Odd Moments in the Distribution of Primes

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A question

Let $\delta > 0$ and let $Q > 1/\delta$. Fix $k \in \mathbb{N}_{\geqslant 2}$. Let $S(k)$ be the number of k-tuples $\left(\frac{a_1}{q_1}, \dots, \frac{a_k}{q_k}\right)$ that satisfy:

- $q_i \in [Q, 2Q]$ for all i
- $\frac{a_i}{q_i} \in (0,1)$ is a fraction in lowest terms with $\left\| \frac{a_i}{q_i} \right\| \leqslant \delta$
- $\sum_{i=1}^{k} \frac{a_i}{q_i} \in \mathbb{Z}$

How big is this set, in terms of δ and Q?

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When $k = 2$, $\frac{a_1}{q_1} + \frac{a_2}{q_2} = 1$ implies that $q_1 = q_2$ and $a_1 = q_1 - a_2$, so there are $Q^2\delta$ solutions.

When k is even, the main term comes from pairing fractions, so that $\frac{a_1}{q_1} = 1 - \frac{a_2}{q_2}$, $\frac{a_3}{q_3} = 1 - \frac{a_4}{q_4}$, and so on, so that $S(k) \sim Q^k \delta^{k/2}$.

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What about when k is odd?

The distribution of primes in short intervals

Motivating Question

Consider intervals of size h, with $h = o(N)$ and $h/\log N \to \infty$ as $N \to \infty$.
What is the distribution of $\pi(n+h) - \pi(n)$ for $n \leqslant N$?
What is the distribution of $\psi(n+h) - \psi(n)$ for $n \leqslant N$?

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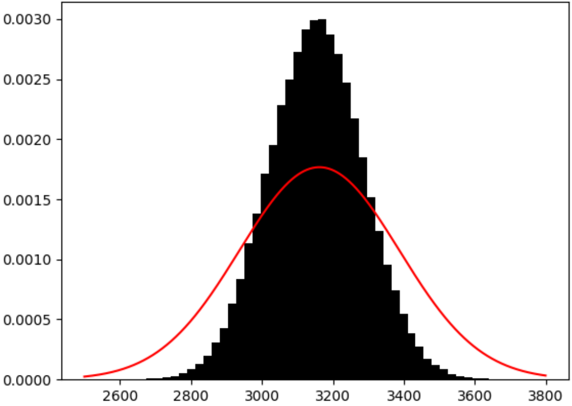
Cramér model answer

If we model the primes by saying each n is independently prime with probability $\frac{1}{\log n}$, then the distribution of $\psi(n+h) - \psi(n)$ would be Gaussian with mean $\sim h$ and variance $\sim h \log N$.

Testing the Cramér model

Here's the (normalized) distribution of $\psi(n+h)-\psi(n)$ for $1 \leqslant n \leqslant 10^7$, with $h=\sqrt{10^7}$. The red line is the Gaussian with mean h and variance $h \log 10^7$.



Hardy–Littlewood conjecture

Hardy–Littlewood Conjecture

Let $\mathcal{D} = \{d_1, \ldots, d_k\}$ be a sequence of distinct integers. As $N \to \infty$,

$$\sum_{n \leqslant N} \prod_{i=1}^{k} \Lambda(n + d_i) = \mathfrak{S}(\mathcal{D})N + o(N)$$

where

$$\mathfrak{S}(\mathcal{D}) = \prod_{p} \frac{1 - \nu_{\mathcal{D}}(p)/p}{(1 - 1/p)^k}$$

for $\nu_{\mathcal{D}}(p)$ is the number of equivalence classes $\bmod p$ occupied by $\mathcal{D}$.

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When $\mathcal{D} = \{0, 2\}$, Hardy–Littlewood predicts the asymptotic number of twin primes, via

$$\sum_{n \leqslant N} \Lambda(n)\Lambda(n + 2) \sim 2 \left(\prod_{p \geqslant 3} \frac{1 - 2/p}{(1 - 1/p)^2} \right) N$$

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When $\mathcal{D} = \{0, 1\}$, $\mathfrak{S}(\mathcal{D}) = 0$, since the factor at $p = 2$ is $\frac{1-2/2}{(1-1/2)^2} = 0$. "Either n or $n+1$ is even, so there are very few consecutive primes."

Variance via Hardy–Littlewood

The Hardy–Littlewood conjectures tell us that the variance is smaller.

$$\frac{1}{N}\sum_{n\leqslant N}\left(\sum_{\ell\leqslant h}\Lambda(n+\ell)-h\right)^2$$

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$$\sim \frac{1}{N}\sum_{n\leqslant N}\sum_{\ell\leqslant h}\Lambda(n+\ell)^2+\frac{2}{N}\sum_{\ell_1<\ell_2\leqslant h}\sum_{n\leqslant N}\Lambda(n+\ell_1)\Lambda(n+\ell_2)-h^2$$

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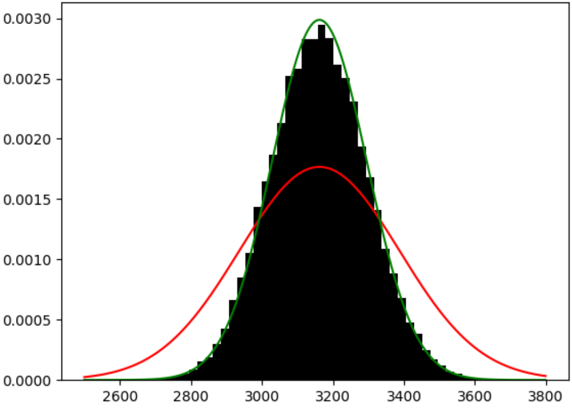
&\sim h\left(\log\frac{N}{h}+B-1\right)
\end{aligned}
$$

The Cramér guess, $h\log N$, is bigger than this!

Testing revisited

Here's the (normalized) distribution of $\psi(n+h) - \psi(n)$ for $1 \leqslant n \leqslant 10^7$, with $h = \sqrt{10^7}$. The red line is the Gaussian with mean h and variance $h \log 10^7$; the green line has mean h and variance $\frac{1}{2} h \log 10^7 + h(B-1)$.



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Sums of Singular Series

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Theorem (Montgomery & Soundararajan, 2004)

For $\mathcal{D} \subseteq \mathbb{N}$, *let* $\mathfrak{S}_0(\mathcal{D}) = \sum_{\mathcal{J} \subset \mathcal{D}} (-1)^{|\mathcal{D} \setminus \mathcal{J}|} \mathfrak{S}(\mathcal{J})$, *and let*

$$R_k(h) := \sum_{\substack{d_1, \ldots, d_k \\ 1 \leqslant d_i \leqslant h \\ \text{distinct}}} \mathfrak{S}_0(\mathcal{D}).$$

Then for any $k \in \mathbb{N}$,

$$R_k(h) = \mu_k(-h \log h + (B+1)h)^{k/2} + O_{k,\varepsilon}(h^{k/2 - 1/(7k) + \varepsilon}).$$

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For k odd, we don't know the asymptotic size of $R_k(h)$; we just have $R_k(h) = O_{k,\varepsilon}(h^{k/2-1/(7k)+\varepsilon})$.

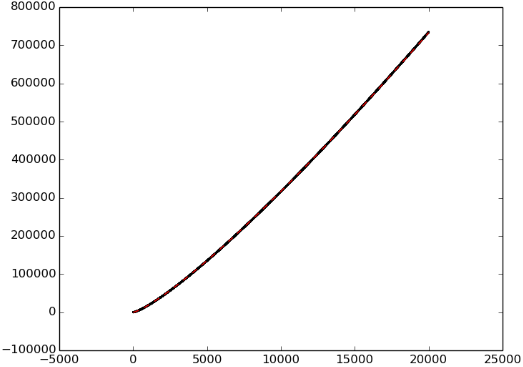
Odd moments: beyond square-root cancellation

Conjecture (K., Lemke Oliver and Soundararajan)

For k odd,

$$R_k(h) \asymp h^{(k-1)/2}(\log h)^{(k+1)/2}$$

For $k = 3$, here is $R_3(h)$ (in black) plotted against $0.373727h(\log h)^2$ (dashed in red).



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Theorem (K.)

$$R_3(h) = O(h(\log h)^5).$$

Techniques: Adding Fractions

$$R_k(h) \approx \sum_{\substack{q_1,\ldots,q_k \\ q_i>1}} \left(\prod_{i=1}^{k} \frac{\mu(q_i)}{\phi(q_i)}\right) \sum_{\substack{a_1,\ldots,a_k \\ 1\leqslant a_i\leqslant q_i \\ (a_i,q_i)=1 \\ \sum_i a_i/q_i\in\mathbb{Z}}} E\left(\frac{a_1}{q_1}\right)\cdots E\left(\frac{a_k}{q_k}\right),$$

where $E(\alpha) = \sum_{m=1}^{h} e(m\alpha)$.
$E(\alpha)$ is about h if $\|\alpha\| \leqslant \frac{1}{h}$ and small otherwise.

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Intuition for $k = 3$: about $\phi(q_1)\frac{1}{h}$ choices for a_1, and $\phi(q_2)\frac{1}{h}$ choices for a_2.

The problem in $\mathbb{F}_q[t]$

primes in $\mathbb{Z}$	irreducible polynomials in $\mathbb{F}_q[t]$
$\|n\|$	$\|F(t)\| := q^{\deg F}$
interval $(n, n+h)$	$I(F(t), h) := \{G(t) : \|F - G\| < h\}$ $h = q^{\ell}$
$\mathfrak{S}(\mathcal{D}) = \prod_p \frac{1 - \nu_p(\mathcal{D})/p}{(1-1/p)^k}$	$\mathfrak{S}(\mathcal{D}) = \prod_P \frac{1 - \nu_P(\mathcal{D})/\|P\|}{(1-1/\|P\|)^k}$

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$R_k(h) = \sum_{\substack{1 \leqslant d_1, \dots, d_k \leqslant h \\ \text{distinct}}} \mathfrak{S}_0(d_1, \dots, d_k)$	$R_k(h) = \sum_{\substack{D_1, \dots, D_k \in \mathbb{F}_q[t] \\ 1 \leqslant \lvert D_i\rvert \leqslant h \\ \text{distinct}}} \mathfrak{S}_0(D_1, \dots, D_k)$
$R_k(h) \asymp h^{(k-1)/2} (\log h)^{(k+1)/2}$ for k odd	$R_k(h) \asymp h^{(k-1)/2} (\log h)^{(k+1)/2}$ for k odd

Function field benefits

In $\mathbb{F}_q[t]$,

$$R_k(h) \approx \sum_{\substack{Q_1,\dots,Q_k \in \mathbb{F}_q[t] \\ |Q_i|>1 \\ \text{monic}}} \prod_{i=1}^{k} \frac{\mu(Q_i)}{\phi(Q_i)} \sum_{\substack{A_1,\dots,A_k \\ |A_i|<|Q_i| \\ (A_i,Q_i)=1 \\ \sum_i A_i/Q_i=0}} E\left(\frac{A_1}{Q_1}\right)\cdots E\left(\frac{A_k}{Q_k}\right)$$

- For $\alpha(t) = \frac{F(t)}{G(t)} \in \mathbb{F}_q(t)$, $E(\alpha) = \sum_{M \in I(0,h)} e(M\alpha)$
- For $\mathrm{res}(\alpha)$ the coefficient of $\frac{1}{t}$ in the Laurent series expansion of α, $e(\alpha) = \exp(2\pi i \cdot \mathrm{tr}(\mathrm{res}(\alpha)))$

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Lemma (Hayes, 1966)

Let $\alpha \in \mathbb{F}_q(t)$ with $|\alpha| \leqslant \frac{1}{q}$. Then

$$E(\alpha) = \begin{cases} h & \text{if } |\alpha| < \frac{1}{h} \\ 0 & \text{if } |\alpha| \geqslant \frac{1}{h}. \end{cases}$$

Function field results

Theorem (K.)

Fix q. As $h \to \infty$,

$$R_3(h) = O(h(\log h)^{19/2})$$

and for all $\varepsilon > 0$,

$$R_5(h) = O_\varepsilon(h^{2+\varepsilon}).$$

Techniques: Fifth moment bound

We want to bound

$$R_5(h) \approx h^5 \sum_{\substack{Q_1,\dots,Q_k \\ |Q_i|>1 \\ \text{monic}}} \prod_{i=1}^{k} \frac{\mu(Q_i)}{\phi(Q_i)} \# \left\{ A_i \bmod Q_i : \left| \frac{A_i}{Q_i} \right| < -h, (A_i, Q_i) = 1, \sum_i \frac{A_i}{Q_i} = 0 \right\}.$$

Techniques: Fifth moment bound

We want to bound

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The adaptation of the work of Montgomery–Soundararajan and Montgomery–Vaughan gives a sharp enough bound for the sum restricted over all terms $Q_1, \ldots, Q_k$ except those such that:

- $|Q_i| \geqslant h$ for all i
- no three Q_i's are equal
- for any i, j, either $Q_i = Q_j$ or $\left|\frac{Q_i}{(Q_i, Q_j)}\right| \geqslant h$ and $|(Q_i, Q_j)| < h/2$

Techniques cont'd: Fifth moment bound

For a tuple $Q_1, \dots Q_5$,

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implies that, possibly after reordering,

- $\left|\frac{Q_2}{(Q_2, Q_1)}\right| \geqslant h$ and
- $\left|\frac{Q_3}{(Q_3, Q_1 Q_2)}\right| \geqslant h/2.$

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Strategy: Count options for A_1/Q_1, then remaining options for A_2/Q_2 after accounting for what has already been determined, then remaining options for A_3/Q_3.

An application

Lemke Oliver and Soundararajan (2016) conjecture that consecutive primes in arithmetic progressions exhibit biases. With $\pi(x_0) = 10^8$, they have the following data:

a	b	$\pi(x_0; 10, (a, b))$	a	b	$\pi(x_0; 10, (a, b))$
1	1	4623042	7	1	6373981
	3	7429438		3	6755195
	7	7504612		7	4439355
	9	5442345		9	7431870
3	1	6010982	9	1	7991431
	3	4442562		3	6372941
	7	7043695		7	6012739
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Showing that sums of three-term singular series are small improves their heuristic.

Further questions

Question 1.

Let $\delta > 0$ and $Q > 1/\delta$. For k odd, what is

$$\#\left\{q_1, \ldots, q_k \in [Q, 2Q], 1 \leqslant a_i \leqslant q_i : \left\| \frac{a_i}{q_i} \right\| \leqslant \delta, \sum_i \frac{a_i}{q_i} \in \mathbb{Z} \right\}?$$

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Question 2.

For $\delta > 0, Q > 1/\delta$, let $J_1, \ldots, J_k \subseteq [0, 1]$ be intervals with $|J_i| \geqslant \delta$. What is

$$\#\left\{q_1, \ldots, q_k \in [Q, 2Q], 1 \leqslant a_i \leqslant q_i : \frac{a_i}{q_i} \in J_i, \sum_i \frac{a_i}{q_i} \in \mathbb{Z}\right\}?$$

Thank you!