Solution of a System

→ Existence/Uniqueness: If f is Lipschitz then
$\dot{x}(t) = f(x(t))$, $x(\cdot):[0,T] \to \mathbb{R}$ has a unique solution for all T.

Lipschitz: A function $f: \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz if
$\exists \lambda > 0$ s.t. $\forall x, \hat{x} \in \mathbb{R}^n \quad \|f(x) - f(\hat{x})\| \leq \lambda \|x - \hat{x}\|$

→ All differentiable functions with bounded derivatives are Lipschitz

→ Linear functions are always Lipschitz

→ State solution:

$$x(t) = \underbrace{\Phi(t)x_0}_{\substack{\text{zero input}\\ \text{transition}\\ \text{(ZIT)}}} + \underbrace{\int_0^t \Phi(t-\tau)Bu(\tau)\,d\tau}_{\substack{\text{zero state}\\ \text{transition}\\ \text{(ZST)}}}, \quad \Phi(t) = e^{At}$$

Total transition

→ Output solution:

$$y(t) = \underbrace{C\Phi(t)x_0}_{\substack{\text{zero input}\\ \text{response}\\ \text{(ZIR)}}} + \underbrace{\int_0^t C\Phi(t-\tau)Bu(\tau)\,d\tau + Du(t)}_{\substack{\text{zero state}\\ \text{response}\\ \text{(ZSR)}}}$$

Total response

1. $\Phi(0) = I$
2. $\frac{d}{dt}\Phi(t) = A\Phi(t)$
3. $\Phi(-t) = [\Phi(t)]^{-1}$
4. $\Phi(t_1 + t_2) = \Phi(t_1)\Phi(t_2)$

System in state space form

$$\dot{x}(t) = Ax(t) + Bu(t) = f(x(t), u(t))$$
$$y(t) = Cx(t) + Du(t) = h(x(t), u(t))$$

→ Time invariant: dynamics do not explicitly depend on time
$\dot{x}(t) = f(x(t), u(t))$, $y(t) = h(x(t), u(t))$

→ Autonomous: time invariant and no input, $u(t) = 0$
$\dot{x}(t) = f(x(t))$, $y(t) = h(x(t))$

→ linear: f and h are linear

$$\Rightarrow \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

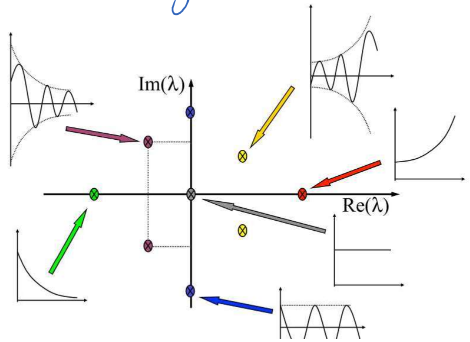
Stability

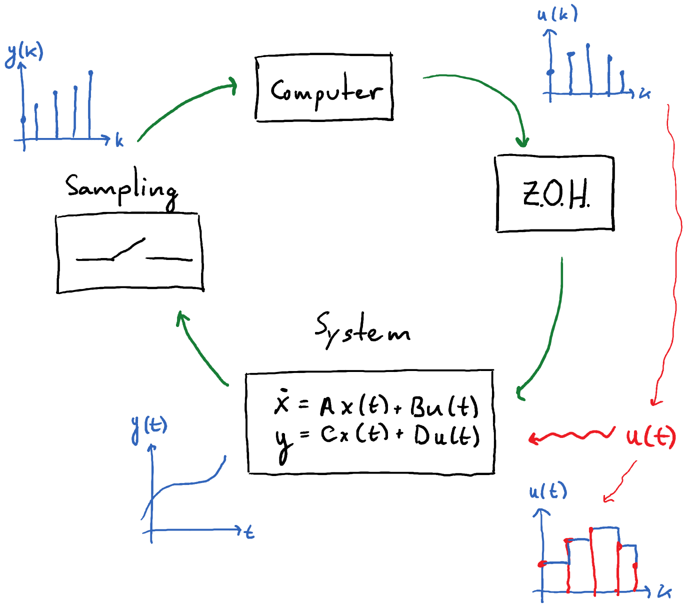
→ stable: The system is stable if
$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. if $\|x_0\| < \delta \Rightarrow \|x(t)\| \leq \varepsilon \ \forall t \geq 0$
otherwise the system is called unstable

→ Asymptotically stable: $\|x(t)\| \to 0$ for $t \to \infty$

→ For diagonalizable matrix A with eigenvalues $\bar{\lambda}_i$

- $\mathrm{Re}\{\bar{\lambda}_i\} < 0 \Rightarrow$ Asymptotically stable
- $\mathrm{Re}\{\bar{\lambda}_i\} \leq 0 \Rightarrow$ stable
- $\exists i$ with $\mathrm{Re}\{\bar{\lambda}_i\} > 0 \Rightarrow$ unstable





Continuous-time (CT)

$$\dot x(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$
$$u(t) = u_k \ (= \text{constant}) \quad t \in [kT, kT+T]$$

→ consider: $t \in [kT, kT+T]$

$$\Rightarrow x(t) = e^{A(t-kT)}x(kT) + \int_{kT}^{t} e^{A(t-\tau)} B u(\tau)\, d\tau$$

$$x(kT+T) = e^{AT}x(kT) + \int_{kT}^{kT+T} e^{A((k+1)T-\tau)} B\, d\tau \cdot u(kT)$$

$$x(kT+T) = \underbrace{e^{AT}}_{A} x(kT) + \underbrace{\int_{0}^{T} e^{A(T-\tau)} B\, d\tau}_{B} \cdot \underbrace{u(kT)}_{u_k}$$

$$\rightarrow y(kT) = \underbrace{C}_{C} x(kT) + \underbrace{D}_{D}\, \underbrace{u(kT)}_{u_k}$$

Discrete-time (DT)

$$x_{k+1} = Ax_k + Bu_k$$
$$y_k = Cx_k + Du_k \qquad k = 0, 1, \dots, N-1$$

Solution: $x_k = A^k \hat{x}_0 + \sum_{i=0}^{k-1} A^{k-i+1} B u_i$

$$y_k = Cx_k + Du_k$$

here is $A = e^{\tilde{A}T}$, where $\hat{A}$ is the A-matrix of the CT-System

Discrete-time (DT)

$$x_{k+1} = Ax_k + Bu_k$$
$$y_{k+1} = Cx_k + Du_k$$

Assume A diagonalizable

↳ compute eigenvalues λ_i

- $|\lambda_i| < 1 \Rightarrow$ Assymptotically stable
- $|\lambda_i| \leq 1 \Rightarrow$ stable
- $|\lambda_i| > 1 \Rightarrow$ unstable

Continuous-time (CT)

$$\dot x(t) = \bar{A}x(t) + \bar{B}u(t)$$
$$y(t) = \bar{C}x(t) + \bar{D}u(t)$$

Assume $\bar{A}$ diagonalizable

↳ compute eigenvalues $\bar{\lambda}_i$

- $\mathrm{Re}\{\bar{\lambda}_i\} < 0 \Rightarrow$ Assymptotically stable
- $\mathrm{Re}\{\bar{\lambda}_i\} \leq 0 \Rightarrow$ stable
- $\mathrm{Re}\{\bar{\lambda}_i\} > 0 \Rightarrow$ unstable

Discrete-time (DT)

$$A = e^{\bar{A}T}, \quad B = 0$$
$$A = e^{\bar{A}T} = e^{W\bar{\Lambda}W^{-1}T}$$
$$= W e^{\bar{\Lambda}T} W^{-1} = W \Lambda W^{-1}$$
$$\Rightarrow \Lambda = \begin{bmatrix} e^{\bar{\lambda}_1 T} & & 0 \\ & e^{\bar{\lambda}_2 T} & \\ 0 & & \ddots \end{bmatrix}$$
$$= \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \end{bmatrix}$$

Continuous-time (CT)

$$\bar{A} = W\bar{\Lambda}W^{-1} \rightarrow \bar{\Lambda} = \begin{bmatrix} \bar{\lambda}_1 & & 0 \\ & \bar{\lambda}_2 & \\ 0 & & \ddots \end{bmatrix}$$

$$\bar{\lambda}_i = \begin{cases} \bar{\sigma}_i \in \mathbb{R} & \text{if } \bar{\lambda}_i \in \mathbb{R} \\ \bar{\sigma}_i + j\bar{\omega}_i \in \mathbb{C} & \text{if } \bar{\lambda}_i \in \mathbb{C} \end{cases}$$

$$\dot x(t) = \bar{A}x(t), \quad B = 0$$

↳ sample every T

$$\lambda_i = e^{\bar{\lambda}_i T} = \begin{cases} e^{\bar{\sigma}_i T} & \text{if } \bar{\lambda}_i \in \mathbb{R} \\ e^{\bar{\sigma}_i T} \cdot e^{j\bar{\omega}_i T} \in \mathbb{C} & \text{if } \bar{\lambda}_i \in \mathbb{C} \end{cases} \Rightarrow |\lambda_i| = \begin{cases} |e^{\bar{\sigma}_i T}| \\ |e^{\bar{\sigma}_i T}|\,|e^{j\bar{\omega}_i T}| \end{cases} = |e^{\bar{\sigma}_i T}| = \begin{cases} < 1, & \text{if } \bar{\sigma}_i < 0 \Rightarrow \text{assymptotically stable} \\ \leq 1, & \text{if } \bar{\sigma}_i \leq 0 \Rightarrow \text{stable} \\ > 1, & \text{if } \bar{\sigma}_i > 0 \Rightarrow \text{unstable} \end{cases}$$

$\Longrightarrow$ if CT un(stable) $\Rightarrow$ DT un(stable)

Linear Algebra

→ **Inverse Matrix:** The inverse of a matrix $A \in \mathbb{R}^{n\times n}$ is a matrix $A^{-1} \in \mathbb{R}^{n\times n}$ with

$$\boxed{AA^{-1} = A^{-1}A = I}$$

- A is invertible if $\det(A) \neq 0$
- If it exists, it is unique

special case 2x2 Matrix → $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \underbrace{\frac{1}{ad-bc}}_{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

→ **Determinant**

i. 2x2 Matrix → $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \det(A) = ad - bc$

ii. 3x3 Matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \Rightarrow \det(A) = a\cdot e\cdot i + b\cdot f\cdot g + c\cdot d\cdot h - g\cdot e\cdot c - h\cdot f\cdot a - i\cdot d\cdot b$$

$$\left[\begin{array}{ccc|cc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array}\right]$$

$- \quad - \quad -$

$+ \quad + \quad +$

→ **Exponential Matrix e^{At}**

- definition: $e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}$

i. Diagonalizable Matrix → $A = W\Lambda W^{-1}$

$$\Rightarrow e^{At} = \sum_{k=0}^{\infty} \frac{(W\Lambda W^{-1}t)^k}{k!} = W \sum_{k=0}^{\infty} \frac{(\Lambda t)^k}{k!} W^{-1} = We^{\Lambda t}W^{-1}$$

$$\text{where} \quad e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & \underline{0} \\ & e^{\lambda_2 t} & \\ \underline{0} & & \ddots \quad e^{\lambda_n t} \end{bmatrix}$$

ii. Nilpotent Matrix → where $A^k = 0$ for some finite k.

$$\Rightarrow e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots + \frac{(At)^{k-1}}{(k-1)!}$$

- A matrix is nilpotent if and only if all its eigenvalues are zero

iii. If $A = A_1 + A_2$ and $A_1A_2 = A_2A_1$

$$\Rightarrow e^{At} = e^{(A_1+A_2)t} = e^{A_1t}\cdot e^{A_2t} = e^{A_2t}\cdot e^{A_1t}$$

→ example: $A = \begin{bmatrix} k & 1 & 1 \\ 0 & k & 1 \\ 0 & 0 & k \end{bmatrix} = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

→ **Eigenvalues**

- λ_i for which $\det(\lambda I - A) = 0$
- Diagonalizable

$$A = W\Lambda W^{-1} \rightarrow W = \begin{bmatrix} \underline{\lambda_1} & \underline{\lambda_2} & \underline{\lambda_3} \cdots \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \quad \lambda_n \end{bmatrix}$$

→ **Change of variables**

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \longrightarrow \hat{x}(t) := Tx(t)$$

$(T \in \mathbb{R}^{n\times n},\ T \text{ invertible})$

$$\Downarrow$$

$$\begin{aligned} \dot{\hat{x}}(t) &= \hat{A}x(t) + \hat{B}u(t) = TAT^{-1}\hat{x}(t) + TBu(t) \\ y(t) &= \hat{C}x(t) + \hat{D}u(t) = CT^{-1}\hat{x}(t) + Du(t) \end{aligned}$$

$$\boxed{\hat{A} = TAT^{-1}, \quad \hat{B} = TB, \quad \hat{C} = CT^{-1}, \quad \hat{D} = D}$$

Controllability

Definition: The system is controllable over $[0,t]$ if for all $x(0)=x_0 \in \mathbb{R}^n$ initial conditions and all terminal $x_1 \in \mathbb{R}^n$ conditions there exists an input $u(\cdot):[0,t] \to \mathbb{R}^m$ such that $x(t)=x_1$ [output irrelevant → we only define it for states]

⇒ For any x_0, x_1 we can find a $u(\cdot):[0,t]\to\mathbb{R}^m$ such that $x(0)=0$ and $x(t)=x_1$ [but not necessarily keep it at x_1]

Controllability gramian

$$W_c(t) := \int_0^t e^{A\tau} BB^T e^{A^T\tau}\, d\tau$$

Controllability matrix

$$P = [B \;\; AB \; \cdots \; A^{n-1}B] \in \mathbb{R}^{n\times nm}$$

System controllable ⇒ $W_c(t)$ invertible ⇔ P full rank (n)

(invertibe ⇔ full rank ⇔ det ≠ 0)

→ Assume the system is controllable. Given $x_1 \in \mathbb{R}^n$ and $t > 0$, the input that drives the system from $x(0)=0$ to $x(t)=x_1$ and has minimum energy is given by

$$u_m(\tau) = B^T e^{A^T(t-\tau)} W_c(t)^{-1} x_1 \quad \text{for } \tau \in [0,t]$$

– Energy used in this process: $J = x_1^T W_c^{-1}(t)\, x_1$

Observability

Definition: The system is observable over $[0,t]$ if given $u(\cdot):[0,t]\to\mathbb{R}^m$ and $y(\cdot):[0,t]\to\mathbb{R}^p$ we can uniquely determine the value of $x(\cdot):[0,t]\to\mathbb{R}^n$

Observability gramian

$$W_o(t) = \int_0^t e^{A^T\tau} C^T C e^{A\tau}\, d\tau$$

Observability matrix

$$Q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathbb{R}^{np\times n}$$

System controllable ⇒ $W_c(t)$ invertible ⇕ P full rank (n)

(invertibe ⇔ full rank ⇔ det ≠ 0)

→ Given the initial conditions of y and u we can find $x(0)$

$$\begin{bmatrix} y(0) \\ \dot y(0) \\ \vdots \\ y^{(n-1)}(0) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x(0) + \begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \cdots & 0 \\ CAB & CB & \cdots & 0 \\ CA^{n-2}B & CA^{n-3}B & \cdots & D \end{bmatrix} \begin{bmatrix} u(0) \\ \dot u(0) \\ \vdots \\ u^{(n-1)}(0) \end{bmatrix}$$

$$Y = Qx(0) + KU \qquad Y \in \mathbb{R}^{np}, K \in \mathbb{R}^{np\times nm}, U \in \mathbb{R}^{nm}$$

↳ solve for $x(0)$

Laplace Transform

→ Convert time function $f(t)$ to a complex variable function $F(s)$

$f:\mathbb{R}\to\mathbb{R}$, $F:\mathbb{C}\to\mathbb{C}$ $\qquad f(t) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} F(s) \qquad F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t) e^{-st}\, dt$

$e^{-at} f(t)$	$F(s+a)$		$\delta(t)$	1
$\frac{d}{dt} f(t)$	$sF(s) - f(0)$		1	$1/s$
$(f * g)(t)$	$F(s)G(s)$		$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$		$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) \quad \text{and} \quad \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

$$\dot x(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$
$x(t)\in\mathbb{R}^n$, $u(t)\in\mathbb{R}^m$, $y(t)\in\mathbb{R}^p$, $t\in\mathbb{R}$

$\mathcal{L}\{LTI\}$

$$X(s) = (sI-A)^{-1}x_0 + (sI-A)^{-1}BU(s)$$
$$Y(s) = CX(s) + DU(s) = G(s)U(s)$$
$$G(s) = C(sI-A)^{-1}B + D$$
$X(s)\in\mathbb{C}^n$, $U(s)\in\mathbb{C}^m$, $Y(s)\in\mathbb{C}^p$, $s\in\mathbb{C}$

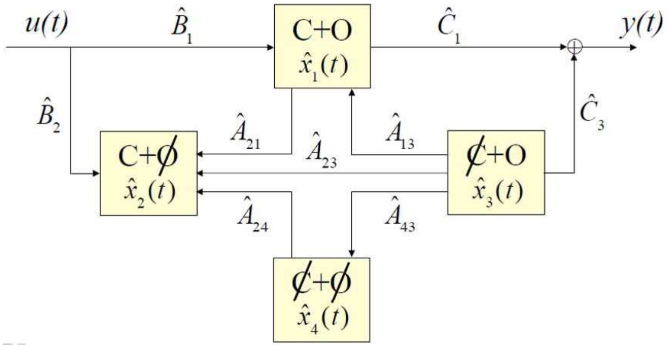
Kalman Decomposition

→ There exists a change of coordinates $T \in \mathbb{R}^{n \times n}$ invertible such that

$$\hat{x}(t) = T\,x(t) = \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \\ \hat{x}_3(t) \\ \hat{x}_4(t) \end{bmatrix} \begin{matrix} \leftarrow \text{controllable \& observable} \\ \leftarrow \text{controllable \& unobservable} \\ \leftarrow \text{uncontrollable \& observable} \\ \leftarrow \text{uncontrollable \& unobservable} \end{matrix}$$

$$\hat{A} = TAT^{-1} = \begin{bmatrix} \hat{A}_{11} & 0 & \hat{A}_{13} & 0 \\ \hat{A}_{21} & \hat{A}_{22} & \hat{A}_{23} & \hat{A}_{24} \\ 0 & 0 & \hat{A}_{33} & 0 \\ 0 & 0 & \hat{A}_{43} & \hat{A}_{44} \end{bmatrix}, \quad \hat{B} = TB = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{C} = CT^{-1} = \begin{bmatrix} \hat{C}_1 & 0 & \hat{C}_3 & 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} \hat{A}_{11} & 0 \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}, \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix} \right) \text{ controllable} \qquad \left(\begin{bmatrix} \hat{A}_{11} & \hat{A}_{13} \\ 0 & \hat{A}_{33} \end{bmatrix}, \begin{bmatrix} \hat{C}_1 & \hat{C}_3 \end{bmatrix} \right) \text{ observable}$$



Detectability

→ **Definition:** The System is detectable if all eigenvalues of $\hat{A}_{22}$ and $\hat{A}_{44}$ in the Kalman Decomposition have negative real part

(Can design observer for observable part with overall observation error decaying to zero)

Stabilizability

→ **Definition:** The system is stabilizable if all eigenvalues of $\hat{A}_{33}$ and A_{44} in the Kalman Decomposition have negative real part

(Can design controller for controllable part which ensures overall system asymptotically stable)

1. Find $\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4$ states

$\hat{x}_1 \Rightarrow$ state x_i with $B_i \neq 0$, $C_i \neq 0$
$\hat{x}_2 \Rightarrow$ state x_i with $B_i = 0$, $C_i \neq 0$
$\hat{x}_3 \Rightarrow$ state x_i with $B_i \neq 0$, $C_i = 0$
$\hat{x}_4 \Rightarrow$ state x_i with $B_i \neq 0$, $C_i \neq 0$

→ Example:

$$\dot{x}(t) = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} x_1 \text{ is controllable \& observable} \Rightarrow x_1 \text{ is } \hat{x}_1 \\ x_2 \text{ is uncontrollable \& observable} \Rightarrow x_2 \text{ is } \hat{x}_3 \end{matrix}$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t)$$

2. Find the $\hat{A}_{ij}$ values (in the example we can only find $\hat{A}_{ij}$ with $i,j \in \{1,3\}$) → $\hat{A}_{11} = 2$, $\hat{A}_{13} = 1$, $\hat{A}_{31} = 0$, $\hat{A}_{33} = 0$

3. Find conclusion for stabilizability / detectability

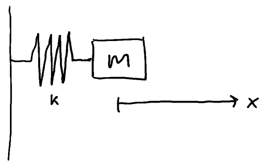
→ Since we don't know $\hat{A}_{22}$ or $\hat{A}_{44}$ we cannot prove detectability but $\hat{A}_{33} = 0 \Rightarrow$ not stabilizable

* This only works if the matrix is already the Kalman decomposition or all other values satisfy the conditions for the decomposition

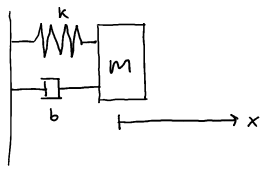
$\hat{A}_{12} = \hat{A}_{14} = \hat{A}_{31} = \hat{A}_{32} = \dots \overset{!}{=} 0$

i. Mechanical systems

– Translational motion



Spring force $F_k = -kx$

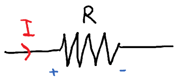


Spring force $F_k = -kx$

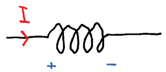
Damper force $F_b = -b\dot{x}$

$$m\ddot{x} = -kx - b\dot{x}$$

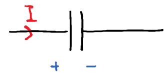
ii. Electrical systems



$V = R \cdot I$

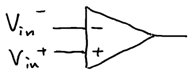


$V = L \frac{dI}{dt}$



$I = C \frac{dV}{dt}$

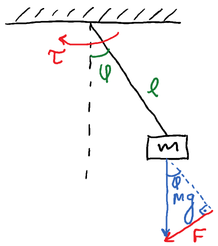
Op-Amp



$V_{out} = M\,(V_{in}^{+} - V_{in}^{-})$

with $M \to \infty$

iii. Rotational motion



$\sum \tau = I \cdot \ddot{\varphi}, \quad I = \int_M r^2\, dm$

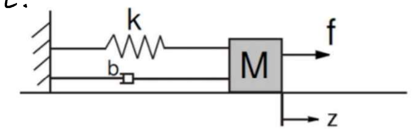
$\tau = F \cdot \ell$

(point mass $\Rightarrow I = m\ell^2$)

$$F = mg\sin(\varphi) = \frac{-m\ell^2\ddot{\varphi}}{\ell} \Rightarrow m\ell^2\ddot{\varphi} = -mg\ell\sin(\varphi)$$

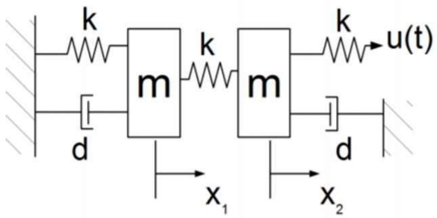
Examples:

i.



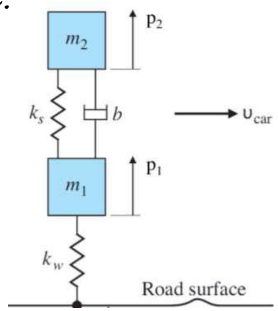
$$M\ddot{z} = f - kz - b\dot{z}$$

ii.



$$\begin{cases} m\ddot{x}_1 = -kx_1 - d\dot{x}_1 + k(x_2 - x_1) \\ m\ddot{x}_2 = -k(x_2 - x_1) + k(u - x_2) - d\dot{x}_2 \end{cases}$$

iii.



$$\begin{cases} m_1\ddot{p}_1 = -k_w p_1 + k_s(p_2 - p_1) + b(\dot{p}_2 - \dot{p}_1) - m_1 g \\ m_2\ddot{p}_2 = k_s(p_1 - p_2) + b(\dot{p}_1 - \dot{p}_2) - m_2 g \end{cases}$$

Transfer Function

→ $Y(s) = G(s)\,U(s)$ → $G(s) = \dfrac{(s-z_1)(s-z_2)\cdots(s-z_k)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \in \mathbb{C}$ zeros poles

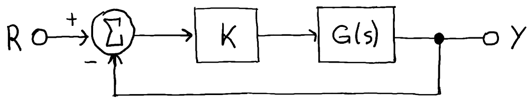
Stability [no zero-pole cancellation]

↳ Zero-pole cancellation ⇒ not observable/controllable

- Assymptotically stable: $\mathrm{Re}\{p_i\} < 0 \ \forall i$
- unstable: $\exists i$ s.t. $\mathrm{Re}\{p_i\} > 0$
- stable:
  - → Single poles: $\mathrm{Re}\{p_i\} \leq 0 \ \forall i$
  - → Repeated poles: $\mathrm{Re}\{p_i\} \leq 0 \ \forall i$ $\exists i$ s.t. $\mathrm{Re}\{p_i\} = 0$

only stable if A has full rank ↵

→ Special case: Feedback Systems



$$E(s) = \frac{1}{1+KG(s)}R(s) = H(s)R(s) = \frac{1}{F(s)}R(s)$$

Nyquist stability criterion: Consider D-Curve, map it under $F(s) = 1 + KG(s)$ ⇒ L-Curve

1. N = number of times L curve encircles (0,0) in clockwise direction
2. Z = Number of closed loop poles (poles of H(s), zeros of F(s)) with positive real part
3. P = Number of open loop poles (poles of G(s), zeros of H(s)) with positive real part

System is stable ⇒ N = Z − P

→ As $s_0 \in \mathbb{C}$ moves around, $G(s_0) \in \mathbb{C}$ also moves

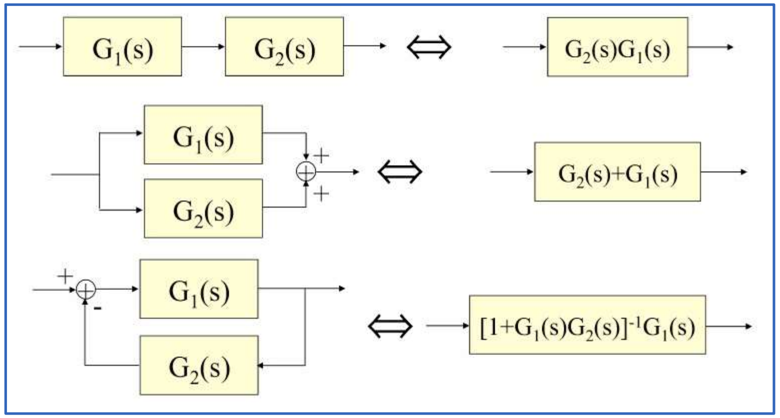
→ If s_0 travels around a closed curve $C \subseteq \mathbb{C}$, $G(s_0) \in \mathbb{C}$ will travel around another closed curve $L \subseteq \mathbb{C}$

[we normally use the D-Curve as C → L = Nyquist-diagram]

Principle of the argument: Assume that the curve C does not pass through any poles or zeros of G(s). Let

1. N = number of times L encircles (0,0) in the clockwise direction
2. Z = number of zeros of G(s) encircled by C
3. P = number of poles of G(s) encircled by C

Then N = Z − P



↳ Alternatives:

- Map D-Curve with KG(s) instead of 1+KG(s)
  → count encirclements of (−1, 0)
- Map D-Curve with G(s) instead of 1+KG(s)
  → count encirclements of $(-1/K, 0)$

Nonlinear Systems

→ We cannot bring the system to the normal state form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

$$\boxed{\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t), u(t)) \end{aligned}} \quad \begin{aligned} & x(t) \in \mathbb{R}^n, \ u(t) \in \mathbb{R}^m, \ y(t) \in \mathbb{R}^p \\ & f: \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n, \ h: \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^p \end{aligned}$$

→ autonomous ⇒ $\dot{x}(t) = f(x(t))$, $y(t) = h(x(t))$

→ Equilibrium: $\hat{x} \in \mathbb{R}^n$ is calle an equilibrium point of the system if it satisfies

$$\boxed{f(\hat{x}) = 0}$$

- We can shift equilibria with a change of coordinates

$$w(t) := x(t) - \hat{x} \in \mathbb{R}^n \longrightarrow \dot{w}(t) = \dot{x}(t) = f(x(t)) = f(w(t) + \hat{x}) =: \hat{f}(w(t))$$

now equilibrium $\hat{x}$ of $f(x(t))$ is at $\hat{w} = 0$ for $\hat{f}(w(t))$

→ Linearization: We want to locally study an equilibrium $\hat{x}$

$$\dot{x}(t) = f(x(t)), \quad f(\hat{x}) = 0$$

- Taylor expansion ⟶ $f(x) = \overset{0}{f(\hat{x})} + A(x - \hat{x}) + \cdots = A(x - \hat{x})$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad f(x) = \begin{bmatrix} f_1(x_1, x_2, \ldots) \\ f_2(x_1, x_2, \ldots) \\ \vdots \\ f_n(x_1, x_2, \ldots) \end{bmatrix}, \quad A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \in \mathbb{R}^{n \times n}$$

- We can investigate the stability by analyzing $A\big|_{x=\hat{x}}$

i. Locally asymptotically stable → All eigenvalues have negative real part

ii. Unstable → At least one eigenvalue with positive real part

iii. Inconclusive → If there are imaginary/zero eigenvalues

- This is called Lyapunov first or Lyapunov indirect method

→ Lyapunov functions

- function of the state $V(x) = \frac{1}{2} x^T Q x$

- Assume there exists an open set $S \subseteq \mathbb{R}^n$ with $\hat{x} \in S$ and a differentiable function $V(\cdot): \mathbb{R}^n \to \mathbb{R}$.

1. $V(\hat{x}) = 0$
2. $V(x) > 0, \ \forall x \in S$ with $x \neq \hat{x}$
3. $\frac{d}{dt} V(x(t)) \leq 0, \ \forall x \in S$
4. $\frac{d}{dt} V(x(t)) < 0, \ \forall x \in S$

1.–3.: stable

1.–4.: locally asymptotically stable

* if $S = \mathbb{R}^n$ ⇒ „globally asymptotically stable"

$V(x)$ = Lyapunov function

- Derivative along trajectory ⇒ Lie derivative

$$\boxed{\frac{d}{dt} V(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(x(t)) \cdot \frac{dx_i(t)}{dt} = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(x(t)) \cdot f_i(x(t)) = \nabla V(x(t)) f(x(t))}$$

→ Energy and Power

- „Energy like" function: $V(x) = \frac{1}{2} x^T Q x \qquad Q = Q^T > 0$
- „Power" $V(x_{k+1}) = \frac{1}{2} x_{k+1}^T Q x_{k+1}$

$$= \frac{1}{2} x_k^T (A^T Q A) x_k + \frac{1}{2} u_k^T B^T Q B u_k + \frac{1}{2} u_k^T B^T Q A x_k + \frac{1}{2} x_k^T A^T Q B u_k$$

$$\rightarrow V(x_{k+1}) - V(x_k) = \frac{1}{2} x_k^T (A^T Q A - Q) x_k = -\frac{1}{2} x_k^T R x_k$$

$$\Rightarrow R = -(A^T Q A - Q)$$

- If $R = R^T > 0$ then energy decreases all the time

- $|\lambda_i| < 1 \ \forall i$ if and only if for all $R = R^T > 0$ the equation $(A^T Q A - Q) = -R$ has a unique solution with $Q = Q^T > 0$