MAASS WAVEFORMS

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1. THE LAPLACIAN OPERATOR

Two weeks ago, we talked about Hecke eigenforms, namely functions on the upper half-plane that are eigenvectors for Hecke operators. In this case we'll be considering a different operator—in fact, one that is so different it is differential—namely, the Laplacian.

Definition 1.1. The *non-Euclidean Laplacian* Δ is a second-order differential operator acting on functions on the upper half-plane, given by

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

Fact 1.2. The non-Euclidean Laplacian is invariant under the action of $\mathrm{SL}(2, \mathbb{R})$. Explicitly, if $g \in \mathrm{SL}(2, \mathbb{R})$ and f is any smooth function on $\mathcal{H}$, then

$$\Delta(f \circ g) = \Delta(f) \circ g.$$

Definition 1.3. A *Maass form* for $\Gamma(1) = \mathrm{SL}(2, \mathbb{Z})$ is a smooth function f on $\mathcal{H}$ satisfying the following three conditions:

(i) For all $\gamma \in \Gamma(1)$, $f(\gamma(z)) = f(z)$.
(ii) f is an eigenfunction of Δ.
(iii) For some $N \in \mathbb{N}$, $f(x + iy) = O(y^N)$ as $y \to \infty$.

A Maass form f is a *Maass cusp form* if it also satisfies:

(iv) The following integral is 0 for all $z \in \mathcal{H}$.

$$\int_0^1 f(z + x) \mathrm{dx} = 0.$$

Equivalently, the 0th Fourier coefficient of f is 0.

2. EISENSTEIN SERIES

One of the most important examples of Maass forms are the Eisenstein series $E(z, \nu + \frac{1}{2})$, where

$$E(z, s) = \pi^{-s} \Gamma(s) \frac{1}{2} \sum_{m,n \in \mathbb{Z}}^{*} \frac{y^s}{|mz + n|^{2s}}.$$

(We write this as $E(z, \nu + \frac{1}{2})$ for functional equation reasons.) This Eisenstein series has two properties that the previous Eisenstein series we've seen, given by

$$E_k(z) = \frac{1}{2} \sum_{m,n \in \mathbb{Z}}^{*} (mz + n)^{-k}$$

does not. Firstly, $E(z,s)$ is not holomorphic as a function of z, and secondly, it is *automorphic* rather than modular. In other words,

$$E(\gamma(z),s) = E(z,s)$$

for $\gamma \in \mathrm{SL}(2,\mathbb{Z})$.

Theorem 2.1. *$E(z,\nu+\frac{1}{2})$ is a Maass form.*

For the proof, we will need the Fourier expansion of $E(z,\nu+\frac{1}{2})$.

Proposition 2.2. *The Eisenstein series $E(z,s)$ has Fourier expansion*

$$E(z,s) = \sum_{r=-\infty}^{\infty} a_r(y,s)e^{2\pi i r x},$$

where

$$a_0(y,s) = \pi^{-s}\Gamma(s)\zeta(2s)y^s + \pi^{s-1}\Gamma(1-s)\zeta(2-2s)y^{1-s}$$

and for $r \neq 0$,

$$a_r(y,s) = 2|r|^{s-1/2}\sigma_{1-2s}(|r|)\sqrt{y}K_{s-1/2}(2\pi|r|y),$$

with K_s the K-Bessel function

$$K_s(y) = \frac{1}{2}\int_0^\infty e^{-y(t+t^{-1})/2}t^s\frac{\mathrm{d}t}{t}$$

and $\sigma_{1-2s}(r) = \sum_{m|r} m^{1-2s}$.

Seems like a mess, but the K-Bessel function has some good properties. In particular it decays really fast; $|K_s(y)| \leq e^{-y/2}K_{\mathrm{re}(s)}(2)$ for $y > 4$, and it's even in s, so $K_s(y) = K_{-s}(y)$. Also, by integrating by parts we arrive at the recurrence

$$K_s(y) = \frac{y}{2s}(K_{s+1}(y) - K_{s-1}(y)).$$

Proof. Since $E(z,\nu+\frac{1}{2})$ is automorphic, it satisfies condition (i) by definition. For fixed ν it is polynomial in z, so it satisfies condition (iii) as well. Condition (ii) is the hard part. If $\mathrm{Re}(\nu) > 1/2$, then the series definition of $E(z,s)$ converges and we don't have to pass to the analytic continuation, so we can compute the Laplacian directly from the series. First, we note that

$$\Delta y^{\nu+1/2} = -y^2(\nu+1/2)(\nu-1/2)y^{\nu-3/2} = (1/4-\nu^2)y^{\nu+1/2}.$$

Thus $y^{\nu+1/2}$ is an eigenfunction of Δ with eigenvalue $(1/4-\nu^2)$. The Laplacian is invariant under the action of $\mathrm{SL}(2,\mathbb{R})$, so for all $\gamma \in \mathrm{SL}(2,\mathbb{R})$, $\mathrm{im}(\gamma(z))^{\nu+1/2} = \frac{y^{\nu+1/2}}{|cz+d|^{2\nu+1}}$ is also an eigenfunction, with the same eigenvalue. Thus any sum of these is also an eigenfunction with the same eigenvalue, so in particular

$$\Delta E(z,\nu+1/2) = (1/4-\nu^2)E(z,\nu+1/2),$$

when $\mathrm{Re}(\nu) > 1/2$.

For the rest of the values of ν, we'll consider the Fourier expansion (see, I told you we'd need it). We'll have the same approach of taking things term by term; in this case, we'll prove that each Fourier coefficient $a_r(y, \nu+1/2)e^{2\pi irx}$ is an eigenfunction of Δ with eigenvalue $1/4-\nu^2$. When $r=0$ we have

$$a_0(y, \nu+1/2) = A(\nu+1/2)y^{\nu+1/2} + B(\nu+1/2)y^{-\nu+1/2}.$$

Both $y^{\nu+1/2}$ and $y^{-\nu+1/2}$ are eigenfunctions with eigenvalue $1/4-\nu^2$, so a_0 must be as well.

For the terms $r \neq 0$, we want to show that

$$\Delta\sqrt{y}K_\nu(2\pi|r|y)e^{2\pi irx} = (1/4-\nu^2)\sqrt{y}K_\nu(2\pi|r|y)e^{2\pi irx}.$$

Via some calculus (including doing a change of variables $2\pi|r|y \to y$), this is equivalent to

$$\left\{y^2\frac{\partial^2}{\partial y^2} + y\frac{\partial}{\partial y} - (y^2+\nu^2)\right\}K_\nu(y) = 0.$$

This is known as *Bessel's differential equation (of the second kind)*, which is a good context clue that it should hold for Bessel functions. And in fact, differentiating under the integral of

$$K_s(y) = \frac{1}{2}\int_0^\infty e^{-y(t+t^{-1})/2}t^s\frac{\mathrm{d}t}{t}$$

gives exactly the result we want (along with applying our integration-by-parts relation several times).

This completes the proof. □

But wait! As we were doing that example, a differential equation cropped up that we happened to have a solution to. Solutions to differential equations have uniqueness properties, so this should let us characterize Maass forms more generally. Let's instead begin with an arbitrary Maass form f. Since $f(\gamma(x)) = f(x+1) = f(x)$ when $\gamma = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, we know that f has a Fourier expansion

$$f(z) = \sum_{r=-\infty}^{\infty} a_r(y)e^{2\pi irx}.$$

We then have the following result:

Theorem 2.3. *Let $\nu \in \mathbb{C}$ be such that the eigenvalue of Δ on f is $1/4-\nu^2$. Then if $r \neq 0$, the Fourier coefficient $a_r(y)$ is given by*

$$a_r(y) = c_r\sqrt{y}K_v(2\pi|r|y),$$

for some constant c_r.

Proof. For fixed $r \neq 0$, we can define $k : \mathbb{R}_+ \to \mathbb{C}$ by $a_r(y) = \sqrt{y}k(2\pi|r|y)$. Doing the same calculus as before, the eigenfunction condition is the same as requiring that k is a solution to Bessel's equation above. Bessel's equation has two solutions, K_ν and I_ν, where K_ν decays rapidly and I_ν grows exponentially. Our strategy for seeing that is to cite our troubles away, but here's some intuition to keep in mind while we do that. If y is large, the terms in Bessel's equation with a y^2 coefficient dominate, so it looks very much like the equation $\frac{\mathrm{d}^2k}{\mathrm{d}y^2} - k = 0$, which has one solution $k = e^{-y}$ that decays rapidly and one solution $k = e^y$ which grows exponentially.

In any case, we have a growth condition on the Fourier coefficients of a Maass form, so we must have $a_r(y)$ a constant multiple of $\sqrt{y}K_\nu(2\pi|r|y)$. □

This means that we completely understand the Maass cusp forms; if f is a Maass cusp form, then

$$f(z) = \sum_{r=-\infty, r\neq 0}^{\infty} a_r \sqrt{y} K_\nu(2\pi|r|y) e^{2\pi i r x},$$

with $a_r \in \mathbb{C}$.

3. L-FUNCTIONS OF MAASS FORMS

We've characterized (cusp) Maass forms as "determined countably many constant coefficients," which is starting to look like something we can slot into an L-function. One final step we can take is to consider the antiholomorphic $\iota : \mathcal{H} \to \mathcal{H}$ given by $\iota(x+iy) = -x+iy$. This involution preserves spaces of Maass forms with a given eigenvalue, so we can restrict our view to Maass forms that are also eigenforms with respect to $\circ\iota$. We have two options, since $\iota^2 = 1$; either $f \circ \iota = f$ or $f \circ \iota = -f$. In the first case, we say that f is *even*, and in the second case, we say that f is *odd*.

Now we really have coefficients indexed by $\mathbb{N}$, so it's natural to define the L-function

$$L(s, f) = \sum_{n=1}^{\infty} a_n n^{-s},$$

where we're assuming that f is a cusp form and is either odd or even. This L-function (which can be proven to be convergent for $\mathrm{Re}(s) > 1$ using the Rankin-Selberg method) has analytic continuation to the complex plane, as well as a functional equation. These results are important, but we've seen similar things in this seminar, so I'm just going to state the functional equation here, rather than proving it.

Proposition 3.1. *Let f be a Maass cusp form with eigenvalue $\frac{1}{4} - \nu^2$. Let $\varepsilon = 0$ if f is even and -1 if f is odd. Let*

$$\Lambda(s, f) = \pi^{-s} \Gamma\left(\frac{s+\varepsilon+\nu}{2}\right) \Gamma\left(\frac{s+\varepsilon-\nu}{2}\right) L(s, f).$$

Then $\Lambda(s, f)$ has analytic continuation to all s and satisfies

$$\Lambda(s, f) = (-1)^\varepsilon \Lambda(1-s, f).$$

In any case, now we have an L-function! This is great news. In many cases the strategy for understanding arithmetic sequences is to slot them into some sort of L-function, and then to glean properties about the arithmetic sequences from what we can understand about the L-function. Since Maass forms aren't holomorphic, the bounding constraints on the Fourier coefficients are weaker, which broadens the scope of their power.

4. FURTHER DIRECTIONS

Here's one example of Maass forms coming from a different direction. Maass constructed certain Maass forms from $\mathrm{GL}(1)$ of a quadratic extension, which in some sense is predicted by Langlands theory.

Remark 4.1. One question you could ask is if all Maass forms are of this type. The answer to that question comes from looking at the spectrum of the Laplacian operator.

Since the Laplacian is positive semidefinite, its eigenvalues of the form $\lambda = \frac{1}{4} - \nu^2$ must be real and nonnegative, which in turn means that ν is either purely imaginary or real of absolute value $\leq 1/2$. If $\lambda < 1/4$, i.e. $\nu \in \mathbb{R}_{>0}$, then ν is an *exceptional eigenvalue*; in 1965, Selberg showed that there are no exceptional eigenvalues with respect to the group $\mathrm{SL}(2, \mathbb{R})$ and conjectured that for other congruence subgroups there are also none. This is precisely analogous to the Ramanujan conjecture for Fourier coefficients of modular forms.

For larger eigenvalues, the spectrum of Δ is discrete when $\Gamma \diagdown \mathcal{H}$ is compact. In this case the Selberg trace formula (analogous to the Poisson summation formula) gives an estimate for the density of these eigenvalues; that estimate shows that there have to be more than the eigenforms we just mentioned Maass constructing.

Let's see a little bit about how this construction works. Let F be a real quadratic field with discriminant D and (narrow) class number one. We start with a Hecke character ψ of F, i.e. a map $\psi : R/I \to \mathbb{C}$, where R is the ring of integers of F and $I \subseteq R$ is an ideal. The associated Maass form is then

$$\theta_\psi(z) = \begin{cases} \sum_J \psi(J)\sqrt{y}K_\nu(2\pi N(J)y)\cos(2\pi N(J)x) \text{ if } \varepsilon = 0 \\ \sum_J \psi(J)\sqrt{y}K_\nu(2\pi N(J)y)\sin(2\pi N(J)x) \text{ if } \varepsilon = 1, \end{cases}$$

where, loosely speaking, ε is a sign of the Hecke character.

This vague exposition has skipped over many details, but the punchline is the following result:

Theorem 4.2 (Mass, 1949)**.** *The function θ_ψ is a Maass cusp form for the group $\Gamma_0(D)$.*

References

[1] Bump, Dan, 1998. Automorphic Forms and Representations, Cambridge Studies in Advance Mathematics, 55, Cambridge University Press.