HOCHSCHILD HOMOLOGY

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1. HOCHSCHILD HOMOLOGY: DEFINITION AND EXAMPLES

We're going to first take a relatively specific example and run with it as long as we can; keep in mind that this can be generalized.

For now, let k be a unital ring, let A be a k-algebra, and let M be an A-bimodule. We consider the module $C_n(A, M)$ defined by

$$C_n(A, M) = M \otimes A^{\otimes n},$$

where all tensors are taken over k. We then have a boundary map, defined as follows.

Definition 1.1. The *Hochschild boundary* is the k-linear map $b : M \otimes A^{\otimes n} \to M \otimes A^{\otimes n-1}$ given by the formula

$$b(m, a_1, \ldots, a_n) = (ma_1, a_2, \ldots, a_n) + \sum_{i=1}^{n-1} (-1)^i (m, a_1, \ldots, a_i a_{i+1}, \ldots, a_n) + (-1)^n (a_n m, a_1, \ldots, a_{n-1}).$$

We will sometimes decompose b in terms of operators $d_i : M \otimes A^{\otimes n} \to M \otimes A^{\otimes n-1}$ given by

$$d_i(m, a_1, \ldots, a_n) = (m, a_1, \ldots, a_i a_i + 1, \ldots, a_n),$$

with appropriate alterations for d_0 and d_n, namely

$$d_0(m, a_1, \ldots, a_n) = (ma_1, a_2, \ldots, a_n)$$

and

$$d_n(m, a_1, \ldots, a_n) = (a_n m, a_1, \ldots, a_{n-1}).$$

For notational simplicity, we will sometimes write a_0 in place of m to denote the element of M.

Fact 1.2. The Hochschild boundary is a boundary map, i.e. $b \circ b = 0$.

Definition 1.3. The *Hochschild complex* is the resulting complex, given by

$$C(A, M) = C_*(A, M) : \cdots \to M \otimes A^{\otimes n} \xrightarrow{b} M \otimes A^{\otimes n-1} \xrightarrow{b} \cdots \cdots \xrightarrow{b} M \otimes A \xrightarrow{b} M,$$

where the module $M \otimes A^{\otimes n}$ is in degree n. The *nth Hochschild homology group* of A with coefficients in M, denoted $H_n(A, M)$, is the nth homology group of the Hochschild complex $(C_*(A, M), b)$. The direct sum

Proposition 1.4. *$H_*(A, M)$ is functorial in M and in A (in a certain sense).*

Proof. The fact that it is functorial in M is straightforward; a bimodule homomorphism $f : M \to M'$ induces a map $f_* : H_*(A, M) \to H_*(A, M')$ via

$$f_*(m, a_1, \dots, a_n) = (f(m), a_1, \dots, a_n).$$

Hochschild homology is also functorial in A in the following sense. Let $g : A \to A'$ be a k-algebra map and M' an A'-bimodule. Then the module M' can be considered as an A-bimodule via g, giving a map $g_* : H_*(A, M') \to H_*(A', M')$ defined by

$$g_*(m, a_1, \dots, a_n) = (m, g(a_1), \dots, g(a_n)).$$

□

The most important case is when $M = A$.

Example 1.5. When taking $M = A$, we write $C_*(A) = C_*(A, A)$, and $HH_*(A) = H_*(A, A)$. Any k-algebra map $f : A \to A'$ induces a homomorphism $f_* : HH_n(A) \to HH_n(A')$. In this case, HH_n is a covariant functor from associative k-algebras to k-modules which respects the product. In other words, it satisfies

$$HH_n(A \times A') = HH_n(A) \oplus HH_n(A').$$

For an A-bimodule M, the group $H_0(A, M)$ is given by

$$H_0(A, M) = M_A = M/\{am - ma \mid a \in A, m \in M\},$$

according to the chain complex definition. If $[A, A']$ is the submodule of A generated by all $[a, a'] = aa' - a'a$, then we further have that $HH_0(A) = A/[A, A]$. If A is commutative, then $HH_0(A) = A$.

Example 1.6. Let $A = k$. Then the Hochschild complex for $M = k$ is

$$\cdots \to k \xrightarrow{1} k \xrightarrow{0} \cdots \xrightarrow{1} k \xrightarrow{0} k.$$

Thus $HH_0(k) = k$, and $HH_n(k) = 0$ for all $n > 0$.

Hochschild homology has a lot of properties that we would want, you know, a homology theory to have. We can define *relative Hochschild homology classes*:

Remark 1.7. We can define Hochschild homology in *a lot* of generality. For most of our purposes k is tacitly assumed to be commutative and unital, but none of the above explicitly said that k should be commutative (gasp), and much of it works in cases where k is not unital. One fun side effect of the noncommutative setting is that we get a *Morita equivalence*, i.e. it is generally true for all r that $H_n(M_r(A), M_r(M)) \cong H_n(A, M)$, where $M_r(-)$ is the ring of $r \times r$ matrices with coefficients in the appropriate module. The Morita equivalence is pretty natural and runs pretty deep.

2. Kähler Differentials

For A unital and commutative, let $\Omega^1_{A/k}$ be the A-module of *Kähler differentials*, generated by the k-linear symbodls da for all $a \in A$, so that $d(\lambda a + \mu b) = \lambda da + \mu db$, with the relation that

$$d(ab) = a(db) + b(da).$$

Note that for all $u \in k$, $du = 0$.

Proposition 2.1. *Let A be a unital, commutative ring. Then there is a canonical isomorphism $HH_1(A) \cong \Omega^1_{A/k}$. If M is a symmetric bimodule, then $H_1(A, M) \cong M \otimes_A \Omega^1_{A/k}$.*

Proof. Since A is commutative, the map $b : A \otimes A \to A$ is trivial. Thus $HH_1(A)$ is the quotient of $A \otimes A$ by the relation

$$ab \otimes c - a \otimes bc + ca \otimes b = 0,$$

since that's precisely the image of the map $b : A \otimes A \otimes A \to A \otimes A$. The map $HH_1(A) \to \Omega^1_{A/k}$ is defined by sending the class of $a \otimes b$ to adb. Note then that

$$\begin{aligned} ab \otimes c - a \otimes bc + ca \otimes b &\mapsto (ab)dc - ad(bc) + (ca)db \\ &= (ab)dc - ((ab)dc + (ac)db) + (ca)db = 0, \end{aligned}$$

so this map is well-defined because of the Kähler relation. But this map is an isomorphism, since the inverse $adb \mapsto a \otimes b$ is also a well-defined module homomorphism, and these are inverses of each other.

The same proof extends to the bimodule case. □

3. Differential Forms

Hochschild (co)homology is closely related to derivations and differential forms (as hinted at by the Kähler example). We'll restrict to the case where k is a field, A is a commutative and unital k-algebra, and M is a unitary A-module.

Definition 3.1. A *derivation* of A with values in M is a k-linear map $D : A \to M$ such that

$$D(ab) = a(Db) + (Da)b$$

for all $a, b \in A$. The module of all derivations is denoted $\mathrm{Der}(A, M)$.

Any element $u \in A$ defines a derivation $ad(u)$ called an *inner derivation* via

$$ad(u)(a) = [u, a] = ua - au.$$

An inner derivation can be extended to $C_n(A, M)$ via the formula

$$ad(u)(a_0, \ldots, a_n) = \sum_{0 \le i \le n} (a_0, \ldots, a_{i-1}, [u, a_i], a_{i+1}, \ldots, a_n),$$

which commutes with the Hochschild boundary.

Proposition 3.2. *Let $h(u) : C_n(A, M) \to C_{n+1}(A, M)$ be the map defined by*

$$h(u)(a_0, \ldots, a_n) = \sum_{0 \le i \le n} (-1)^i (a_0, \ldots, a_i, u, a_{i+1}, \ldots, a_n).$$

Then

$$bh(u) + h(u)b = -ad(u),$$

so $ad(u)_* : H_n(A, M) \to H_n(A, M)$ is the zero map.

Proof. Let h_i be the insertion of u after the ith component, so that $h(u) = \sum_{0 \le i \le n} (-1)^i h_i$. We can similarly write $b = \sum_{i=0}^n (-1)^i d_i$ as above, and then get that

$$d_i h_i - d_i h_{i-1} : (a_0, \ldots, a_n) \mapsto (a_0, a_1, \ldots, a_{i-1}, -[u, a_i], a_{i+1}, \ldots, a_n).$$

Then
$$h(u)b + bh(u) = d_0h_0 - d_{n+1}h_n + \sum_i (d_ih_i - d_ih_{i-1}) = -ad(u),$$
as desired. The last fact follows from homological algebra in this setting. □

Note that the symmetric group S_n acts on $C_n(A, M)$ on the left by permuting the indices of the A coordinates. This extends linearly to an action of the group algebra $k[S_n]$ on $C_n(A, M)$.

Definition 3.3. The *antisymmetrization element* $\varepsilon_n \in k[S_n]$ is the element given by
$$\varepsilon_n = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\sigma.$$

The *antisymmetrization map* is the map
$$\varepsilon_n : M \otimes \Lambda^n A \to C_n(A, M)$$
sending $a_0 \otimes a_1 \wedge \cdots \wedge a_n \mapsto \varepsilon_n(a_0, \ldots, a_n)$.

There is another boundary map on the $M \otimes \Lambda^n A$ side of the world, known as the *Chevalley-Eilenberg* map $\delta : M \otimes \Lambda^n A \to M \otimes \Lambda^{n-1} A$ which is given by the formula

$$\begin{aligned}\delta(a_0 \otimes a_1 \wedge \cdots \wedge a_n) = &\sum_{i=1}^{n} (-1)^i [a_0, a_i] \otimes a_1 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge a_n \\ &+ \sum_{1 \le i < j \le n} (-1)^{i+j-1} a_0 \otimes [a_i, a_j] \wedge a_1 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge \hat{a}_j \wedge \cdots \wedge a_n.\end{aligned}$$

Proposition 3.4. *The following square is commutative.*

$$\begin{array}{ccc} M \otimes \Lambda^n A & \xrightarrow{\varepsilon_n} & C_n(A, M) \\ \downarrow \delta & & \downarrow b \\ M \otimes \Lambda^{n-1} A & \xrightarrow{\varepsilon_{n-1}} & C_{n-1}(A, M) \end{array}$$

Definition 3.5. The A-module of *differential n-forms* is by definition the exterior product
$$\Omega^n_{A/k} = \Lambda^n_A \Omega^1_{A/k}.$$

It is spanned by the elements $a_0 da_1 \wedge \cdots \wedge da_n$ for $a_i \in A$.

We can also define π_n, a map in the other direction, via $\pi_n : C_n(A, M) \to M \otimes_A \Omega^n_{A/k}$ is defined by
$$\pi_n(a_0, \ldots, a_n) = a_0 da_1 \ldots da_n.$$
There is a lemma (computation) taht says that $\pi_n \circ b = 0$.

Theorem 3.6. *Let A be a commutative k-algebra and M an A-module.*

- *The antisymmetrization map induces a canonical map $\varepsilon_n : M \otimes_A \Omega^n_{A/k} \to H_n(A, M)$. If $M = A$, it gives $\varepsilon_n : \Omega^n_{A/k} \to HH_n(A)$.*

- *The map* $\pi_n : H_n(A, M) \to M \otimes_A \Omega^n_{A/k}$ *is functorial in* A *and* M. *If* $M = A$, *it gives* $\pi_n : HH_n(A) \to \Omega^n_{A/k}$.
- *The composite map* $\pi_n \circ \varepsilon_n$ *is multiplication by* $n!$ *on* $M \otimes_A \Omega^n_{A/k}$.
- *(Hochschild-Kostant-Rosenberg) When* A *is finitely presented and smooth over* k, *then these are isomorphisms.*