Q&A

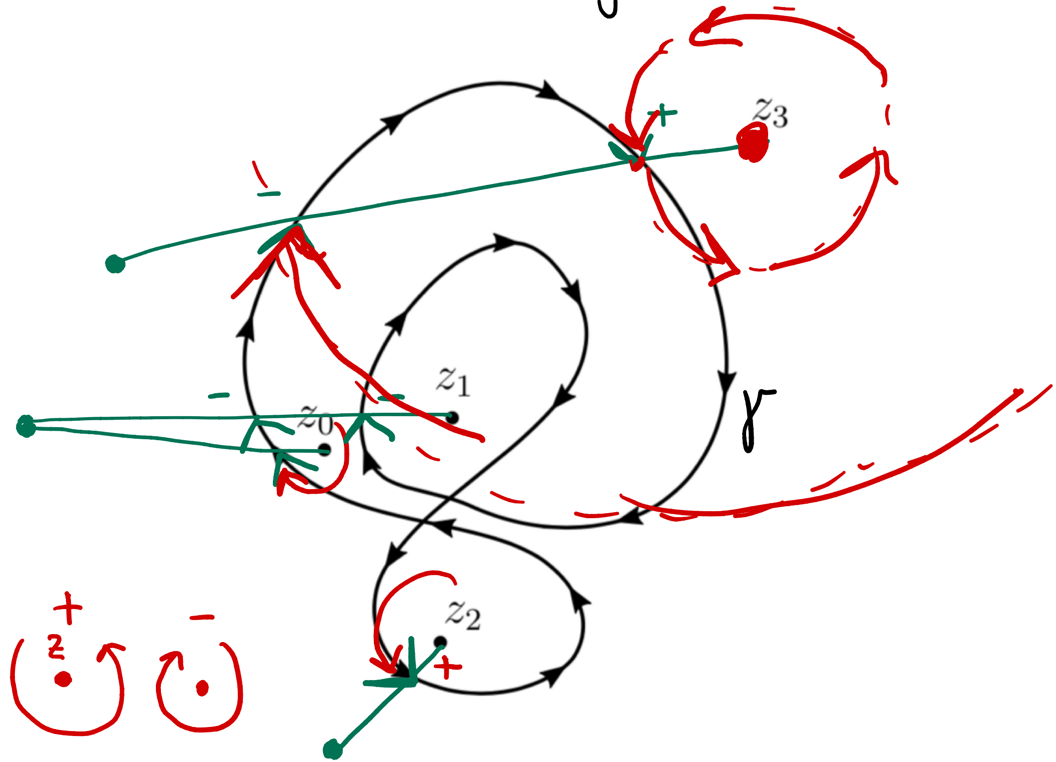
Komplexe Analysis

FS 2020

[Aufgabe 10]

Beispiel:

Bestimme die Windungszahl für z_0, z_1, z_2 und z_3



$\mathrm{Ind}_\gamma(z_0) = -1$

$\mathrm{Ind}_\gamma(z_1) = -2$

$\mathrm{Ind}_\gamma(z_2) = +1$

$\mathrm{Ind}_\gamma(z_3) = 0$

Beispiel:

[Aufgabe 8]

Sei $f: \mathbb{C} \to \mathbb{C}$ eine holomorphe Funktion. Welche der folgenden Funktionen ist im Allgemein **nicht** holomorph?

i. $g(z) = f(z)^3$ **ii.** $g(z) = f(z^4)$ **iii.** $g(z) = f(\bar{z})$ **iv** $g(z) = \boxed{\overline{f(\bar{z})}}$

$f(z) = z$ $f(\bar{z}) = \bar{z}$ ✗

$\overline{\sum_k \bar{z}^k c_k}$

$\sum_k z^k \bar{c}_k$

Generell: $[f(x+yi)]^3$

→ Zusammensetzung von holom. Fkt ist holomorph

$f(z)^n$ f_1, f_2 holomorph $f_1(f_2(z))$ $e^{f(z)}$, $f(e^z)$, $\sin(f(z))$

→ $\bar{z}$, $\mathrm{Re}\{z\}$, $\mathrm{Im}\{z\}$, $|z|$, $\arg(z)$ nicht holomorph

$$\underbrace{\mathrm{Re}\{z\} + i\,\mathrm{Im}\{z\}}_{z}$$

$h(4) \leftarrow$ [Aufgabe 5]

Beispiel:

Gegeben sei $f, g : \mathbb{R} \to \mathbb{C}$, $f(x) = \begin{cases} \frac{1}{6}, & 0 \leq x \leq 6 \\ 0, & \text{sonst} \end{cases}$, $g(x) = \boxed{x^2 - 3ix}$

Berechne $h(t) := (f * g)(t)$ $\qquad \mathbb{R} \to \mathbb{C}$

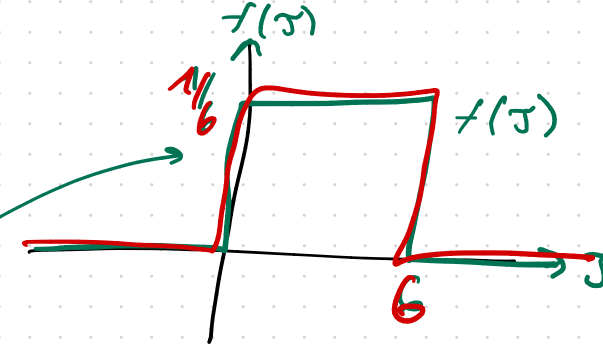
Generell: $(f*g)(t) = \underline{\int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau} = \int_{-\infty}^{\infty} g(\tau) \cdot f(\underbrace{t-\tau})\, d\tau$

g Spiegelung + Verschiebung

Laplace: $(f*g)(t) = \int_{0}^{t} f(\tau)\, g(t-\tau)\, d\tau$

1. g spiegeln und verschieben $= \tilde{g}$
2. f mit $\tilde{g}$ multiplizieren
3. $-\infty, \infty$ integrieren

$$f(x) = \begin{cases} \frac{1}{6}, & 0 \leq x \leq 6 \\ 0, & \text{sonst} \end{cases}, \quad g(x) = \boxed{x^2 - 3ix}$$



$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)\, g(t-\tau)\, d\tau$$

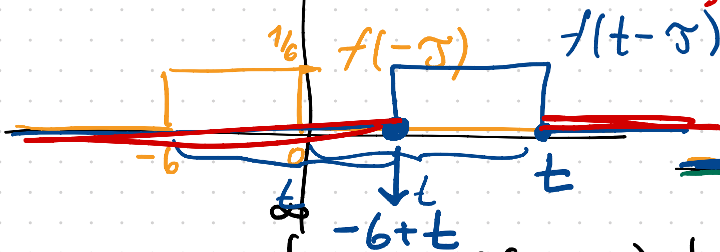
$$= \int_{0}^{6} \frac{1}{6} \left[(t-\tau)^2 - 3i(t-\tau) \right] d\tau$$

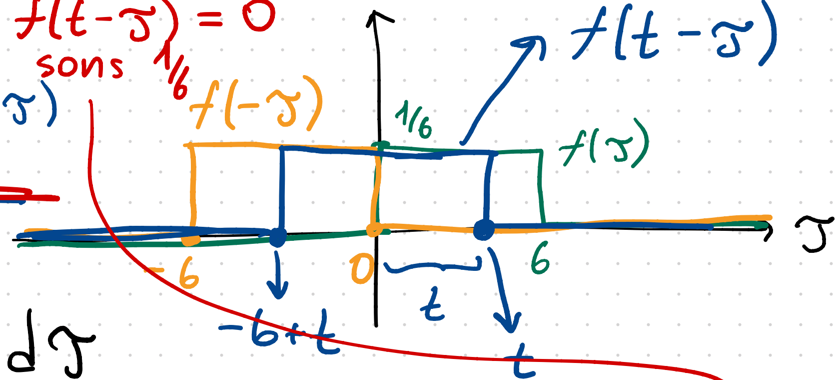
$$\boxed{h(t)}$$

$$= \frac{1}{6} \left[-\frac{1}{3}(t-\tau)^3 + \frac{3i}{2}(t-\tau)^2 \right]_0^6$$

$$= \frac{1}{6} \left[\left(-\frac{1}{3}(t-6)^3 + \frac{3i}{2}(t-6)^2 \right) - \left(-\frac{1}{3} t^3 + \frac{3i}{2} t^2 \right) \right]$$

$\tau > t$
$\tau < -6+t$ $\rightarrow f(t-\tau) = 0$
sons





$$(f*g)(t) = \int_{-\infty}^{\infty} g(\tau) f(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} (\tau^2 - 3i\tau)\, \underbrace{f(t-\tau)}\, d\tau \quad \Rightarrow \quad f(t-\tau) = \begin{cases} \frac{1}{6}, & -6+t < \tau < t \\ 0, & \text{sonst} \end{cases}$$

$$= \int_{\boxed{-6+t}}^{\boxed{t}} \frac{1}{6} (\underbrace{\tau^2 - 3i\tau})\, d\tau \qquad g(\tau) = \tau^3 - 3i\tau$$

$$= \frac{1}{6} \left(\frac{1}{3}\tau^3 - \frac{3i}{2}\tau^2 \right) \Big|_{-6+t}^{t}$$

Beispiel: [Aufgabe 4]

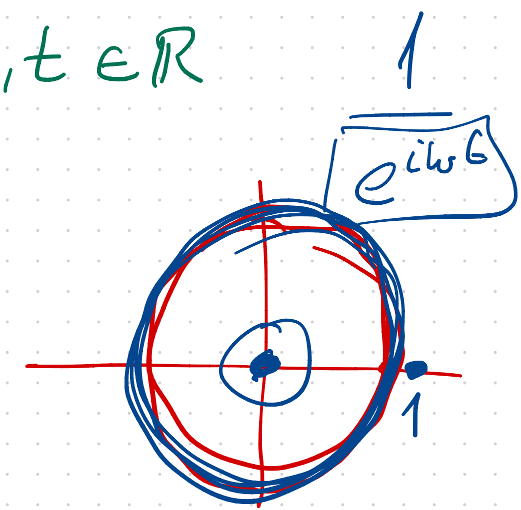
Finde die Fouriertransformation von $f(t) = \frac{1}{1+t^2}$ für $\omega > 0$

$\hat{f}(\omega)$ für $\omega > 0$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\frac{1}{1+t^2}}_{< t^{-2}} \underbrace{e^{-i\omega t}}_{|\cdot| = 1} \, dt$$

$\omega, t \in \mathbb{R}$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt$$



$e^0 = 1$

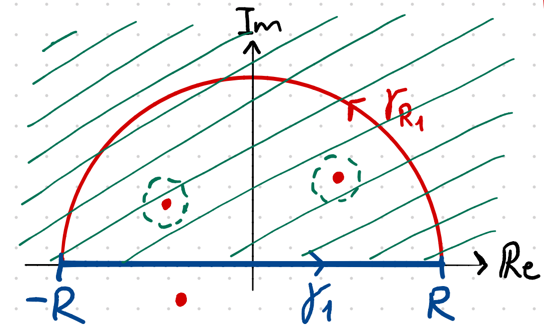
Uneigentliche Integrale

$$\int_a^b \dots = -\int_b^a$$

→ Falls eine Funktion schneller als x^{-2} abfällt

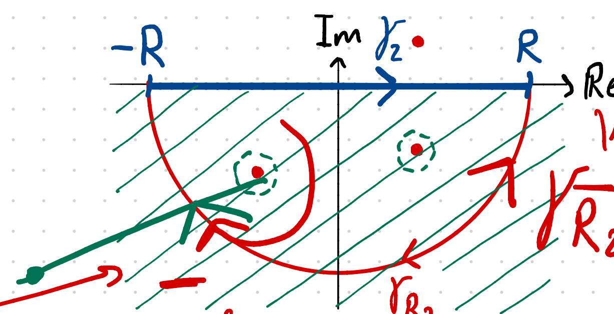
$$2\pi i \sum_{Im>0} Res(f|z_k) = \lim_{R\to\infty} \cancelto{0}{\int_{\gamma_R} f(z)\,dz} + \int_{-\infty}^{\infty} f(z)\,dz$$

$$\boxed{\int_{-\infty}^{\infty} f(z)\,dz = 2\pi i \sum_{Im>0} Res(f|z_k)}$$



$$-2\pi i \sum_{Im<0} Res(f|z_k) = \lim_{R\to\infty} \cancelto{0}{\int_{\gamma_R} f(z)\,dz} + \int_{-\infty}^{\infty} f(z)\,dz$$

$$\boxed{\int_{-\infty}^{\infty} f(z)\,dz = -2\pi i \sum_{Im<0} Res(f|z_k)}$$



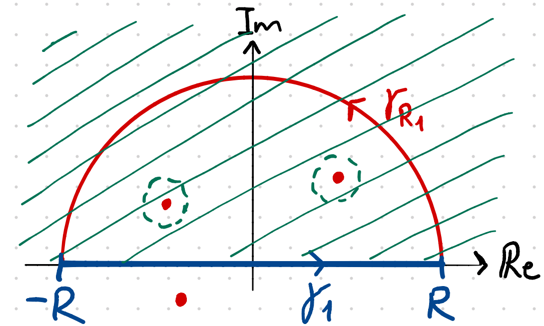
$$\int_{\gamma_{R2}} \dots = -\int_{\overline{\gamma_{R2}}}$$

Uneigentliche Integrale

→ Falls eine Funktion schneller als x^{-2} abfällt

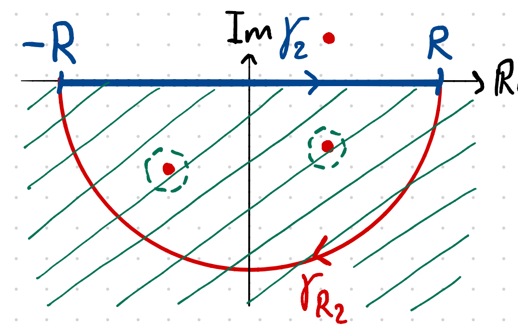
$f: \mathbb{R} \to \mathbb{R}$

$\Rightarrow$ obere oder untere Halbebene



$f: \mathbb{R} \to \mathbb{C}$

$\Rightarrow$ nicht sicher → Abschätzen



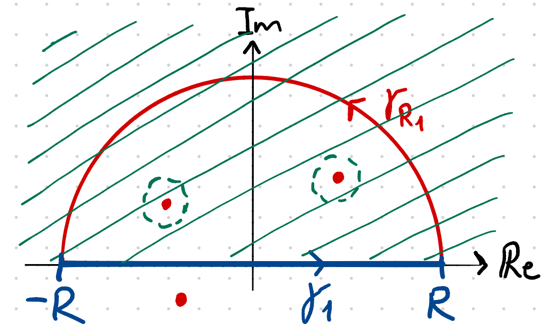
Uneigentliche Integrale

→ Falls eine Funktion schneller als x^{-2} abfällt

$f: \mathbb{R} \to \mathbb{R}$

$\Rightarrow$ obere oder untere Halbebene

Bsp $\left[\int_{-\infty}^{\infty} \frac{1}{t^2+1} \, dt \qquad \underbrace{t}_{\mathbb{R}} \longmapsto \underbrace{\frac{1}{t^2+1}}_{\mathbb{R}} \right]$

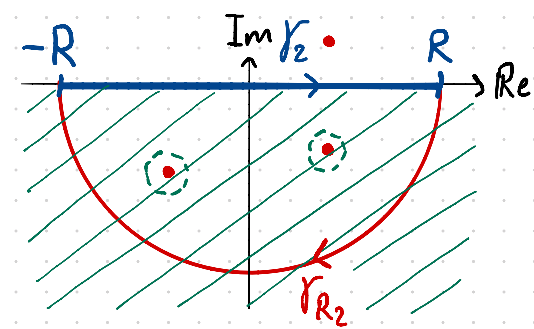


$f: \mathbb{R} \to \mathbb{C}$

$\Rightarrow$ nicht sicher → Abschätzen

Bsp $\left[\int_{-\infty}^{\infty} \frac{1}{1+t^2} e^{-i\omega t} \, dt \qquad \underbrace{t}_{\mathbb{R}} \longmapsto \underbrace{\frac{1}{1+t^2} e^{-i\omega t}}_{\mathbb{C}} \right]$

$\mathbb{R} \to \mathbb{C}$



Beispiel: [Aufgabe 4]

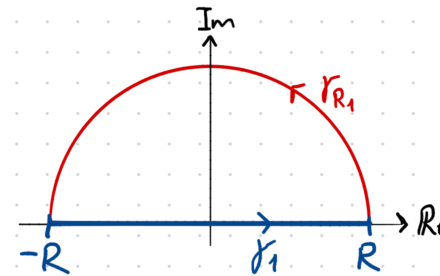
Finde die Fouriertransformation von $f(t)=\frac{1}{1+t^2}$ für $\omega > 0$

$$\sqrt{2\pi}\,\hat{f}(\omega)=\int_{-\infty}^{\infty}\frac{1}{1+t^2}e^{-i\omega t}\,dt$$

$$\int_{\gamma_{R_1}} f(z)\,dz = \int_{\gamma_{R_1}}\frac{1}{1+z^2}e^{-i\omega z}\,dz$$

$\gamma_{R_1}(t)=z=Re^{it}$
$t\in[0,\pi]$

$\dot{\gamma}_{R_1}(t)=Rie^{it}$



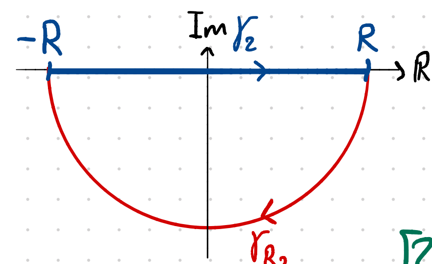
$\gamma_{R_1}(t)=Re^{it}$, $t\in[0,\pi]$

$$=\int_0^\pi \frac{1}{1+(Re^{it})^2}\,e^{-i\omega Re^{it}}\cdot Rie^{it}\,dt$$

$e^{it}=\cos(t)+i\sin(t)$

$-i\omega R(\cos(t)+i\sin(t))=-i\omega R\cos(t)$
$+i\omega R\sin(t)$

$$=\int_0^\pi \frac{1}{1+R^2e^{2it}}\,Rie^{it}\cdot e^{-i\omega R\cos(t)}\cdot e^{i\omega R\sin(t)}\,dt$$



$[2\pi,\pi]$

$\gamma_{R_2}(t)=Re^{it}$, $t\in[0,-\pi]$

$$\left|\int_0^{\pi} \underbrace{\frac{1}{1+R^2e^{2it}}}_{\downarrow}\ \underbrace{|R\,i\,e^{it}|}_{R}\ \underbrace{e^{-i\omega R\cos(t)}}_{1}\ e^{\omega R\sin(t)}\,dt\right|$$

sin(t) für $t\in[0,\pi] \Rightarrow \sin(t)>0$

sin(t) für $t\in[0,-\pi] \Rightarrow \sin(t)<0$

$|R|\ |i|\ |e^{it}|$: $1 \quad 1$

$-i\omega R\cos(t) \in \mathbb{R}$

$$\frac{1}{1+R^2e^{2it}}$$

$$\frac{1}{1-R^2}$$

$$\underbrace{\frac{1}{1+R^2e^{2it}}} \rightarrow \mathcal{O}(R^{-2})$$

$R \rightarrow \mathcal{O}(R)$

$$|R|\,|i|\,\underbrace{|e^{it}|}$$

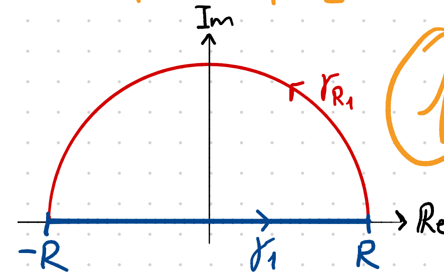
$R \quad 1$

$$\mathcal{O}(R^{-1})$$

$$e^{\omega R\sin(t)}$$

$$\underbrace{e^{i\mathbb{R}}}_{|\cdot|=1}$$

$\omega, R, \sin(t) \in \mathbb{R}$



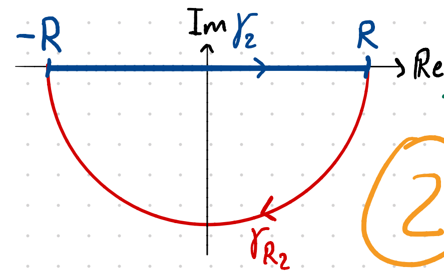
1

$$\gamma_{R_1}(t) = Re^{it},\quad t\in[0,\pi]$$

$e^{R\omega\sin(t)}$: $\boxed{[\omega>0]}$

falls $\sin(t)>0 \Rightarrow \lim\limits_{R\to\infty} e^{R\omega\sin(t)} \rightarrow \infty$ 1

falls $\sin(t)<0 \Rightarrow \lim\limits_{R\to\infty} e^{R\omega\sin(t)} \rightarrow 0$ 2



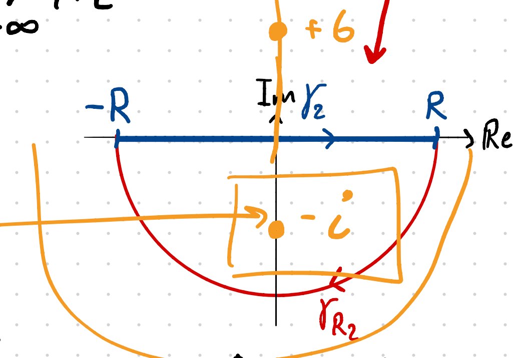
2

$$\gamma_{R_2}(t) = Re^{it},\quad t\in \boxed{\begin{matrix}[0,-\pi]\\ [2\pi,\pi]\end{matrix}}$$

⇒ Untere Halbebene

$$-2\pi i \sum_{Im<0} Res(f|z_k) = \lim_{R\to\infty} \underbrace{\cancel{\int_{\gamma_{R_2}} \frac{1}{1+z^2} e^{-i\omega z}\, dz}}_{\to 0} + \int_{-\infty}^{\infty} \frac{1}{1+t^2} e^{-i\omega t}\, dt$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{1+t^2} e^{-i\omega t}\, dt = -2\pi i \sum_{\boxed{Im<0}} Res(f|z_k)$$



→ Singularitäten → $\underline{1+t^2}=0$ ⇒ $t=\pm i$

$z=-i$ ⇒ Pol 1. Ord.

$z=+i$ ⇒ ~~Pol 1. Ord.~~

→ Residuum

$$Res(f|-i) = \lim_{z\to -i} \cancel{(z+i)} \cdot \frac{e^{-i\omega z}}{\cancel{(z+i)}(z-i)} = \lim_{z\to -i} \frac{e^{-i\omega z}}{z-i} = \frac{e^{-\omega}}{-2i}$$

$$\int_{-\infty}^{\infty} \frac{1}{1+t^2} e^{-i\omega t}\, dt = -2\pi i \sum_{\mathrm{Im}<0} \mathrm{Res}(f|z_k) = \cancel{-2}\pi \cancel{i} \frac{e^{-\omega}}{\cancel{-2i}} = \pi e^{-\omega}$$

$$\Rightarrow \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} \frac{1}{1+t^2} e^{-i\omega t}} = \frac{1}{\sqrt{2\pi}} \pi e^{-\omega} = \sqrt{\frac{\pi}{2}}\, e^{-\omega}$$

$[\omega > 0]$

$$F\{f\}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$$

$$F^{-1}\{\hat{f}\}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{+i\omega t}\, d\omega$$

[Aufgabe 7]

Beispiel:

a) Finde die reelle Fourierreihe von x^2 für $x \in [-1,1]$

b) Berechne $\sum_{k=1}^{\infty} \frac{1}{k^2}$

komplex

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi i n t}{T}}$$

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(x)\, e^{-\frac{2\pi i n x}{T}}\, dx$$

$$a_n = c_n + c_{-n}$$

$$b_n = i(c_n - c_{-n})$$

reell

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{T}\right) + b_n \sin\left(\frac{2\pi n t}{T}\right)$$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(x) \cos\left(\frac{2\pi n x}{T}\right) dx \quad (n \geq 0)$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(x) \sin\left(\frac{2\pi n x}{T}\right) dx \quad (n \geq 1)$$

$$\left[\begin{array}{lll} f \text{ ungerade} & \Rightarrow & a_k = 0 \ \forall k \\ f \text{ gerade} & \Rightarrow & b_k = 0 \ \forall k \end{array}\right]$$

Satz von Parseval

komplex

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi i n t}{T}}$$

reell

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{T}\right) + b_n \sin\left(\frac{2\pi n t}{T}\right)$$

f 2π-periodisch

$$\boxed{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx = \sum_{k=-\infty}^{\infty} |c_k|^2 = \frac{a_0^2}{4} + \frac{1}{2}\sum_{k=1}^{\infty}\left(|a_k|^2 + |b_k|^2\right)}$$

[komplex] [reell]

[Aufgabe 7]

Beispiel:

a) Finde die reelle Fourierreihe von x^2 für $x \in [-1,1]$

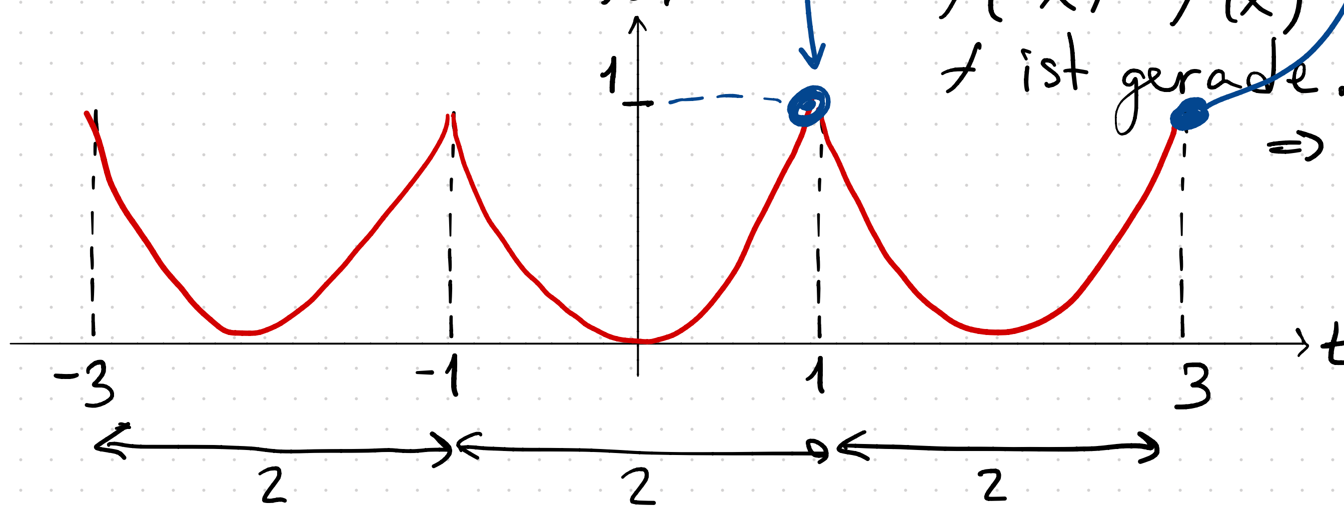
a_k, b_k

$T=2$

b) Berechne $\sum_{k=1}^{\infty} \frac{1}{k^2}$

$\boxed{f(1)=1}$

$\boxed{f(3)=1}$



$f(-x) = f(x)$

f ist gerade!

$\Rightarrow b_k = 0$

a) Finde die reelle Fourierreihe von x^2 für $x \in [-1,1]$

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\cos\left(\frac{2\pi k x}{T}\right)dx \quad (k \geq 0) \qquad b_k = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\sin\left(\frac{2\pi k x}{T}\right)dx \quad (k \geq 1)$$

$\rightarrow f$ gerade $\Rightarrow b_k = 0 \;\forall k$ $\qquad \int_{-a}^{+a} g(x)\,dx = 2\int_0^a g(x)\,dx$

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} x^2 \cos\left(\frac{2\pi}{T}kx\right)dx = \int_{-1}^{1} \underbrace{x^2}_{g}\underbrace{\cos(\pi k x)}_{g}\,dx = 2\int_0^1 x^2\cos(\pi k x)\,dx$$

$[T=2]$

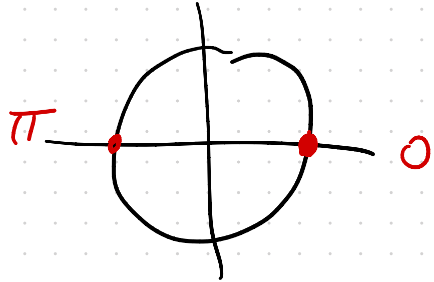
$\sin(\pi k) = 0$
$\sin(0) = 0$

P.I.
$$= 2\left(\cancel{x^2 \frac{1}{\pi k}\sin(\pi k x)\Big|_0^1} - \int_0^1 2x \frac{1}{\pi k}\sin(\pi k x)\,dx\right)$$

P.I.
$$= -\frac{4}{\pi k}\int_0^1 x\sin(\pi k x)\,dx = \boxed{-\frac{4}{\pi k}\left(-\frac{1}{\pi k}\right.} x\cos(\pi k x)\Big|_0^1 + \cancel{\int_0^1 \frac{1}{\pi k}\cos(\pi k x)\,dx}$$

$$= \frac{4}{\pi^2 k^2}\cos(\pi k) = \boxed{\frac{4}{\pi^2 k^2}(-1)^k}$$

$\cos(\pi k) = (-1)^k$



$\cos(0) = 1$

$\cos(\pi) = -1$

$(-1)^k$

$\cos(\pi k) = (-1)^k$

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} x^2\,dx = \int_{-1}^{1} x^2\,dx = \frac{2}{3}$$

$$\boxed{a_0 = \frac{2}{3}}$$

a) Finde die reelle Fourierreihe von x^2 für $x \in [-1,1]$

$$\boxed{f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2\pi k t}{T}\right) + b_k \sin\left(\frac{2\pi k t}{T}\right)}$$

$$a_0 = \frac{2}{3} \qquad a_k = \frac{4(-1)^k}{\pi^2 k^2} \qquad b_k = 0 \ \forall k$$

$$\boxed{f(t) = \frac{1}{3} + \sum_{k=1}^{\infty} \frac{4(-1)^k}{\pi^2 k^2} \cos(\pi k t)}$$

b) Berechne $\sum_{k=1}^{\infty} \frac{1}{k^2}$ ~~Parseval~~

$$a_0 = \frac{2}{3}$$

$$a_k = \frac{4(-1)^k}{\pi^2 k^2}$$

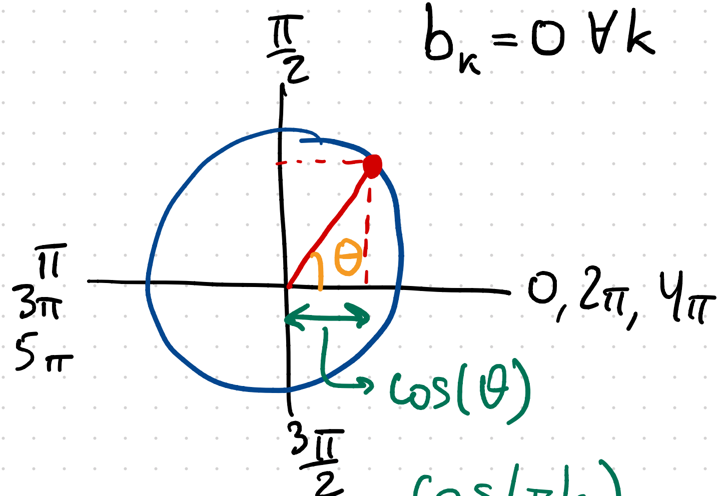
$$b_k = 0 \ \forall k$$

$\cos(\pi k)$

$$f(t) = \frac{1}{3} + \sum_{k=1}^{\infty} \frac{4\boxed{(-1)^k}}{\pi^2 k^2} \cos(\pi k t)$$

$$f(1) = \frac{1}{3} + \sum_{k=1}^{\infty} \frac{4\cancel{(-1)^k}}{\pi^2 k^2} \overbrace{\cos(\pi k)}^{\cancel{(-1)^k}}$$

$$1 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2}$$



$\cos(\pi k)$

$k \in \mathbb{Z}$

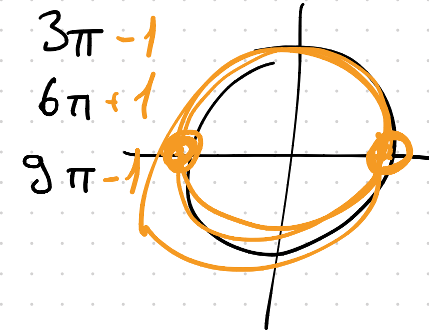
$\cos(0) = 1$
$\cos(\pi) = -1$
$\cos(2\pi) = 1$
$\cos(3\pi) = -1$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \left(1 - \frac{1}{3}\right) \frac{\pi^2}{4} = \frac{2}{3} \cdot \frac{\pi^2}{4} = \boxed{\frac{\pi^2}{6}}$$

$$\frac{3}{3} - \frac{1}{3} = \frac{2}{3}$$

$$f(t) = \frac{1}{3} + \sum_{k=1}^{\infty} \frac{4\boxed{(-1)^k}}{\pi^2 k^2} \cos(\pi k t)$$

$\cos(\pi k)$



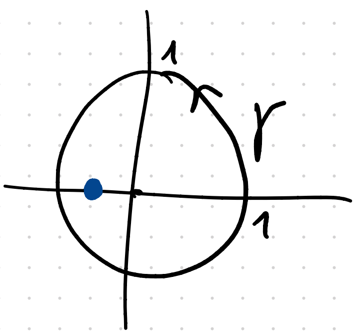
$$f(3) = \frac{1}{3} + \sum_{k=1}^{\infty} \frac{4\cancel{(-1)^k}}{\pi^2 k^2} \underbrace{\cos(3\pi k)}_{\cancel{(-1)^k}}$$

$$\boxed{1 = \frac{1}{3} + \sum_{k=1}^{\infty} \frac{4}{\pi^2 k^2}}$$

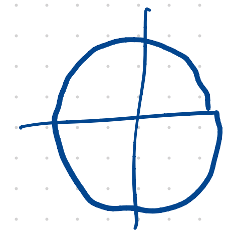
<u>Kein Parseval</u>

Integrale mit exp, sin, cos

$$\int_{|z|=1} f(z)\,dz \xrightarrow[t\in[0,1]]{\gamma(t):=e^{2\pi i t}} \int_0^1 f(\underbrace{e^{2\pi i t}}_{\gamma(t)})\cdot\underbrace{2\pi i\, e^{2\pi i t}}_{\gamma'(t)}\,dt$$



$$\xrightarrow[t\in[0,2\pi]]{\gamma(t):=e^{it}} \int_0^{2\pi} f(\underbrace{e^{it}}_{\gamma(t)})\cdot\underbrace{i e^{it}}_{\gamma'(t)}\,dt$$



$1+2z=0$

$z=-\frac{1}{2}$

$(b-a)=k2\pi$

$b=2\pi$

$a=0$

$$2\pi i \operatorname{Res}(f|-\tfrac{1}{2}) = \int_{|z|=1} \frac{z}{1+2z}\,dz$$

$\boxed{\text{Beispiel}}$

$$\int_0^{2\pi} \frac{e^{it}}{1+2e^{it}}\cdot i e^{it}\,dt$$

$$\underline{z=e^{it}}$$
$t\in[0,2\pi]$

Integrale mit exp, sin, cos

$$\int_{|z|=1} f(z)\,dz \xrightarrow[t\in[0,1]]{\gamma(t):=e^{2\pi i t}} \int_0^1 f(\underbrace{e^{2\pi i t}}_{\gamma(t)})\cdot \underbrace{2\pi i e^{2\pi i t}}_{\gamma'(t)}\,dt$$

$$\xrightarrow[t\in[0,2\pi]]{\gamma(t):=e^{it}} \int_0^{2\pi} f(\underbrace{e^{it}}_{\gamma(t)})\cdot \underbrace{i e^{it}}_{\gamma'(t)}\,dt$$

$$\frac{e^{\cdot t}+e^{\cdot -t}}{2}$$

Bedingung

→ f hat nur $e^{\cdot t}$, $\cos(\cdot t)$, $\sin(\cdot t)$

→ Grenzenintervall = Vielfaches der Periode

$$\boxed{e^{it} := z}$$

Beispiel:

$$\left(\int_0^{4\pi} \frac{1}{2+e^{it}}\, dt = \right.$$

$\hookrightarrow T = 2\pi$

$4\pi - 0 = 4\pi = 2 \cdot 2\pi$

$z = e^{it}$

$2 \times (|z| = 1)$

Beispiel: $\mathrm{Res}(f|0) = \lim_{z\to 0} \cancel{z} \cdot \frac{1}{(2+z)(\cancel{z})} = \lim_{z\to 0} \frac{1}{2+z} = \frac{1}{2}$

$$\int_0^{4\pi} \frac{1}{2+e^{it}}\,dt = \int_{2\times(|z|=1)} \frac{1}{2+z}\,\frac{1}{iz}\,dz = \frac{1}{i}\int_{\underline{\underline{2\times}}(|z|=1)} \frac{1}{(2+z)z}\,dz = \frac{1}{\cancel{i}}\,2\pi\cancel{i}\,\mathrm{Res}(f|0)\cdot 2$$

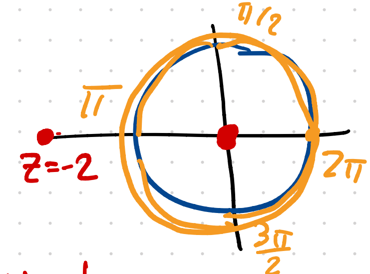
$$= 2\cancel{2}\pi\cdot\frac{1}{\cancel{2}} = \pi\cdot 2$$

$$= \underline{2\pi}$$

$$\boxed{\mathrm{Ind}_\gamma(z=0) = 2}$$

Variablenwechsel

$z = -2$
$z = 0$



1. Variable $z = e^{it}$

2. Grenzen 0 bis 4π für e^{it} entspricht 2 Umkreisungen um den Einheitskreis $\to 2\times(|z|=1)$

3. Integrationslänge $\frac{dz}{dt} = ie^{it} = iz \Rightarrow dt = \frac{1}{iz}\,dz$

$$dz \longleftrightarrow dt$$

$$\boxed{z = e^{it}}$$

$$\int \cdots\cdots \boxed{dt} \qquad dt = \cdots\cdots$$

$$\frac{dz}{dt} = \frac{d}{dt}\left(e^{it}\right) = ie^{it}$$

$$= iz$$

$$\frac{dz}{dt} = iz$$

$$dz = iz \cdot dt$$

$$dt = \frac{1}{iz}\, dz$$

[Aufgabe 6]

Beispiel:

Berechne c_{49} der 1-periodischen Funktion $f(t) = \frac{1}{4-e^{2\pi i t}}$

$$\rightarrow c_k = \frac{1}{T}\int_0^T \frac{1}{4-e^{2\pi i t}} e^{-\frac{i2\pi}{T}kt}\,dt = \int_0^1 \frac{1}{4-e^{2\pi i t}} \underbrace{e^{-2\pi i k t}}_{\frac{1}{(e^{2\pi i t})^k}}\,dt$$

$\rightarrow z = e^{2\pi i t}$, $t \in [0,1] \Rightarrow$ 1 Umkreisung um den Einheitskreis

Variablenwechsel:

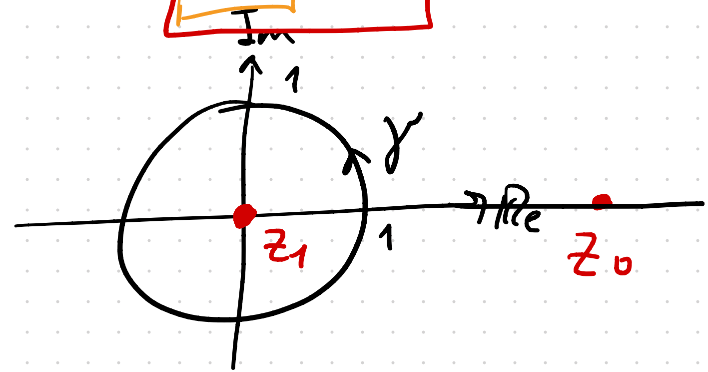
1. Variable: $z = e^{2\pi i t}$
2. Grenzen: für $t \in [0,1]$ für $z = e^{2\pi i t} \Rightarrow$ Einheitskreis (Ind = 1)
3. Integrationslänge: $\frac{dz}{dt} = 2\pi i e^{2\pi i t} = 2\pi i z \Rightarrow dt = \frac{1}{2\pi i z}dz$

$$z = e^{2\pi i t}, \quad |z| = 1, \quad dt = \frac{1}{2\pi i z}\, dz$$

$$\frac{1}{z}\cdot\frac{1}{2\pi i}\cdot\frac{i}{i} = \boxed{\frac{1}{z}\,\frac{-i}{2\pi}}$$

$$\Rightarrow \int_0^1 \frac{1}{4-e^{2\pi i t}}\, \underbrace{e^{-2\pi i k t}}\, dt = \int_{|z|=1} \frac{1}{4-z}\,\frac{1}{z^k}\cdot \boxed{\frac{1}{2\pi i z}\, dz}$$

$$\left(e^{-2\pi i t}\right)^k = \frac{1}{\left(e^{2\pi i t}\right)^k}$$



$$= \frac{i}{2\pi} \int_{|z|=1} \frac{1}{(z-4)\, z^{\boxed{k+1}}}\, dz = -\frac{\cancel{i}}{\cancel{2\pi}} \cdot \cancel{2\pi}\cancel{i}\, \underbrace{\sum_k \mathrm{Res}(f|z_k)}_{\mathrm{Res}(f|0)}$$

$z_0 = 4$

$z_1 = 0 \longrightarrow$ Pol $(k+1)$-te Ordnung

$$\frac{i}{2\pi}\int_{|z|=1}\frac{1}{(z-4)\,z^{k+1}}\,dz$$

$$\frac{1}{z-4}=\boxed{(z-4)^{-1}}$$

$$\mathrm{Res}(f|0)=\frac{1}{k!}\lim_{z\to 0}\frac{d^k}{dz^k}\left[\cancel{z^{k+1}}\frac{1}{(z-4)\cancel{z^{k+1}}}\right]=\frac{1}{k!}\lim_{z\to 0}\frac{d^k}{dz^k}\Big(\underbrace{(z-4)^{-1}}_{g(z)}\Big)$$

$$\left.\begin{aligned}
g(z)&=+1(z-4)^{-1}\\
g'(z)&=-1(z-4)^{-2}\\
g^{(2)}(z)&=+1\,2\cdot 1(z-4)^{-3}\\
g^{(3)}(z)&=-1\,3\cdot 2\cdot 1(z-4)^{-4}\\
g^{(4)}(z)&=4\cdot 3\cdot 2\cdot 1(z-4)^{-5}
\end{aligned}\right\}\quad g^{(k)}(z)=(-1)^k k!\,(z-4)^{-(k+1)}$$

$|m=k+1)$ Pol m-te Ordnung:

$$\boxed{\mathrm{Res}(f|z_i)=\frac{1}{(m-1)!}\lim_{z\to z_i}\frac{\partial^{m-1}}{\partial z^{m-1}}\left[(z-z_i)^m f(z)\right]}$$

$$\mathrm{Res}(f|0) = \frac{1}{k!} \lim_{z\to 0} \frac{d^k}{dz^k}\left[\cancel{z^{k+1}} \frac{1}{(z-4)\cancel{z^{k+1}}}\right] = \frac{1}{k!}\lim_{z\to 0} \boxed{\frac{d^k}{dz^k}\Big(\underbrace{(z-4)^{-1}}_{g(z)}\Big)}$$

$$g^{(k)}(z) = (-1)^k k!\,(z-4)^{-(k+1)}$$

$$\Rightarrow \mathrm{Res}(f|0) = \frac{1}{\cancel{k!}} \lim_{z\to 0} (-1)^k \cancel{k!}\,(z-4)^{-(k+1)}$$

$$= (-1)^k(-4)^{-(k+1)}$$

$$\mathrm{Res}(f|0) = (-1)^k \, (-4)^{-(k+1)}$$

$$\Rightarrow C_{49} = \frac{i}{2\pi} \int_{|z|=1} \frac{1}{(z-4)\, z^{k+1}} \, dz = \boxed{\frac{-\cancel{i}}{\cancel{2\pi}} \cdot \cancel{2\pi i}} \underbrace{\sum_k \mathrm{Res}(f|z_k)}$$

$\rightarrow \mathrm{Res}(f|0)$

$$= \cancel{-} \cancel{(-1)^{49}} \, \cancel{(-1)}^{-50} \cdot 4^{-50} = \frac{1}{4^{50}}$$

$+$

$$(-4)^{-50} = \boxed{(-1)^{-50}} \cdot 4^{-50}$$

$$\boxed{e^{it}} \rightarrow z = e^{it}$$

$r = 1$

$$\int_0^{2\pi} \frac{\boxed{2e^{it}}}{3-\boxed{2e^{it}}}\,dt \qquad z = \underline{2}e^{it}$$

$[0,2\pi]$ für $2e^{it}$ $\Rightarrow$ Kreis mit Radius 2 am Ursprung (1 Umkreisung)

$$\frac{dz}{dt} = 2ie^{it} = i\cdot z \Rightarrow dt = \frac{1}{iz}\,dz$$

$$\boxed{= \int_{|z|=2} \frac{z}{3-z}\,\frac{1}{iz}\,dz}$$

[Aufgabe 1]

Beispiel:

Bestimme $\int_0^{2\pi} \frac{\cos(3\theta)}{5-4\cos(\theta)}\, d\theta$

$\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}$

$$= \int_0^{2\pi} \frac{\frac{e^{3i\theta}+e^{-3i\theta}}{2}}{5-4\,\frac{e^{i\theta}+e^{-i\theta}}{2}}\, d\theta = \int_0^{2\pi} \frac{\frac{(e^{i\theta})^3+(e^{-i\theta})^3}{2}}{5-4\,\frac{e^{i\theta}+e^{-i\theta}}{2}}\, d\theta \qquad e^{i\theta}$$

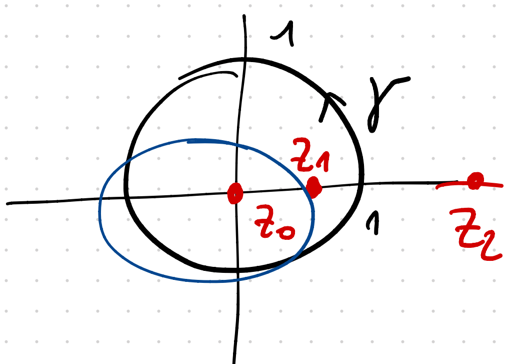
$e^{-i\theta} = \frac{1}{e^{i\theta}}$

→ **Variablenwechsel:**

1. Variable $z = e^{i\theta}$
2. Grenzen $[0,2\pi]$ für $e^{i\theta} \Rightarrow$ Einheitskreis (1x)
3. Integrationslänge $\frac{dz}{dt} = ie^{i\theta} = iz \Rightarrow dt = \frac{1}{iz}\, dz$

$$\int_0^{2\pi} \frac{\dfrac{(e^{i\theta})^3+(e^{-i\theta})^3}{2}}{5-4\,\dfrac{e^{i\theta}+e^{-i\theta}}{2}}\,d\theta = \int_{|z|=1} \frac{\dfrac{\frac{z^3\cdot z^3}{z^3}+\frac{1}{z^3}}{2}}{5-4\left(\dfrac{z+\frac{1}{z}}{2}\right)}\cdot\frac{1}{iz}\,dz$$

$$= \int_{|z|=1} \frac{z^6+1}{2iz^3\underbrace{(5z-2z^2-2)}}\,dz = -\frac{1}{4i}\int_{|z|=1} \frac{z^6+1}{z^3\left(z-\frac{1}{2}\right)(z-2)}\,dz$$



$z_0 = 0$ → Pol 3. Ordnung

$z_1 = \frac{1}{2}$

$z_2 = 2$ } Pol 1. Ordnung

i. $\mathrm{Res}(f|0) = \frac{1}{2!} \lim_{z\to 0} \frac{d^2}{dz^2}\left[z^3 \frac{z^6+1}{(z-\frac{1}{2})(z-2) z^3} \right] = \dots = \frac{21}{4}$

ii. $\mathrm{Res}(f|\frac{1}{2}) = \lim_{z\to 1/2} \cancel{(z-\tfrac{1}{2})} \frac{z^6+1}{z^3(z-2)\cancel{(z-\frac{1}{2})}} = \dots = -\frac{65}{12}$

$$\int_0^{2\pi} \frac{\cos(3\theta)}{5-4\cos(\theta)}\,d\theta = -\frac{1}{\cancel{4i}} \cdot \cancel{2}^{\frac{1}{2}} \pi \cancel{i} \left(\mathrm{Res}(f|0) + \mathrm{Res}(f|\tfrac{1}{2}) \right)$$

$$= -\frac{\pi}{2}\left(\frac{21}{4} - \frac{65}{12} \right) = \frac{\pi}{12}$$

$$\cos(\dots) = \mathrm{Re}\{e^{i\dots}\}$$

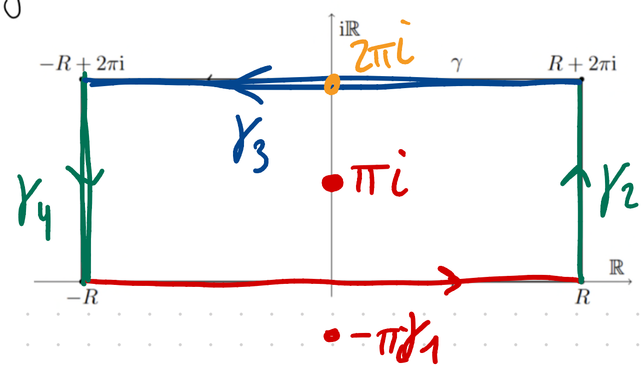
$$\frac{\mathrm{Re}\{\dots\}}{5 - 4\,\mathrm{Re}\{\dots\}} \frac{1}{iz}$$

$$\mathrm{Res}(f|\dots) = \frac{1}{2!} \lim_{z\to\dots} \frac{d^2}{dz^2}\left[\dots \frac{\boxed{\mathrm{Re}\{\dots\}}}{5 - \boxed{4\,\mathrm{Re}\{\dots\}}}\right]$$

$$\boxed{\int \frac{\dots \mathrm{Re}\{\dots\}}{\dots \mathrm{Re}\{\dots\}}} = \mathrm{Re}\left\{ \int \frac{\dots z}{z} \frac{1}{iz}\, dz \right\}$$

Aufgabe 3

Sei $0 < a < 1$. Berechnen Sie das Integral $\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx$ indem Sie über den Weg γ integrieren



$$f(z) := \frac{e^{az}}{1+e^z}$$

$$2\pi i \sum_k \mathrm{Res}(f(z_k)) = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz + \int_{\gamma_4} f(z)\,dz$$

γ_1: $$\int_{\gamma_1} f(z)\,dz = \int_{-R}^{R} f(x)\,dx$$

$$\lim_{R\to\infty} \int_{\gamma_1} f(z)\,dz = \int_{-\infty} \frac{e^{ax}}{1+e^x}\,dx \longleftarrow \text{was wir suchen!}$$

γ_3: $\lim\limits_{R\to\infty} \int_{\gamma_3} f(z)\,dz$ $\qquad \gamma_3(t) = t + 2\pi i, \; t \in [R, -R] \quad [\dot{\gamma}_3(t) = 1]$

$$\int_{\gamma_3} f(z)\,dz = \int_R^{-R} \frac{e^{a(t+2\pi i)}}{1+e^{t+2\pi i}} 1 \cdot dt = -\int_{-R}^{R} \frac{e^{at} \cdot e^{2\pi i a}}{1 + e^t \cdot e^{2\pi i}}\,dt \quad e^{2\pi i} \to 1$$

$$= -e^{2\pi i a} \int_{-R}^{R} \frac{e^{at}}{1+e^t}\,dt$$

was wir suchen!

$$\lim_{R\to\infty} \int_{\gamma_3} f(z)\,dz = -e^{2\pi i a} \boxed{\int_{-\infty}^{\infty} \frac{e^{at}}{1+e^t}\,dt}$$

$$\gamma_2 : \quad \gamma_2(t) = R + 2\pi i t, \quad t \in [0,1] \qquad \dot{\gamma}_2(t) = 2\pi i$$

$$\int_{\gamma_2} f(z)\,dz = \int_0^1 \frac{e^{a(R+2\pi i t)}}{1+e^{R+2\pi i t}} \cdot 2\pi i \, dt = \int_0^1 \frac{e^{aR} \cdot e^{2\pi i t a}}{1+e^{R} \cdot e^{2\pi i t}} 2\pi i \, dt$$

$$\lim_{R\to\infty} \int_{\gamma_2} f(z)\,dz = \lim_{R\to\infty} \left| \int_0^1 \frac{|e^{aR}| \, \overbrace{|e^{2\pi i t a}|}^{1\cdot 1 = 1}}{|1+e^{R} e^{2\pi i t}|} \cdot \left|\underbrace{2\pi i}_{1\cdot 1=1}\right| dt \right| = 0$$

$$\leq \lim_{R\to\infty} \int_0^1 \frac{e^{\boxed{aR}}}{\boxed{1+e^{R} e^{2\pi i t}}} \cdot 2\pi$$

$1 \cdot 1 = 1$ $\quad O(e^{aR})$ $\quad O(e^{R})$

$$\boxed{0 < a < 1}$$

$$\frac{O(e^{aR})}{O(e^{R})} = O\left(e^{(a-1)R}\right)$$

$0 < a < 1 \Rightarrow -1 < a-1 < 0$

$\gamma_4: \quad \gamma_4(t) = -R + 2\pi i t, \quad t \in [1,0] \quad \dot{\gamma}_4(t) = 2\pi i$

$$\int_{\gamma_4} f(z)\,dz = \int_1^0 \frac{\underbrace{e^{-aR}}_{\to 0} \cdot e^{a2\pi i t} \cdot e^R}{1 + \underbrace{e^{-R}}_{\to 0} \cdot e^{2\pi i t} \cdot e^R} \cdot 2\pi i \, dt = 0$$

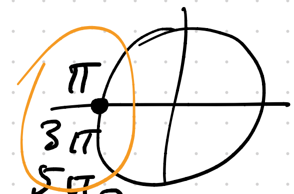
$$\lim_{R\to\infty}\left(2\pi i\sum_k \mathrm{Res}(f|z_k)\right) = \lim_{R\to\infty}\left(\int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz + \int_{\gamma_4} f(z)\,dz\right)$$

$$2\pi i\sum_k \mathrm{Res}(f|z_k) = \int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^x}\,dx + 0 - e^{2\pi i a}\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^x}\,dx + 0$$

<u>Singularitäten:</u> $1+e^x=0 \Rightarrow e^x=-1$

$$\downarrow$$

$$e^{i(\pi+2\pi k)} = -1$$



$\rightarrow (\pi+2\pi k) = \varphi$

$$x = i(\pi+2\pi k),\quad k\in\mathbb{Z}$$

$$z = \pi i,\ 3\pi i,\ 5\pi i,\ -\pi i,\ -3\pi i\ -5\pi i\ \dots$$

$$\mathrm{Res}(f|\pi i), \quad f(z) = \frac{e^{az}}{1+e^{z}} \qquad e^{az} \to h(z_0), \quad 1+e^{z} \to g(z)$$

$$f(z) = \frac{h(z)}{g(z)} \quad \text{mit} \quad h(z_0) \neq 0$$

$$\Rightarrow \mathrm{Res}(f|z_0) = \frac{h(z_0)}{g'(z_0)}$$

$$\mathrm{Res}(f|\pi i) = \boxed{\lim_{z\to\pi i} (z-\pi i)\frac{e^{az}}{1+e^{z}}} \quad \text{l'Hop.}$$

$$g(z) = 1+e^{z}$$

$$g'(z) = e^{z}$$

$$g'(z) = e^{\pi i} = -1$$

$$\mathrm{Res}(f|\pi i) = \frac{h(z_0)}{g'(z_0)} = \frac{e^{a\cdot\pi i}}{-1} = -e^{a\pi i}$$

$$\lim_{R\to\infty}\left(2\pi i \sum_k \mathrm{Res}(f|z_k)\right) = \lim_{R\to\infty}\left(\int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \int_{\gamma_3} f(z)\,dz + \int_{\gamma_4} f(z)\,dz\right)$$

$$2\pi i \sum_k \mathrm{Res}(f|z_k) = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx + 0 - e^{2\pi i a}\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^x}\,dx + 0$$

$$\mathrm{Res}(f|\pi i) = -e^{a\pi i}$$

$$\Rightarrow 2\pi i\left(-e^{a\pi i}\right) = 1\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^x}\,dx - e^{2\pi i a}\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^x}\,dx$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^x}\,dx = \frac{-2\pi i e^{a\pi i}}{1-e^{2\pi i a}}$$

Log, log

$$z = re^{i\varphi} = re^{i\varphi + 2\pi i k},\ k \in \mathbb{Z}$$

$$= \ln(|z|) + i\varphi + 2\pi i k,\ k \in \mathbb{Z}$$

$\times$ $\ln(a \cdot b) = \ln(a) + \ln(b)$

$\times$ $a \ln(b) = \ln(b^a)$

Hauptwert $\mathrm{Log}(z)$

$$\boxed{\varphi \in [-\pi, \pi]}$$

$\mathrm{Log}(a \cdot b) = \mathrm{Log}(a) + \mathrm{Log}(b)$

log

$$\log(-1+i) + \log(-1+i) = 2\log(-1+i) = 2\left(\log(\sqrt{2}) + \frac{3\pi}{4} i\right) = \log(2) + \frac{3\pi}{2} i$$

$$\log(-1+i) + \log(-1+i) = \log((-1+i)(-1+i)) = \log(-2i) = \log(2) - \frac{\pi}{2} i$$

$\neq$

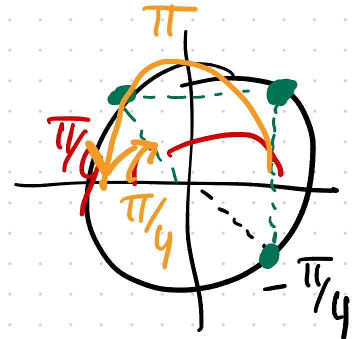
$2\log(-1+i)$ → $\boxed{\mathrm{Re}<0}$

$\mathrm{Im}>0$

$r = \sqrt{1^2+1^2} = \sqrt{2}$

$\varphi = \arctan\left(\frac{1}{-1}\right) = -\cancel{\frac{\pi}{4}}$ $\quad \frac{3\pi}{4}$

arctan $\in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



$\pi - \frac{\pi}{4} = \frac{3\pi}{4}$

	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
sin	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
tan	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

Aufgabe 12 Finde $Y(s)$

$$\mathcal{L}\left\{\frac{d^2}{dx^2}y(x) + 2\underbrace{\frac{d}{dx}y(x)} + y(x)\right\} = \mathcal{L}\left\{e^{-5x}\sin(x)\right\},\quad y(0)=1,\ \boxed{y'(0)=2}$$

$$\Rightarrow \underbrace{\mathcal{L}\{y''(x)\}} + 2\underbrace{\mathcal{L}\{y'(x)\}} + \underbrace{\mathcal{L}\{y(x)\}}_{Y(s)} = \mathcal{L}\{e^{-5}\sin(x)\}$$

$$2\cdot\left(s\cdot\mathcal{L}\{y(x)\} - \underbrace{\boxed{y(0)}}_{1}\right)$$

$$\left.\frac{d}{dx}y(x)\right|_{x=0} = y'(0) = 2$$

$$\mathcal{L}\left\{\boxed{\frac{d}{dx}}\underbrace{\frac{d}{dx}y(x)}_{f(x) := \frac{d}{dx}y(x)}\right\} = \mathcal{L}\left\{\frac{d}{dx}f(x)\right\} = s\cdot F(s) - f(0)$$

$$\frac{d}{dt}f(t) \;\circ\!\!-\!\!\bullet\; sF(s) - f(0)$$

$$t^n e^{-at} \;\circ\!\!-\!\!\bullet\; \frac{n!}{(s+a)^{n+1}}$$

$$\Rightarrow \underbrace{\mathcal{L}\{y''(x)\}}_{\downarrow} + 2\underbrace{\mathcal{L}\{y'(x)\}}_{2\cdot(Y(s)-1)} + \underbrace{\mathcal{L}\{y(x)\}}_{Y(s)} = \mathcal{L}\{e^{-5}\sin(x)\}$$

$$\mathcal{L}\left\{\frac{d}{dx}\underbrace{\frac{d}{dx}y(x)}_{f(x) := \boxed{\frac{d}{dx}y(x)}}\right\} = \mathcal{L}\left\{\frac{d}{dx}f(x)\right\} = s\cdot F(s) - \overset{2}{f(0)} = sF(s) - 2 = s(sY(s)-1) - 2$$

$$= s^2Y(s) - s - 2$$

$$\boxed{F(s)} = \mathcal{L}\left\{\frac{d}{dx}y(x)\right\} = sY(s) - \underbrace{y(0)}_{1} = \boxed{sY(s) - 1}$$

$$\left[\begin{array}{lcl} \frac{d}{dt}f(t) & \circ\!\!-\!\!\bullet & sF(s) - f(0) \\ t^n e^{-at} & \circ\!\!-\!\!\bullet & \frac{n!}{(s+a)^{n+1}} \end{array}\right]$$

$$\Rightarrow \underbrace{\mathcal{L}\{y''(x)\}}_{s^2 Y(s) - s - 2} + 2\underbrace{\mathcal{L}\{y'(x)\}}_{2\cdot(Y(s)-1)} + \underbrace{\mathcal{L}\{y(x)\}}_{Y(s)} = \underbrace{\mathcal{L}\{e^{-5x}\sin(x)\}}_{\vdots}$$

$$s^2 Y(s) - s - 2 + 2Y(s) - 2 + Y(s) = \mathcal{L}\{e^{-5x}\sin(x)\}$$

$$Y(s)(s^2+3) - s - 4 = \mathcal{L}\{e^{-5}\sin(x)\}$$

e^{-x}

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

Ende?