreis mit $r=1$)

$$\boxed{f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{it} + a)\, dt}$$

→ $r = 1$



$$f(a) = \frac{1}{2\pi} \int_\gamma f(z)\, dz$$

Maximumsprinzip

→ Realteil bzw Imaginärteil einer holomorphen Funktion besitzt keine *lokale* Maxima oder Minima, ausser wenn sie konstant ist

$$f(a) = \frac{1}{2\pi} \lim_{r \to 0} \int_0^{2\pi} f(\cancel{re^{it}}^{\,0} + a)\, dt = \frac{1}{2\pi} \int_0^{2\pi} f(a)\, dt = \frac{1}{\cancel{2\pi}} \cdot f(a) \cdot \cancel{2\pi} = f(a)$$

Integralformel von Cauchy

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{\boxed{z-z_0}} dz$$

$$\gamma(t) = z_0 + re^{it}, \quad t \in [0, 2\pi]$$

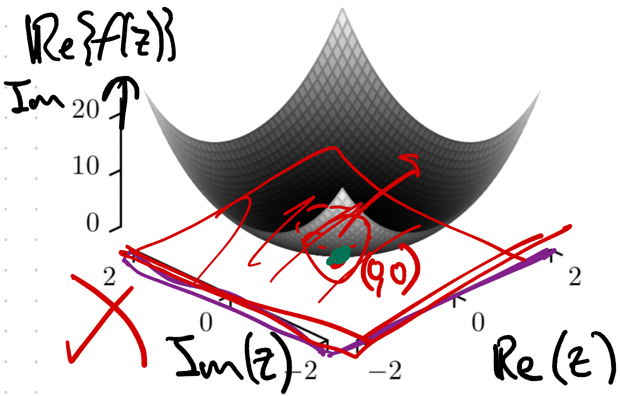
$$\dot{\gamma}(t) = rie^{it}$$

$$f(z_0) = \frac{1}{2\pi \cancel{i}} \int_0^{2\pi} \frac{f(z_0 + re^{it})}{\cancel{z_0} + \cancel{re^{it}} - \cancel{z_0}} \cdot \cancel{rie^{it}}\, dt$$

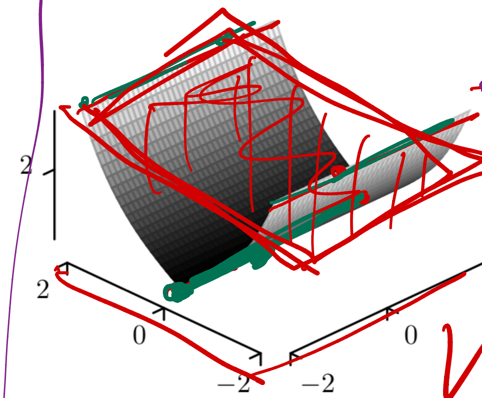
$$\boxed{= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt} \rightarrow \text{Mittelwertsatz}$$

Beispiel:

[FS 2019]



i.



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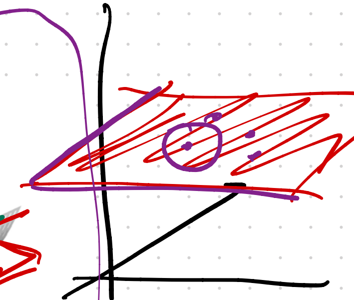

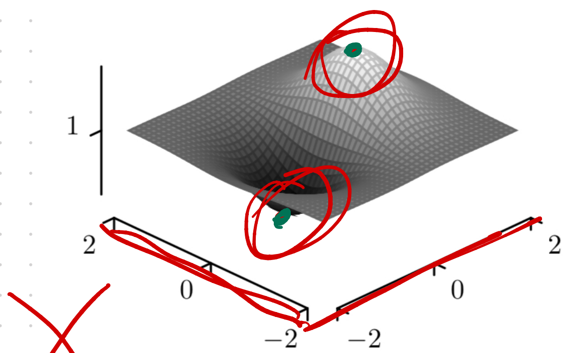

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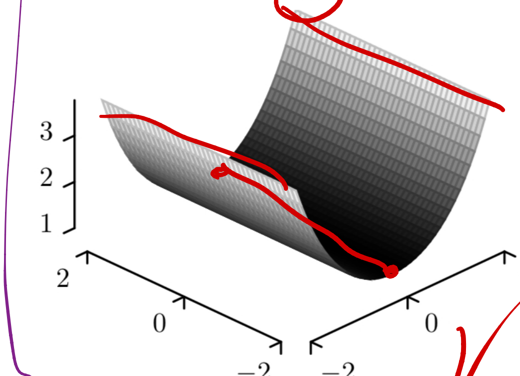


iv.

Residuensatz

Reihen

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z - z_0)^k$$

Taylor

sin, cos, exp, log

Konvergenzradius ∞

Geometrische Reihe

PBZ

$$\frac{1}{(z-a)(z-b)} = \boxed{\frac{A}{z-a}} + \boxed{\frac{B}{z-b}}$$

$$\boxed{\frac{1}{1-q}} = \sum_{\boxed{k=0}}^{\infty} q^k , \quad \boxed{|q| < 1} = \sigma$$

$\dots z$

Konvergenzradius

Beispiel: $f(z) = \dfrac{1}{z^2 - 5z + 6}$, $z_0 = 0$ [FS 2018]

$$z = \frac{-(-5) \pm \sqrt{25-24}}{2} \Rightarrow \begin{array}{l} z_1 = 2 \\ z_2 = 3 \end{array}$$

$$f(z) = \frac{1}{(z-2)(z-3)} = \frac{A}{z-2} + \frac{B}{z-3}$$

$$A(z-3) + B(z-2) = 1$$

$$\begin{array}{lccccl} z=3 & 0 & + & B & = 1 & \Rightarrow \boxed{B=1} \\ z=2 & -A & + & 0 & = 1 & \Rightarrow \boxed{A=-1} \end{array}$$

$$f(z) = -\frac{1}{z-2} + \frac{1}{z-3}$$

$$f(z) = \underbrace{-\frac{1}{z-2}}_{1} + \underbrace{\frac{1}{z-3}}_{2} \qquad \frac{1}{1-z} = \sum_{k=0}^{\infty} z^k, \quad |z|<1$$

1 $$-\frac{1}{z-2} = -\frac{\frac{1}{2}}{\frac{z}{2}-1} = \frac{1}{2}\cdot \boxed{\frac{1}{1-\boxed{\frac{z}{2}}}} = \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^k, \quad \boxed{|z|<2} = \sigma_1$$

GR

$|\cdot|<1$

$$\left|\frac{z}{2}\right|<1 \Rightarrow |z|<2$$

$$\boxed{\sum_{k=0}^{\infty}\frac{z^k}{2^{k+1}}}$$

2 $$\frac{1}{z-3} = \frac{\frac{1}{3}}{\frac{z}{3}-1} = -\frac{1}{3}\boxed{\frac{1}{1-\boxed{\frac{z}{3}}}} = -\frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{z}{3}\right)^k, \quad \boxed{|z|<3} = \sigma_2$$

GR

$|\cdot|<1$

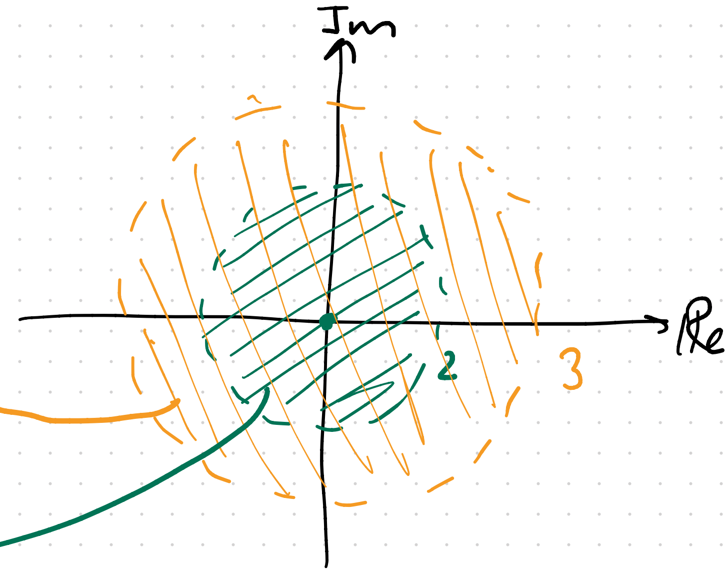
$$\left|\frac{z}{3}\right|<1 \Rightarrow |z|<3$$

$$\boxed{-\sum_{k=0}^{\infty}\frac{z^k}{3^{k+1}}}$$

$$f(z) = -\frac{1}{z-2} + \frac{1}{z-3}$$

$$= \boxed{\sum_{k=0}^{\infty} \frac{z^k}{2^{k+1}}} - \boxed{\sum_{k=0}^{\infty} \frac{z^k}{3^{k+1}}}$$

$\sigma_1 = 2$ $\qquad$ $\sigma_2 = 3$



wo konvergiert Σ_1 und Σ_2? Bei $|z| < 2 = \sigma_2$

$\min(\sigma_2, \sigma_3)$

$$\frac{1}{1-\dots z} = \dots \frac{1}{1-\boxed{\frac{\dots}{z}}} = \sum_{k=0}^{\infty} \left(\frac{\dots}{z}\right)^k,$$

$$\left|\frac{\dots}{z}\right| < 1 \Rightarrow \boxed{|z| > \dots}$$

Residuum

$\rightarrow c_{-1}$ von $f(z) = \sum_{k=-\infty}^{\infty} c_k (z-z_0)^k$ $\rightarrow$ also das was $\frac{1}{z-z_0}$ multipliziert

$\rightarrow$ Singularität z_0

i. Pol 1. Ordnung: 2 Methoden

1. $\boxed{\operatorname{Res}(f|z_i) = \lim_{z \to z_i} (z-z_i) f(z)}$

2. $f(z) = \frac{h(z)}{g(z)}$ mit $h(z_i) \neq 0$

$\Rightarrow \boxed{\operatorname{Res}(f|z_i) = \frac{h(z_i)}{g'(z_i)}}$

ii. Pol m-te Ordnung:

iii. Wesentliche Singularität:

Residuum

$\rightarrow c_{-1}$ von $f(z) = \sum_{k=-\infty}^{\infty} c_k (z-z_0)^k \rightarrow$ also das was $\frac{1}{z-z_0}$ multipliziert

i. Pol 1. Ordnung: 2 Methoden

1. $\boxed{\mathrm{Res}(f|z_i) = \lim_{z \to z_i} (z-z_i) f(z)}$

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ii. Pol m-te Ordnung:

$$\boxed{\mathrm{Res}(f|z_i) = \frac{1}{(m-1)!} \lim_{z \to z_i} \frac{\partial^{m-1}}{\partial z^{m-1}} \left[(z-z_i)^m f(z) \right]}$$

iii. Wesentliche Singularität:

Residuum

→ c_{-1} von $f(z) = \sum_{k=-\infty}^{\infty} c_k (z-z_0)^k$ → also das was $\frac{1}{z-z_0}$ multipliziert

i. Pol 1. Ordnung: 2 Methoden

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iii. Wesentliche Singularität: Laurententwicklung

→ Wesentlich: Generell exp, cos, sin, log mit Argument $\approx \frac{1}{z^a}$

Bsp.: $e^{\frac{1}{z-3}} \Rightarrow$ ωE mit EP $z=3$

$a=1$

Bsp.: $e^{1/z} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{z}\right)^k = \sum_{k=-\infty}^{0} \frac{1}{(-k)!} z^k \Rightarrow z=0$, Wesentlich

$z=0 \Longrightarrow EP=0$

$[EP\ z=0]\ (-k)! = k!$

→ Polynom / trigonometrische Funktion im Nenner

Bsp.: $\frac{1}{(z-3)^1(z-2)^2}$

$z=3$ Ordn. 1

$z=2$ Ordn. 2

Bsp.: $\frac{1}{z\sin(z)}$

sin(z) hat NS 1. Grades

$\Rightarrow \frac{1}{\sin(z)}$ hat POLE Ordn. 1

$\boxed{z=\pi\cdot k}$, $k\in\mathbb{Z}$

$\Rightarrow k=0 \Rightarrow z=0$ Pol / Ordn $\boxed{2}$

$z=0$

$\frac{1}{z} = \frac{1}{z}$ Ordn. 1

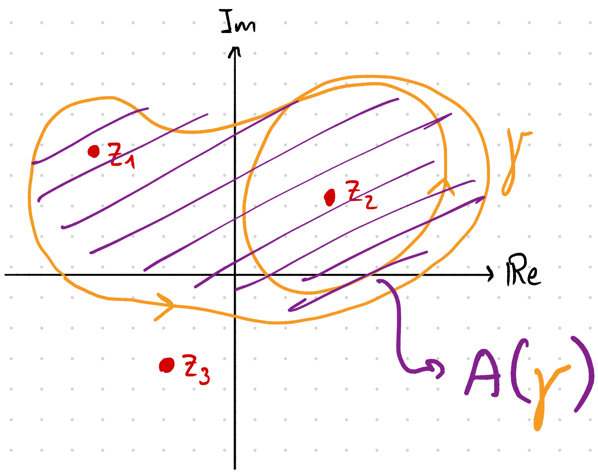
$\frac{1}{z\cdot z}$ Ordn 1

$\left.\begin{array}{l}\sin(z)\ \text{Pol 1 Ordn. bei } z=0 \\ \frac{1}{z}\ \text{Pol 1 Ordn. bei } z=0\end{array}\right\} \frac{1}{z\cdot\sin(z)} \Rightarrow$ Pol / Ordn $\boxed{2}$

Residuensatz

$f: U \subset \mathbb{C} \to \mathbb{C}$, f holomorph

- Singularität von f

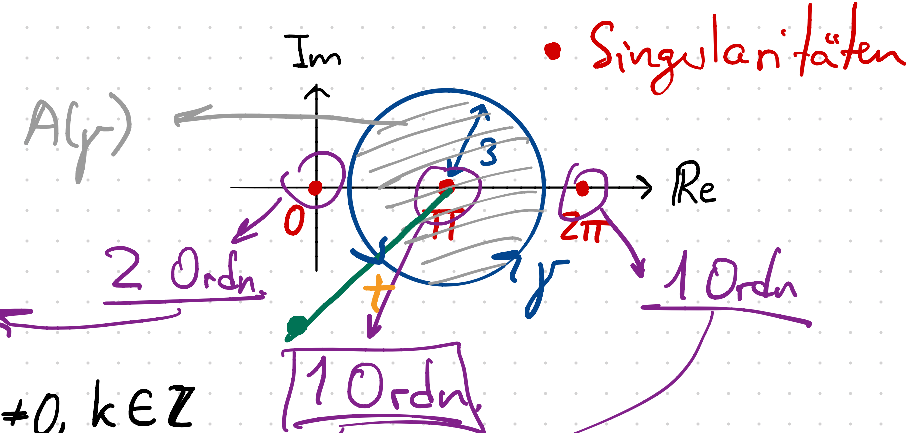


$$\boxed{\int_\gamma f(z)\,dz = 2\pi i \sum_k \mathrm{Ind}_\gamma(z_k) \cdot \mathrm{Res}(f \mid z_k)}$$

$\forall$ Sing $\in A(\gamma)$

C_{-1} von der LE mit EP $z = z_k$

Beispiel: $\displaystyle\int_{|z-\pi|=3} \frac{1}{z\sin(z)}\,dz$



→ Pol 2. Ordnung bei $z=0$

→ Pol 1. Ordnung bei $z=\pi k \neq 0,\ k\in\mathbb{Z}$

$$\int_{|z-\pi|=3} \frac{1}{z\sin(z)}\,dz = \underbrace{2\pi i \sum_k \mathrm{Res}(f|z_k)\cdot \mathrm{Ind}_\gamma(z_k)}_{z=\pi,\ \mathrm{Ind}_\gamma(\pi)=1} = 2\pi i\,\mathrm{Res}(f|\pi)$$

l'Hôp.

$$\mathrm{Res}(f|\pi) = \lim_{z\to\pi} \frac{(z-\pi)}{z\sin(z)} \overset{\text{"}\frac{0}{0}\text{"}}{=} \lim_{z\to\pi} \frac{1}{\underbrace{\sin(z)}_{0} + \underbrace{z}_{\pi}\underbrace{\cos(z)}_{-1}} = \boxed{-\frac{1}{\pi}}$$

$$\left.\begin{aligned}(f\cdot g)' &= f'g + fg' \\ f=z \quad & g=\sin(z)\end{aligned}\right\}$$

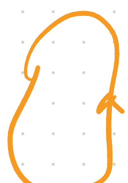
$$\int_{|z-\pi|=3} \frac{1}{z\sin(z)}\,dz = 2\pi i\,\underbrace{\mathrm{Res}(f|\pi)}_{-\frac{1}{\pi}} = 2\cancel{\pi} i\left(-\frac{1}{\cancel{\pi}}\right) = \boxed{-2i}$$

hebbar $\Rightarrow \lim\limits_{z\to z_0} f(z)$ konvergiert Sing. bei $z_0 \Rightarrow$ hebbar

$$\Rightarrow f(z) = \sum_{k=-\infty}^{\infty} c_k (z-z_0)^k \quad \text{mit } c_k = 0 \text{ für } k<0$$

$$= \sum_{k=0}^{\infty} c_k (z-z_0)^k \qquad \boxed{c_{-1}=0} \quad \boxed{\mathrm{Res}(f|z_0)=0}$$

Sing. auf den Rand $\Rightarrow \cancel{\int}$

$\int \Rightarrow$  γ $z\in\gamma \rightsquigarrow \boxed{f(z)} \rightsquigarrow \sum$

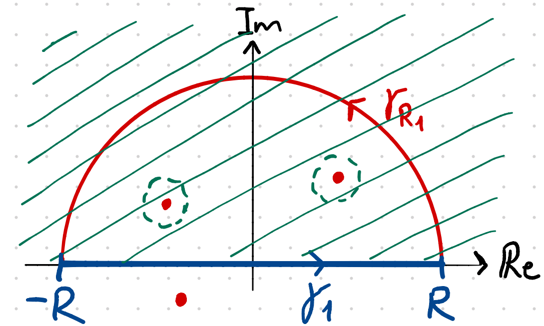
Uneigentliche Integrale $\int_{-\infty}^{\infty}$

$f: \mathbb{R} \to \mathbb{R}$

→ Falls eine Funktion schneller als x^{-2} abfällt

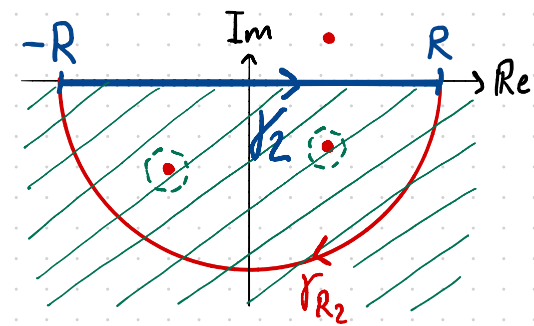
$$2\pi i \sum_{Im>0} Res(f|z_k) = \lim_{R\to\infty} \underbrace{\cancel{\int_{\gamma_R} f(z)\,dz}}_{\to 0} + \int_{-\infty}^{\infty} f(z)\,dz$$

$$\boxed{\int_{-\infty}^{\infty} f(z)\,dz = 2\pi i \sum_{Im>0} Res(f|z_k)}$$



$$-2\pi i \sum_{Im<0} Res(f|z_k) = \lim_{R\to\infty} \underbrace{\cancel{\int_{\gamma_R} f(z)\,dz}}_{\to 0} + \int_{-\infty}^{\infty} f(z)\,dz$$

$$\boxed{\int_{-\infty}^{\infty} f(z)\,dz = -2\pi i \sum_{Im<0} Res(f|z_k)}$$

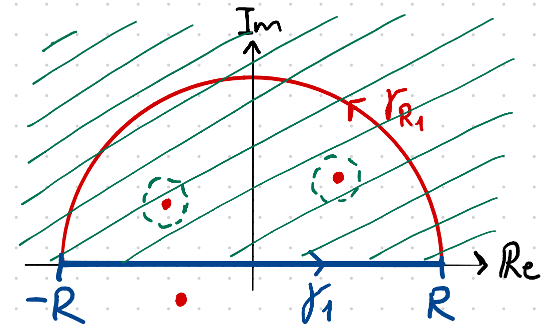


Uneigentliche Integrale

→ Falls eine Funktion schneller als x^{-2} abfällt

$f: \mathbb{R} \to \mathbb{R}$

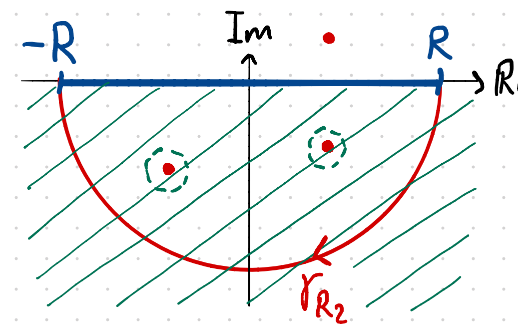
⇒ obere oder untere Halbebene



$f: \mathbb{R} \to \mathbb{C}$

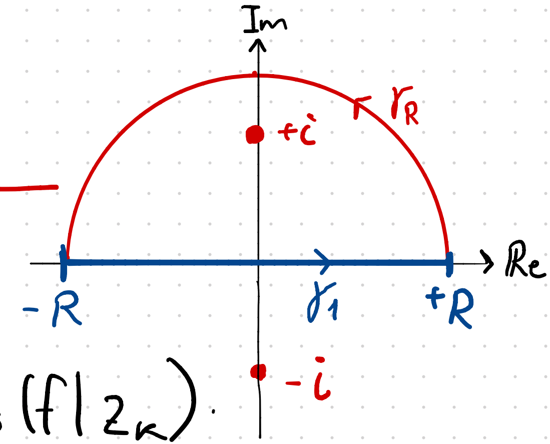
⇒ nicht sicher → Abschätzen

Bsp.: $\int_{-\infty}^{\infty} \frac{1}{t^2+1} e^{-it} \, dt$ [Q&A Session]



Beispiel: $\displaystyle\int_{\boxed{-\infty}}^{\boxed{\infty}} \frac{1}{x^2+1}\,dx$ $\quad f < x^{-2}$ ✓ $\quad f: \mathbb{R} \to \mathbb{R}$ ✓

$$\gamma_R(t) = Re^{\pi i t}, \quad t \in [0,1]$$



$$\lim_{R\to\infty}\left(\int_{\gamma_R} f(z)\,dz\right) + \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{\mathrm{Im}>0} \mathrm{Res}(f|z_k)$$

$\to 0$

$$\lim_{R\to\infty} \int_{\gamma_R} f(z)\,dz = \lim_{R\to\infty}\left|\int_0^1 \left|\frac{1}{\underbrace{(Re^{\pi i t})^2 + 1}_{1}}\right| \cdot R\pi i e^{\pi i t}\, dt\right| \leq \lim_{R\to\infty} \int_0^1 \frac{1}{R^{\cancel{2}}} \cancel{R}\,\pi\, dt$$

$\downarrow \quad 1 \quad 1$

$R^2 \to R$

$$= 0$$

$$\frac{1}{a+b} \leq \frac{1}{|a|+|b|}$$

$$|a+b| \leq |a|+|b|$$

$$\int_{\gamma_R} f(z)\,dz = \left| \int_0^1 \left|f(Re^{\pi i t})\right| \cdot \left|R\pi i e^{\pi i t}\right| dt \right|$$

$$\underbrace{|f(Re^{\pi i t})|}_{1\cdot 1=1} \to \boxed{O(R^{-2})} \to \underline{\underline{O}}(R^{-1})$$

$$|R\pi i e^{\pi i t}| \to R\pi \cdot 1 \to O(R) \to \boxed{O(R^{-1})}$$

$$\approx C$$

$$R \to \infty \Rightarrow \int = 0$$

$$f(z) \sim x^{-1}$$

$$f(z) = \frac{1}{z^2+1} \Rightarrow \frac{1}{(z+i)^1(z-i)^1}$$

$$\mathrm{Res}(f|i) = \lim_{z \to i} \cancel{(z-i)} \frac{1}{(z+i)\cancel{(z-i)}} = \frac{1}{2i}$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \cdot \mathrm{Res}(f|i) = 2\pi i \cdot \frac{1}{2i} = \underline{\pi}$$

Bis jetzt:

1. Komplexe Zahlen
   → Kartesische Form ⇄ Polarform
   → De Moivre

2. Funktionen
   → CR-Gleichungen

3. Integrale
   → Parametrisierung
   → Residuensatz

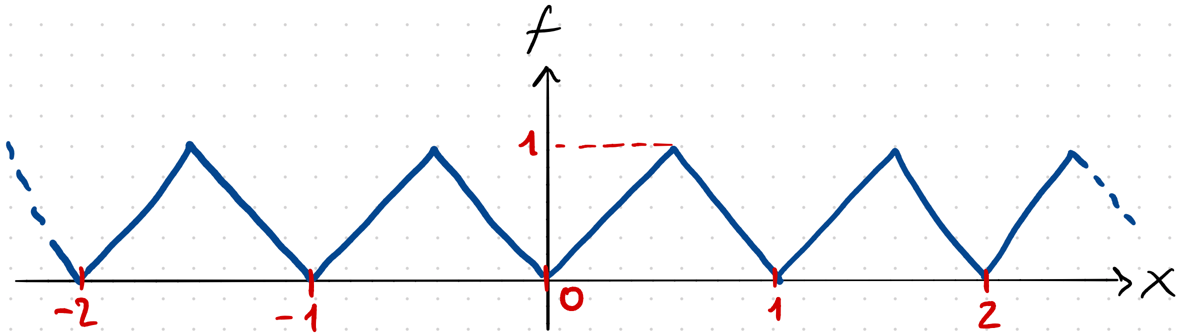
→ Satz von Cauchy
(Mittelwertsatz
Maximumsprinzip)

Fourier-Reihen

Fourierreihen

→ f Periodisch $\Rightarrow \exists T > 0$ s.d. $f(x+\boxed{k}T) = f(x),\ k \in \mathbb{Z}$

kleinste T = Fundamentalperiode



→ Darstellung von periodische Funktionen durch trigonometrische Funktionen

exp(i·), cos, sin

$$\sum_k e^{it\cdot\frac{2\pi}{T}\cdot k}\cdot c_k = \sum a_k \cos\left(\frac{2\pi}{T}kt\right) + b_k \sin\left(\frac{2\pi}{T}kT\right)$$

komplex

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi i n t}{T}}$$

$$C_n = \frac{1}{T} \int_{\boxed{-T/2}}^{\boxed{T/2}} f(x)\, e^{-\frac{2\pi i n x}{T}}\, dx$$

T-periodisch

T 2T

0 1T

$$a_n = c_n + c_{-n}$$

$$b_n = i(c_n - c_{-n})$$

$$\int_{-T/2}^{T/2} = \int_{0}^{T} = \int_{3T}^{4T} = \dots$$

reell

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{T}\right) + b_n \sin\left(\frac{2\pi n t}{T}\right)$$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(x) \cos\left(\frac{2\pi n x}{T}\right) dx \quad (n \geq 0)$$

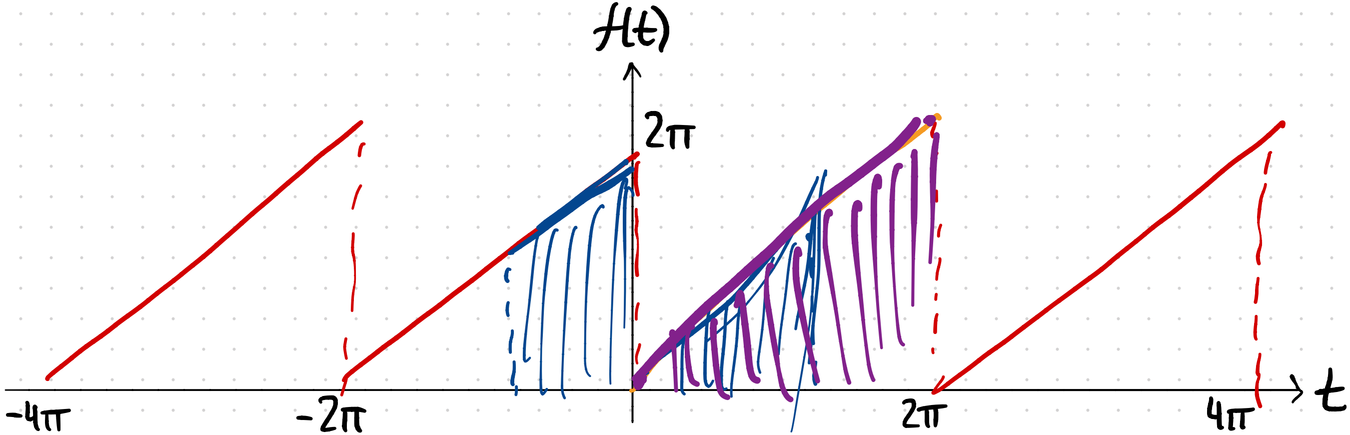
$$a_0 = \frac{2}{T} \int_{-T/2}^{T/2} f(x)\, dx$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(x) \sin\left(\frac{2\pi n x}{T}\right) dx \quad (n \geq 1)$$

$$\int_a^b = (b-a) = T$$

$$b_0 = \frac{2}{T} \int_{-T/2}^{T/2} f(x) \cdot 0 = 0$$

Beispiel: Berechne die Fourierreihe von $\tilde{f}(t) := t, \; t \in [0, 2\pi]$



a) Finde die komplexe Reihe (c_n)

b) Finde die reelle Reihe $(a_n \,\&\, b_n)$

Beispiel: a) Finde die komplexe Reihe (c_n)

$$c_n = \frac{1}{2\pi}\int_{0\ (-T/2)}^{T\ (T/2)} f(t)\, e^{-int} = \frac{1}{2\pi}\int_0^{2\pi} t e^{-int}\,dt \overset{\text{P.I.}}{=} \frac{1}{2\pi}\left(\frac{1}{-in} t e^{-int}\Big|_0^{2\pi} - \int_0^{2\pi} \frac{1}{-in} e^{-int}\,dt\right)$$

$$= \frac{i}{n} \rightarrow n=0\,?$$

$$c_0 = \frac{1}{2\pi}\int_0^{2\pi} t\,dt = \pi$$

$$C_n = \begin{cases} \frac{i}{n}, & n \neq 0 \\ \pi, & n = 0 \end{cases}$$

Beispiel: b) Finde die reelle Reihe (a_n & b_n)

$$c_n = \begin{cases} \frac{i}{n}, & n \neq 0 \\ \pi, & n = 0 \end{cases}$$

→ $a_n = c_n + c_{-n}$

→ $b_n = i(c_n - c_{-n})$

$$a_n = c_n + c_{-n} = \cancel{\frac{i}{n}} + \cancel{\frac{i}{-n}} = 0$$

$$a_0 = c_0 + \underset{\overset{\wr}{0}}{c_{-0}} = \pi + \pi = 2\pi$$

$$\boxed{a_n = \begin{cases} 0, & n \neq 0 \\ 2\pi, & n = 0 \end{cases} \qquad b_n = -\frac{2}{n}}$$

$$b_n = i(c_n - c_{-n}) = i\Big(\underbrace{\frac{i}{n} - \frac{i}{-n}}_{+\frac{2i}{n}}\Big) = -\frac{2}{n}$$

Satz von Parseval

komplex

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi i n t}{T}}$$

reell

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{T}\right) + b_n \sin\left(\frac{2\pi n t}{T}\right)$$

Periode $T = 2\pi$

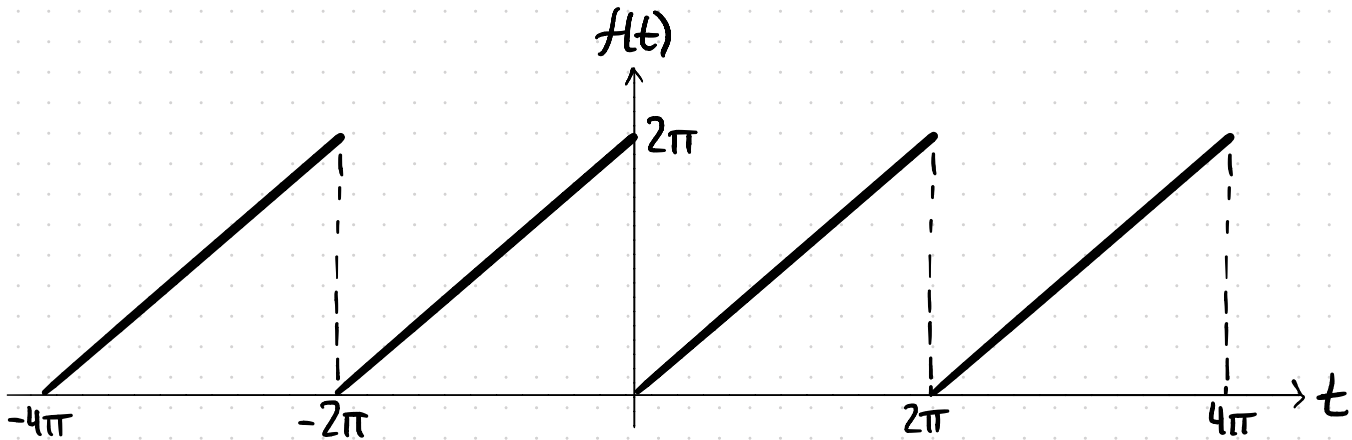
$$\boxed{\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \sum_{k=-\infty}^{\infty} |c_k|^2 = \frac{a_0^2}{4} + \frac{1}{2} \sum_{k=1}^{\infty} \left(|a_k|^2 + |b_k|^2\right)}$$

$f(x) \cdot \overline{f(x)} = f(x)^2$

[komplex]

[reell]

Beispiel: Berechne $\sum_{k=1}^{\infty} \frac{1}{k^2}$ mit Hilfe der Fourierreihe von $\tilde{f}(t) := t,\ t \in [0, 2\pi]$ und Satz von Parseval



Satz von Parseval: $$\underbrace{\frac{1}{2\pi} \int_0^{2\pi} |f(t)|^2 \, dt}_{1} = \underbrace{\sum_{k=-\infty}^{\infty} |c_k|^2}_{2}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

$$c_k = \begin{cases} \pi, & k=0 \\ \frac{i}{k}, & \text{sonst} \end{cases} \Longleftrightarrow a_k = \begin{cases} 2\pi, & k=0 \\ 0, & \text{sonst} \end{cases}$$

$$b_k = -\frac{2}{k} \quad (k \neq 0)$$

① $$\frac{1}{2\pi} \int_{\underset{0 \to 0}{-T/2}}^{\overset{T/2 \to T}{2\pi}} |f(t)|^2 dt = \frac{1}{2\pi} \int_0^{2\pi} t^2 dt = \frac{1}{2\pi} \frac{1}{3} t^3 \Big|_0^{2\pi} = \boxed{\frac{4}{3}\pi^2}$$

$\updownarrow =$

② $$\frac{a_0^2}{4} + \frac{1}{2} \sum_{\boxed{k=1}}^{\boxed{\infty}} |a_k|^2 + |b_k|^2 = \frac{(2\pi)^2}{4} + \frac{1}{2} \sum_{k=1}^{\infty} 0 + \underbrace{\left(-\frac{2}{k}\right)^2}_{\frac{4}{k^2}}$$

$$= \boxed{\pi^2 + 2 \boxed{\sum_{k=1}^{\infty} \frac{1}{k^2}}}$$

$$① \quad \frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2 \, dt = \frac{a_0^2}{4} + \frac{1}{2}\sum_{k=1}^{\infty} |a_k|^2 + |b_k|^2 \quad ②$$

$$\underline{\frac{4}{3}\pi^2} \qquad\qquad \underline{\pi^2 + 2\sum_{k=1}^{\infty}\frac{1}{k^2}}$$

$$\Rightarrow \sum_{k=1}^{\infty}\frac{1}{k^2} = \left(\frac{4}{3}\pi^2 - \pi^2\right)\frac{1}{2} = \frac{\pi^2}{6}$$

Prüfung

a) Finde die Fourier-Reihe von $\tilde{f}(t)$

↳ in die Definition einsetzen (c_n, a_n, b_n)

↳ Integrale mit $e^{it\cdot}$, $\cos(\cdot)$, $\sin(\cdot)$ (Residuensatz)

b) Berechne die relle/komplexe Reihe

↳ $a_n = c_n + c_{-n}$, $b_n = i(c_n - c_{-n})$ $\quad c_n \rightleftarrows a_n, b_n$

c) Berechne $\sum_k^{\infty} \cdots (k) \approx \sum c_k^2$ oder $\sum a_k^2 + b_k^2$

↳ Parseval

Fourier Transformation

Fouriertransformation

$$\boxed{\mathcal{F}\{f\}(\omega) = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt}$$

→ Inverse FT: $\boxed{\mathcal{F}^{-1}\{\hat{f}\}(t) = f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\omega t}\, d\omega}$

$$\Rightarrow \underbrace{\mathcal{F}^{-1}\{\underbrace{\mathcal{F}\{f\}(\omega)}_{\text{FT}}\}(t)}_{\text{Inv FT}} = f(t)$$

$\frac{1}{\sqrt{2\pi}}$!

Fouriertransformation

$$\mathcal{F}\{f\}(\omega) = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

→ Eigenschaften

1. $f(t) \xrightarrow{\mathcal{F}} \hat{f}(\omega)$

2. $f(t-a) \xrightarrow{\mathcal{F}} e^{-ia\omega} \hat{f}(\omega)$

3. $\frac{d^n}{dt^n} f(t) \xrightarrow{\mathcal{F}} (i\omega)^n \hat{f}(\omega)$

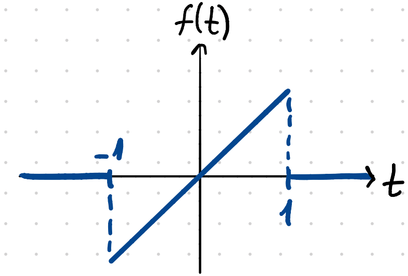
4. $f(at) \xrightarrow{\mathcal{F}} \frac{1}{|a|} \hat{f}\left(\frac{\omega}{a}\right)$

5. $f(t)\, g(t) \xrightarrow{\mathcal{F}} \sqrt{2\pi}\, (f * g)(\omega)$

Faltung

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau$$

Beispiel: Finde die FT von $f(t) = \begin{cases} t, & -1 \le t \le 1 \\ 0, & \text{sonst} \end{cases}$



$$\sqrt{2\pi}\,\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = \int_{-1}^{1} t\, e^{-i\omega t}\, dt \quad \text{P.I.}$$

$$= t\, e^{-i\omega t} \cdot \frac{1}{-i\omega}\Big|_{-1}^{1} - \int_{-1}^{1} \frac{1}{-i\omega}\, e^{-i\omega t}\, dt$$

$$= -\frac{1}{i\omega} \underbrace{\left(e^{-i\omega} + e^{i\omega}\right)}_{2\cos(\omega)} + \underbrace{\frac{1}{i\omega} \cdot \frac{1}{-i\omega}}_{\frac{1}{\omega^2}} \underbrace{e^{-i\omega t}\Big|_{-1}^{1}}_{e^{-i\omega} - e^{i\omega} = -2i\sin(\omega)}$$

$$= 2i\,\frac{1\,\omega}{\omega^2}\cos(\omega) - 2i\,\frac{1}{\omega^2}\sin(\omega)$$

$$\hat{f}(\omega) = \frac{2i}{\sqrt{2\pi}} \left(\frac{\omega\cos(\omega) - \sin(\omega)}{\omega^2} \right)$$

Satz von Plancherel

→ $\hat{f}(\omega)$ ist die FT von $f(t)$ $\rightsquigarrow$ $\hat{f}(\omega) = F\{f\}(\omega)$

$$\boxed{\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 \, d\omega}$$

Beispiel: Berechne $\int_{-\infty}^{\infty} \frac{(x\cos(x)-\sin(x))^2}{x^4}\,dx$

↳ Idee: Plancherel

$$\hat{f}(\omega) = \underbrace{\frac{2i}{\sqrt{2\pi}}\left(\frac{\omega\cos(\omega)-\sin(\omega)}{\omega^2}\right)}$$

$$\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |f(t)|^2 dt$$
$$\quad (1) \qquad\qquad (2)$$

$|i| = 1$

(1) $\displaystyle\int_{-\infty}^{\infty} \left| \frac{2i}{\sqrt{2\pi}} \frac{\omega\cos(\omega)-\sin(\omega)}{\omega^2} \right|^2 d\omega = \int_{-\infty}^{\infty} \frac{4}{2\pi} \frac{(\omega\cos(\omega)-\sin(\omega))^2}{\omega^4}\, d\omega$

$$= \boxed{\frac{2}{\pi}\boxed{\int_{-\infty}^{\infty} \frac{(\omega\cos(\omega)-\sin(\omega))^2}{\omega^4}\,d\omega}}$$

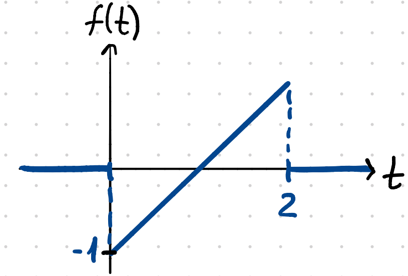
(1) = (2)

(2) $\displaystyle\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-1}^{1} t^2\, dt = \boxed{\frac{2}{3}}$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{(\omega\cos(\omega)-\sin(\omega))^2}{\omega^4}\,d\omega = \frac{\cancel{2}}{3}\cdot\frac{\pi}{\cancel{2}}$$

$$= \frac{\pi}{3}$$

<u>Beispiel</u>: Finde die FT von $\boxed{f(t)} = \begin{cases} t-1, & 0 \le t \le 2 \\ 0, & \text{sonst} \end{cases}$



Vergleich mit $g(t) = \begin{cases} t, & -1 \le t \le 1 \\ 0, & \text{sonst} \end{cases}$

$$f(t) = \boxed{g(t-1)}$$

$$g(t-1) = \begin{cases} t-1, & \overbrace{-1 \le t-1 \le 1}^{0 \le t \le 2} \\ 0, & \text{sonst} \end{cases}$$

$g(t+1)$ → links

$$\hat{f}(\omega) = \widehat{g(t-1)} = e^{-i\omega} \cdot \hat{g}(\omega) = e^{-i\omega} \frac{2i}{\sqrt{2\pi}} \left(\frac{\omega \cos(\omega) - \sin(\omega)}{\omega^2} \right)$$

$a = 1$

$$\left[f(t-a) \xrightarrow{\mathcal{F}} e^{-ia\omega} \hat{f}(\omega) \right]$$

Prüfung

a) Finde die FT von f

↳ in die Definition einsetzen

↳ $\int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$ mit $f(t) \leq t^{-2}$ (Residuensatz)

↳ Partielle Integration

b) Berechne $\int_a^b$ von etwas $\approx \hat{f}(\omega)^2$ → $\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |f(t)|^2 dt$

$$\int_0^{\infty} f(x)\, dx \overset{f \text{ gerade}}{=} \frac{1}{2} \int_{-\infty}^{\infty} f(x)\, dx$$

↳ Plancherel

→ Faltung: $\mathcal{F}\{f(t) \cdot g(t)\}(\omega) = \sqrt{2\pi}\, (\hat{f} * \hat{g})(\omega) \rightsquigarrow \boxed{(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau)\, d\tau}$

Laplace Transformation

Laplacetransformation

$$\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

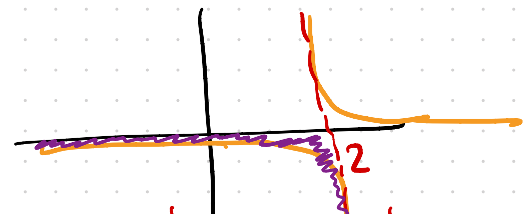
→ **Existenz**

1. f ist von exponentieller Ordnung

   $\exists\, c, s_0 > 0$ s.d. $|f(t)| \leq Ce^{s_0 t}, t > 0$ $\Rightarrow$ $\mathcal{L}$ existiert für $\mathrm{Re}\{s\} > s_0$

2. Integrierbarkeit

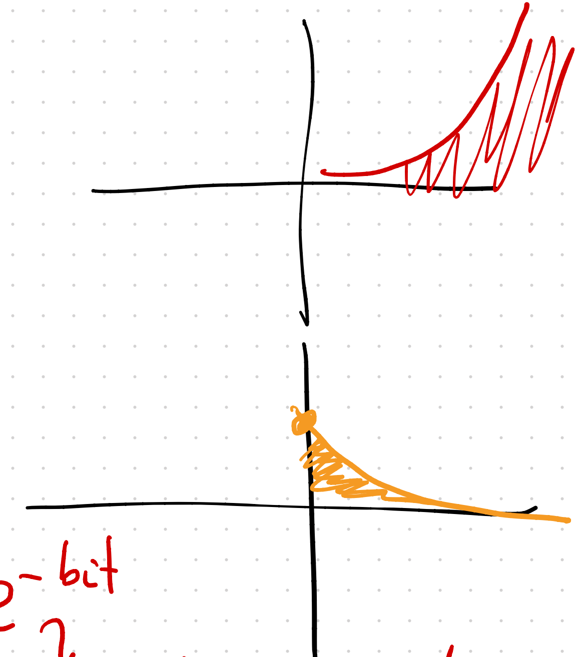
   $\frac{1}{t-2} \to t = 2$



   $\int_0^T |f(t)|\,dt < \infty, T > 0$ $\Rightarrow$ f stetig zwischen 0 und ∞

Warum $\mathrm{Re}\{s\} > s_0$?

→ Nehmen wir mal an, dass $f(t) = e^{s_0 t}$



$s \in \mathbb{C}, \quad s = a + bi$

$$\left| \int_0^\infty \left| e^{s_0 t} \underset{\downarrow\, a+bi}{e^{-st}} \right| dt \right| = \int_0^\infty e^{s_0 t} e^{-at - bit} \, dt$$

$$= \int_0^\infty \underbrace{\left| e^{t(s_0 - a)} \right|} \underbrace{\left| e^{-bit} \right|}_{1} dt$$

$e^{-bit} \to (e^{it})^b \quad |e^{it}|^b = 1^b = 1$, $\underbrace{e^{it}}_{1}$

konvergiert, falls $s_0 - a < 0$

divergiert, falls $s_0 - a \geq 0$

$s_0 - a < 0$

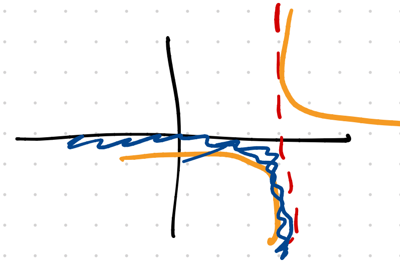
$s_0 < a$

$\mathrm{Re}\{s\} > \mathrm{Re}\{s_0\}$

Beispiel:

→ $f(t) = 1e^{-2t}$ $\quad \Big| \; \boxed{Ce^{s_0 t}} \rightsquigarrow C = 1, \; \boxed{s_0 = -2}$ $\quad \mathcal{L} \; \exists$ für $\mathrm{Re}\{s\} > -2$

→ $f(t) = e^{t^2}$ $\quad t^2 > \underline{s_0 t} \Longrightarrow \mathcal{L} \; \nexists$



→ $f(t) = \frac{1}{t-3} \rightsquigarrow \int_0^T |f(t)| \, dt \not< \infty$ für $t = 3 \Longrightarrow \mathcal{L} \; \nexists$

→ $f(t) = \frac{1}{\sqrt{t}} \rightsquigarrow Ce^{s_0 t}$ $\Big|$ 1. ✓
2. Integrierbarkeit ✓

$$\int_0^T t^{-1/2} = 2\sqrt{t}\,\Big|_0^T = 2\sqrt{T} < \infty \Longrightarrow \mathcal{L} \; \exists \text{ für } \boxed{\mathrm{Re}\{s\} > 0}$$

$f(t) = \log(t)$ $\qquad \int_0^T \log(t) \, dt = \dots < \infty$

Laplacetransformation

$$\boxed{\mathcal{L}\{f\}(s) = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt}$$

→ Eigenschaften

1. $f(t) \circ\!\!-\!\!\bullet\ F(s)$
2. $f(t-a) \circ\!\!-\!\!\bullet\ e^{-as} F(s)$
3. $\frac{d}{dt} f(t) \circ\!\!-\!\!\bullet\ sF(s) - f(0)$
4. $t^n e^{-at} \circ\!\!-\!\!\bullet\ \frac{n!}{(s+a)^{n+1}}$
5. $f(t)\, g(t) \circ\!\!-\!\!\bullet\ (f * g)(s)$

Faltung

$$\boxed{(f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau}$$

Beispiel: Löse $\ddot{y}(t) + \dot{y}(t) = e^t, \quad \dot{y}(0) = y(0) = 0$

$\hookrightarrow \frac{d}{dt} f(t)$

$\mathcal{L}\{y(t)\}(s) = Y(s)$

$$\underbrace{\mathcal{L}\{\frac{d}{dt}\underbrace{\frac{d}{dt}y(t)}_{g(t)}\}(s)}_{\mathcal{L}\{\frac{d}{dt}g(t)\}(s)} + \underbrace{\mathcal{L}\{\frac{d}{dt}y(t)\}(s)}_{sY(s)-y(0)} = \underbrace{\mathcal{L}\{e^t\}(s)}_{\substack{n=0\\a=-1}}$$

$\Rightarrow \frac{0!}{(s-1)^{0+1}} = \frac{1}{s-1}$

$= s\mathcal{L}\{g(t)\}(s) - g(0)$

$g(t) = \frac{d}{dt}y(t)$

$g(0) = \frac{d}{dt}y(t)\Big|_{t=0} = \dot{y}(0)$

$= s\mathcal{L}\{\frac{d}{dt}y(t)\}(s) - \dot{y}(0)$

$= s\left(s\cdot Y(s) - y(0)\right) - \dot{y}(0)$

$=$

$$\left[\begin{array}{lcl} \frac{d}{dt}f(t) & \circ\!\!-\!\!\bullet & sF(s) - f(0) \\ t^n e^{-at} & \circ\!\!-\!\!\bullet & \frac{n!}{(s+a)^{n+1}} \end{array}\right]$$

$$s^2 Y(s) - s\cancelto{0}{y(0)} - \cancelto{0}{\dot{y}(0)} + sY(s) - \cancelto{0}{y(0)} = \frac{1}{s-1} \qquad \dot{y}(0) = y(0) = 0$$

$$\underbrace{s^2 Y(s) + sY(s)}_{\substack{(s^2+s)Y(s) \\ s\cdot(s+1)Y(s)}} = \frac{1}{s-1} \Longrightarrow \mathcal{L}^{-1}\{\underline{Y(s)}\} = \mathcal{L}^{-1}\left\{\underline{\frac{1}{(s-1)\cdot s\cdot(s+1)}}\right\} = \underline{y(t)}$$

$$Y(s) = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1}$$

$$A(s-1)(s+1) + B(s+1)s + Cs(s-1) = 1$$

$s=0$	$-1A$	$+$	0	$+$	0	$=$	1
$s=+1$	0	$+$	$2B$	$+$	0	$=$	1
$s=-1$	0	$+$	0	$+$	$2C$	$=$	1

$$\Longrightarrow \boxed{A = -1} \quad \boxed{B = \frac{1}{2}} \quad \boxed{C = \frac{1}{2}}$$

$$Y(s) = -\frac{1}{s} + \frac{1}{2}\frac{1}{s-1} + \frac{1}{2}\frac{1}{s+1}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = -\underset{\substack{n=0\\a=0}}{\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}} + \boxed{\frac{1}{2}\underset{\substack{n=0\\a=-1}}{\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}} + \frac{1}{2}\underset{\substack{n=0\\a=1}}{\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}}} \qquad t>0$$

$$= -1 + \underbrace{\frac{1}{2}e^{t} + \frac{1}{2}e^{-t}}_{\cosh(t)}$$

$$\frac{1}{2}\frac{1}{s-1} + \frac{1}{2}\frac{1}{s+1} = \frac{s}{s^2-1}$$

$$\cosh(t) \circ\!\!-\!\!\bullet \frac{s}{s^2-1} \qquad = -1 + \cosh(t)$$

$$\left[\begin{array}{lcl} \frac{d}{dt}f(t) & \circ\!\!-\!\!\bullet & sF(s) - f(0) \\ t^n e^{-at} & \circ\!\!-\!\!\bullet & \frac{n!}{(s+a)^{n+1}} \end{array}\right]$$

Prüfung

a) Existiert $\mathcal{L}\{f\}$?

↳ Bedingung für Existenz

→ Faltung: $f(t)\cdot g(t) \circ\!\!-\!\!\bullet (f*g)(s)$

$$\boxed{(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau}$$

b) Finde $\mathcal{L}\{f\}$

↳ In die Definition einsetzen

c) Differentialgleichung

↳ $\frac{d}{dt}f(t) \circ\!\!-\!\!\bullet sF(s)-f(0)$

↳ $t^n e^{-at} \circ\!\!-\!\!\bullet \frac{n!}{(s+a)^{n+1}}$

1. DGL $\ddot{y}(t) + \dots = \dots$
2. $\mathcal{L}\{\ddot{y}(t) + \dots\} = \mathcal{L}\{\dots\}$
3. $Y(s) = \dots$ PBZ
4. $\mathcal{L}^{-1}\{Y(s)\} = y(t) = \mathcal{L}^{-1}\{\dots\}$

Ende?