PRIMES AND REDUCED RESIDUES IN SHORT INTERVALS

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1. Introduction

How many primes are there in an interval in the integers? If the interval is long (say, [0,X]), this question is answered by the prime number theorem. For shorter intervals, the prime number theorem can give us an expectation, but not an asymptotic. The question we want to be asking is: what is the distribution of primes in short intervals?

To be precise, let's consider the following setup. Let $H = H(N)$ be a function of N with $H = o(N)$ and $H/logN \to \infty$ as $N \to \infty$. We'll then study the distribution of $\psi(n+H) - \psi(n)$ for $n \leq N$.

The Cramér model predicts that as N gets large, this distribution becomes approximately normal with mean $\sim H$ and variance $\sim H \log N$. Since the normal distribution is determined by its moments, our goal will be to understand the moments of the number of primes in an interval of size H. Assuming that $H/\log N \to \infty$ but $H \leq N^{1-\delta}$, and relying on an effective form of the Hardy-Littlewood conjecture, we'll see that the distribution approaches normal and get a formula for the variance. In the end, the Cramér prediction for the variance is about correct when $\log H/\log N \to 0$, but there is evidence that it is smaller when $N^{\delta} \leq H \leq N^{1-\delta}$.

Nevertheless, our main task of the moment is just that: computing the moments. We'll spend the majority of time computing moments for a related question, namely for the distribution of reduced residues modulo q in a short interval.

2. The Distribution of Reduced Residues modulo q

Rather than consider the number of primes in an interval, let's fix an integer q and consider the number of reduced residues modulo q in an interval. In particular, we consider

$$m_k(q;h) = \sum_{n=1}^{q} \left(\sum_{\substack{m=1 \\ (m+n,q)=1}}^{h} 1 - h\phi(q)/q \right)^k,$$

the kth centered moment of the number of reduced residues $\pmod q$ in an interval. Montgomery and Vaughan use the following expression for $m_k(q;h)$:

Lemma 2.1.

$$m_k(q;h) = q \left(\frac{\phi(q)}{q} \right)^k V_k(q;h),$$

with

$$V_k(q;h) = \sum_{\substack{d_1,\dots,d_k \\ 1\le d_i\le h}} \sum_{\substack{r_1,\dots,r_k \\ 1<r_i|q}} \left(\prod_{i=1}^k \frac{\mu(r_i)}{\phi(r_i)}\right) \sum_{\substack{a_1,\dots,a_k \\ 1\le a_i<r_i \\ (a_i,r_i)=1 \\ \sum a_i/r_i\in\mathbb{Z}}} e\left(\sum_{i=1}^k \frac{a_i d_i}{r_i}\right).$$

For the moment, let's consider small values of k. When $k=1$, we can't have any terms with $1 \le a_1 < r_1$ and $a_1/r_1 \in \mathbb{Z}$, so the sum is empty. Thus $V_1(q;h) = 0$. In turn we get that $m_1(q;h) = 0$; we knew this already, because we were taking the centered moment!

To understand $k=2$, we will define $E(\alpha) = \sum_{m=1}^h e(m\alpha)$, for any real number α. The conditions on a_1 and a_2 imply that $q_1 = q_2 = a_1 + a_2$, so we get

$$V_2(q;h) = \sum_{\substack{d|q \\ d>1}} \frac{\mu(d)^2}{\phi(d)^2} \sum_{\substack{a=1 \\ (a,d)=1}}^{d} |E(a/d)|^2$$

The sums $E(\alpha)$ are generally useful to simplify the expression, where we have

$$V_k(q;h) = \sum_{\substack{r_1,\dots,r_k \\ 1<r_i|q}} \prod_{i=1}^k \frac{\mu(r_i)}{\phi(r_i)} \sum_{\substack{a_1,\dots,a_k \\ 1\le a_i\le r_i \\ (a_i,r_i)=1 \\ \sum a_i/r_i\in\mathbb{Z}}} \prod_{i=1}^k E(a_i/r_i).$$

Our first goal will be the following theorem, a refinement of Montgomery and Vaughan's results.

Theorem 2.2. *Let $\mu_k = 1\cdot 3\cdots(k-1)$ if k is even and 0 if k is odd. Then*

$$V_k(q;h) = \mu_k V_2(q;h)^{k/2} + O_k\left(h^{k/2-1/(7k)}\left(\frac{q}{\phi(q)}\right)^{2^k+k/2}\right).$$

2.1. **The case when k is odd.**

Proposition 2.3. *Let $k \ge 1$ be fixed and odd. Then*

$$V_k(q;h) \ll h^{k/2-1/(7k)}\left(\frac{\phi(q)}{q}\right)^{-2^k-k/2}.$$

We've already addressed the case when $k=1$, so assume $k\ge 3$. Note that if $F(x) = \min(h, 1/||x||)$, then $|E(x)| \le F(x)$ for all x.

Since

$$V_k(q;h) = \sum_{\substack{r_1,\dots,r_k \\ 1<r_i|q}} \prod_{i=1}^k \frac{\mu(r_i)}{\phi(r_i)} \sum_{\substack{a_1,\dots,a_k \\ 1\le a_i\le r_i \\ (a_i,r_i)=1 \\ \sum a_i/r_i\in\mathbb{Z}}} \prod_{i=1}^k E(a_i/r_i),$$

we can bound it without taking notice of the possible cancellation due to $\mu(r_i)$ or oscillations in the $E(a_i/r_i)$, via

$$V_k(q;h) \ll \sum_{r|q} \sum_{\substack{r_i>1 \\ [r_1,\dots,r_k]=r}} \frac{S(\vec{r})}{\phi(r_1)\cdots\phi(r_k)},$$

with

$$S(\vec{r}) = \sum_{\substack{1\le a_i\le r_i \\ (a_i,r_i)=1 \\ \sum a_i/r_i\in\mathbb{Z}}} F(a_1/r_1)\cdots F(a_k/r_k).$$

To estimate this, we rely on the following Fundamental Lemma of Montgomery and Vaughan, as well as some more specific bounds on $F(x)$.

Lemma 2.4 (Fundamental Lemma)**.** *Let $r_1,\ldots,r_k$ be squarefree integers with $r=[r_1,\ldots,r_k]$, and say that if a prime p divides r, then p divides at least two of the r_is. Then for any complex-valued functions $G_1,\ldots,G_k$ on $(0,1]$, we have*

$$\left|\sum_{\substack{b_1,\ldots,b_k \\ 1\le b_i\le r_i \\ \sum b_i/r_i\in\mathbb{Z}}} \prod_{i=1}^{k} G_i(b_i/r_i)\right| \le \frac{1}{r}\prod_{i=1}^{k}\left(r_i\sum_{b_i=1}^{r_i}|G_i(b_i/r_i)|^2\right)^{1/2}.$$

I won't get into all of the details of the proof here, but it is in broad strokes a proof by induction on k relying on Cauchy-Schwartz.

Using this fundamental lemma, we can get a bound on $S(\vec{r})$ that is almost-but-not-quite good enough without much effort. Specifically, with $[r_1,\ldots,r_k]=r$,

$$\begin{aligned} S(\vec{q}) &= \sum_{\substack{1\le a_i\le r_i \\ (a_i,r_i)=1 \\ \sum a_i/r_i\in\mathbb{Z}}} F(a_1/r_1)\cdots F(a_k/r_k) \\ &\le \frac{1}{r}\prod_{i=1}^{k}\left(r_i\sum_{\substack{1\le a_i\le r_i \\ (a_i,r_i)=1}} F(a_i/r_i)^2\right)^{1/2} \\ &\ll \frac{1}{r}\prod_{i=1}^{k}(r_i^2\min(r_i,h))^{1/2} \\ &\ll r_1\cdots r_k r^{-1}h^{k/2}. \end{aligned}$$

In particular, the power of $h^{k/2}$ here is $k/2$, but we were angling for $k/2-1/(7k)$. So instead, they do significantly more work to get the following lemma, which again relies (in more complicated ways) on the same fundamental lemma.

Lemma 2.5. *Let $k\ge 3$, and let $r_1,\ldots,r_k$ be squarefree numbers with $r_i>1$. Let $r=[r_1,\ldots,r_k]$, let $d=(r_1,r_2)$, and write $d=st$ with $s|r_3\cdots r_k$ and $(t,r_3\cdots r_k)=1$. Moreover, write $r_1=dr_1'$ and $r_2=dr_2'$. Then*

$$S(\vec{r}) \ll r_1\cdots r_k r^{-1}h^{k/2}(T_1+T_2+T_3+T_4)$$

where

$$\begin{aligned} T_1 =& h^{-1/20} \\ T_2 =& d^{-1/4} \qquad \textit{when } r_i > h^{8/9} \textit{ for all } i \\ =& 0 \qquad \textit{otherwise} \\ T_3 =& s^{-1/2} \qquad \textit{when } r_i > h^{8/9} \textit{ for all } i \textit{ and } r_1 = r_2 \\ =& 0 \qquad \textit{otherwise} \\ T_4 =& \left(\frac{1}{r_1 r_2 s h^2} \sum_{\substack{1 \le \tau \le t \\ (\tau, t) = 1}} F\left(\frac{||r_1' s\tau/t||}{r_1' s} \right)^2 F\left(\frac{||r_2' s\tau/t||}{r_2' s} \right)^2 \right)^{1/2} \\ & \textit{when } h^{8/9} < r_i \le h^2 \textit{ for } i = 1, 2, \textit{ and } t > d^{1/2},\, d \le h^{5/9} \\ =& 0 \qquad \textit{otherwise.} \end{aligned}$$

In the interest of time and clarity, I'll omit the proof of this lemma, but let's see how it helps us. Again, we are considering

$$V_k(q; h) \ll \sum_{r|q} \sum_{\substack{r_i > 1 \\ [r_1, \ldots, r_k] = r}} \frac{S(\vec{r})}{\phi(r_1) \cdots \phi(r_k)},$$

with

$$S(\vec{r}) = \sum_{\substack{1 \le a_i \le r_i \\ (a_i, r_i) = 1 \\ \sum a_i / r_i \in \mathbb{Z}}} F(a_1/r_1) \cdots F(a_k/r_k).$$

Consider a fixed k-tuple $\vec{r} = (r_1, \ldots, r_k)$ with $r_i > 1$, $[r_1, \ldots, r_k] = r$, and each prime divisor of r divides at least two r_i's. Now we want to apply Lemma 2.5, but we can choose how to label the r_i's in the application of the lemma; this choice will crucially fail if the r_i's are all equal in pairs.

If $r_i \le h^{8/9}$ for any i, then we automatically have $S(\vec{r}) \ll r_1 \cdots r_k r^{-1} h^{k/2 - 1/20}$, which is plenty, so we are done. Assume that $r_i > h^{8/9}$ for all i, and let $d_{ij} = (r_i, r_j)$. Note that $r_i | \prod_{j \ne i} r_j$, so $r_i | \prod_{j \ne i} d_{ij}$. Thus for each i, there must be some j so that

$$d_{ij} \ge h^{8/(9k-9)}.$$

If there is a pair (i, j) so that this is the case but $d_i \ne d_j$, we take these to be r_1, r_2. Now suppose that we *only* have $d_{ij} \ge h^{8/(9k-9)}$ when $r_i = r_j$. Thus in particular for every i, there exists r_j with $r_i = r_j$, so there exists a triple (r_i, r_j, r_ℓ). In this case, let $r_1 = r_i$ and $r_2 = r_j$. Note also that this argument also applies when k is even, as long as we're not specifically in the case when the r_i's are equal in pairs.

Returning to $V_k(q;h)$, we have as before that

$$V_k(q;h) \ll \sum_{r|q} \sum_{\substack{r_i>1 \\ [r_1,\ldots,r_k]=r}} \frac{S(\vec{r})}{\phi(r_1)\cdots\phi(r_k)}$$
$$\ll h^{k/2} \sum_{r|q} \frac{1}{r} \sum_{r_i>1} \frac{r_1\cdots r_k}{\phi(r_1)\cdots\phi(r_k)}(T_1+T_2+T_3+T_4),$$

with T_1, T_2, T_3, T_4 as in the Lemma. By the Lemma, $T_1 = h^{1/20}$. We've chosen r_1 and r_2 so that when T_2 is nonzero, we have $T_2 = d^{-1/4}$ with $d \geq h^{8/(9k-9)}$, and thus $T_2 \ll h^{-2/(9k-9)}$.

As for T_3, for the terms when T_3 is nonzero, we have $r_1 = r_2$. By our choice of r_1 and r_2, this means that there exists another r_ℓ with $r_\ell = r_1 = r_2$. Thus in particular when we decompose $d = (r_1, r_2)$ as $d = st$, with $s|r_3\cdots r_k$ and $(t, r_3\cdots r_k) = 1$, we get that $s = r_1$ as well, so $s > h^{8/9}$. Thus $T_3 \ll h^{-4/9}$.

In particular, $T_1 + T_2 + T_3 \ll h^{-2/(9k-9)}$. Thus if we only consider the contributions from T_1, T_2, and T_3, we get

$$\begin{aligned} h^{k/2}\sum_{r|q}\frac{1}{r}\sum_{r_i>1}\frac{r_1\cdots r_k}{\phi(r_1)\cdots\phi(r_k)}(T_1+T_2+T_3) &\ll h^{k/2-2/(9k-9)}\sum_{r|q}\frac{1}{r}\sum_{r_i>1}\frac{r_1\cdots r_k}{\phi(r_1)\cdots\phi(r_k)} \\ &= h^{k/2-2/(9k-9)}\prod_{p|q}\left(1+\frac{1}{p}\left(2+\frac{1}{p-1}\right)^k\right) \\ &\ll h^{k/2-2/(9k-9)}\prod_{p|q}\left(1+\frac{2^k}{p}\right) \\ &\ll h^{k/2-2/(9k-9)}\left(\prod_{p|q}\frac{p+1}{p}\right)^{2^k} \\ &\ll h^{k/2-2/(9k-9)}\left(\frac{q}{\phi(q)}\right)^{2^k} \end{aligned}$$

The term for T_4 is dealt with separately in Montgomery-Vaughan. It's a bit complicated and I felt not so enlightening, so here we'll just cite the following fact from Montgomery and Vaughan, which uses Cauchy-Schwarz as well as some results on sums of $F(x)$.

Fact 2.6.

$$h^{k/2}\sum_{r|q}\frac{1}{r}\sum_{r_i>1}\frac{r_1\cdots r_k}{\phi(r_1)\cdots\phi(r_k)}T_4 \ll h^{k/2-1/(7k)}\left(\frac{\phi(q)}{q}\right)^{-2^k-k/2}$$

This constraint is stricter than that on the $(T_1+T_2+T_3)$ term, giving in total that

$$V_k(q;h) \ll h^{k/2-1/(7k)}\left(\frac{\phi(q)}{q}\right)^{-2^k-k/2},$$

as desired.

One important note about this is that the argument for T_4 also carries through in the case where k is even. In fact, the entire argument does, with the exception of the "diagonal" termswhen the r_i's are equal in pairs, which we had to avoid in the treatment of T_3.

2.2. **When k is even: diagonal terms.** In order to understand the even k case, as we've seen, we need to understand specifically the terms with the r_i's equal in pairs and otherwise distinct, since as we've seen, all other terms contribute at most $h^{k/2-1/(7k)}\left(\frac{\phi(q)}{q}\right)^{-2^k-k/2}$.

There are $(k-1)(k-3)\cdots 3\cdot 1=\mu_k$ ways for the r_i's to be paired up. Let's take the pairing to be $r_i=r_{k/2+i}$, and set $b_i=a_i+a_{k/2+i}$. So these terms are precisely given by

$$\mu_k \sum_{r_1\neq\ldots\neq r_{k/2}|q}\prod_{i=1}^{k/2}\frac{\mu(r_i)^2}{\phi(r_i)^2}\sum_{\substack{b_1,\ldots,b_{k/2}\\ 1\leq b_i\leq r_i\\ \sum b_i/r_i\in\mathbb{Z}}}\prod_{i=1}^{k/2}J(b_i,r_i)$$

As we've seen, the terms where the r_i are not distinct contribute to smaller order. This means that we may as well sum over all values r_i, without worrying about distinctness, since it won't affect the answer.

For each value of b_i, we need to take into account all of the possible decompositions into $a_i+a_{i+k/2}$, both of which are relatively prime to r_i. This is exactly what $J(b_i,r_i)$ does. In particular, we have

$$J(b,r)=\sum_{\substack{a=1\\ (a,r)=1\\ (b-a,r)=1}}^{r} E\left(\frac{a}{r}\right)E\left(\frac{b-a}{r}\right)$$

Let's recall from way back when our expression for $V_2(q;h)$, the second moment, and see how we can relate it to J-functions. Specifically,

$$V_2(q;h)=\sum_{1<d|q}\frac{\mu(d)^2}{\phi(d)^2}\sum_{\substack{a=1\\ (a,d)=1}}^{d}|E(a/d)|^2=\sum_{1<d|q}\frac{\mu(d)^2}{\phi(d)^2}J(d,d)$$

So now in our expression for these terms, whenever we have $b_i=r_i$, we get a factor of $V_2(q;h)$. Let's see what we get by factoring those out. We'll assume that $b_i<r_i$ for exactly j values of the i available, and that $b_i=r_i$ for the remaining $k/2-j$ values. Then we get that the terms under consideration are

$$\mu_k \sum_{r_1,\ldots,r_{k/2}|q}\prod_{i=1}^{k/2}\frac{\mu(r_i)^2}{\phi(r_i)^2}\sum_{\substack{b_1,\ldots,b_{k/2}\\ 1\leq b_i\leq r_i\\ \sum b_i/r_i\in\mathbb{Z}}}\prod_{i=1}^{k/2}J(b_i,r_i)=\mu_k\sum_{j=0}^{k/2}\binom{k/2}{j}V_2(q;h)^{k/2-j}W_j(q;h),$$

with $W_0(q;h)=1$ and

$$W_j(q;h)=\sum_{1<r_1,\ldots,r_j|q}\prod_{i=1}^{j}\frac{\mu(r_i)^2}{\phi(r_i)^2}\sum_{\substack{b_1,\ldots,b_j\\ 0<b_i<r_i\\ \sum b_i/r_i\in\mathbb{Z}}}\prod_{i=1}^{j}J(b_i,r_i)$$

Our goal was to show that the main term is precisely the term when $j = 0$, which is $\mu_k V_2(q;h)^{k/2}$. So we need only show that the other terms are smaller.

Using the fact that $|E(\alpha)| \le F(\alpha) = \min(h, 1/||\alpha||)$, one can derive the following bound on sums of $J(b,r)$:

Fact 2.7.

$$\sum_{0<b<r} J(b,r)^2 \ll r^3 \min(r,h).$$

At this point we're in a position to apply something like the Fundamental Lemma that was necessary in the odd case as well; broadly, a bound on $W_j(q;h)$ it is a bound on a sum of a product of functions to $\mathbb{C}$. In fact we'll use the following variant from Montgomery-Soundararajan.

Lemma 2.8. *Let $r_1, \dots, r_k$ be squarefree integers with $r_i > 1$, and let $d = [r_1, \dots, r_k]$. Let $G : (0,1) \to \mathbb{C}$ be a function and let G_0 be a non-decreasing function on the positive integers with*

$$\sum_{a=1}^{r-1} |G(a/r)|^2 \le rG_0(r)$$

for all squarefree integers $r > 1$. Then

$$\left| \sum_{\substack{a_1,\dots,a_k \\ 0<a_i<r_i \\ \sum a_i/r_i \in \mathbb{Z}}} \prod_{i=1}^{k} G(a_i/r_i) \right| \le \frac{1}{d} \prod_{i=1}^{k} r_i G_0(r_i)^{1/2}.$$

Again here, I won't discuss the proof, but it relies on the previous Fundamental Lemma, whose proof in turn relied on repeated applications of Cauchy-Schwarz.

Taking $G_0(r) = Chr^2$ in the Lemma, we can apply it to the J functions, via

$$\sum_{\substack{b_1,\dots,b_j \\ 0<b_i<r_i \\ \sum b_i/r_i \in \mathbb{Z}}} \prod_{i=1}^{j} J(b_i, r_i) \ll \frac{1}{d} \prod_{i=1}^{j} r_i^2 h^{1/2},$$

so

$$\begin{aligned} W_j(q;h) &= \sum_{1<r_1,\dots,r_j|q} \prod_{i=1}^{j} \frac{\mu(r_i)^2}{\phi(r_i)^2} \sum_{\substack{b_1,\dots,b_j \\ 0<b_i<r_i \\ \sum b_i/r_i \in \mathbb{Z}}} \prod_{i=1}^{j} J(b_i, r_i) \\ &\ll h^{j/2} \sum_{d|q} \frac{1}{d} \left(\sum_{r|d} \frac{\mu(r)^2 r^2}{\phi(r)^2} \right)^j \\ &= h^{j/2} \prod_{p|q} \left(1 + \frac{1}{p} \left(1 + \frac{p^2}{(p-1)^2} \right)^j \right) \\ &\ll h^{j/2} \left(\frac{q}{\phi(q)} \right)^{2^j}. \end{aligned}$$

Meanwhile, we can bound $V_2(q;h)$ in these terms via

$$\begin{aligned} V_2(q;h) &\leq \sum_{d|q} \frac{\mu(d)^2}{\phi(d)^2} \sum_{a=1}^{d-1} F(a/d)^2 \\ &\ll h \sum_{d|q} \frac{\mu(d)^2 d}{\phi(d)^2} \\ &= h \prod_{p|q} \left(1 + \frac{p}{(p-1)^2}\right) \ll h \frac{q}{\phi(q)}. \end{aligned}$$

Thus we get that these diagonal terms add to

$$\begin{aligned} &\mu_k \sum_{j=0}^{k/2} \binom{k/2}{j} V_2(q;h)^{k/2-j} W_j(q;h) \\ &\quad = \mu_k V_2(q;h)^{k/2} + O\left(\sum_{j=1}^{k/2} \binom{k/2}{j} V_2(q;h)^{k/2-j} W_j(q;h)\right) \\ &\quad = \mu_k V_2(q;h)^{k/2} + O\left(\sum_{j=2}^{k/2} \binom{k/2}{j} h^{k/2-j} \left(\frac{q}{\phi(q)}\right)^{k/2-j} h^{j/2} \left(\frac{q}{\phi(q)}\right)^{2^j}\right) \\ &\quad = \mu_k V_2(q;h)^{k/2} + O\left(h^{k/2-1} \left(\frac{q}{\phi(q)}\right)^{2^{k/2}}\right), \end{aligned}$$

which is finally exactly what we wanted!

3. The Distribution of Primes

Now we'd like to get from the world of reduced residues mod q to the world of primes; we will outline the connection in broad strokes.

We'd like to study the moments of $\psi(x+h) - \psi(x) - h$ when $x^\delta \leq h \leq x^{1-\delta}$. In order to study these delicate moments, we'll consider a shifted von Mangoldt function (and correspondingly shifted singular series). In particular let $\Lambda_0(n) = \Lambda(n) - 1$, so that

$$\psi(x+h) - \psi(x) - h = \sum_{x<n\leq x+h} \Lambda_0(n)$$

We can state the Hardy-Littlewood prime k-tuple conjecture in terms of this shifted von Mangoldt function:

Conjecture 3.1 (Hardy-Littlewood k-tuples)**.**

$$\sum_{n\leq x} \prod_{i=1}^{k} \Lambda_0(n+d_i) = (\mathfrak{S}_0(\mathcal{D}) + o(1))x$$

as $x \to \infty$.

Here we have

$$\mathfrak{S}_0(\mathcal{D}) = \sum_{\mathcal{J}\subseteq\mathcal{D}} (-1)^{\operatorname{card}\mathcal{J}} \mathfrak{S}(\mathcal{J}) \qquad \text{and} \qquad \mathfrak{S}(\mathcal{D}) = \sum_{\mathcal{J}\subseteq\mathcal{D}} \mathfrak{S}_0(\mathcal{J}),$$

where $\mathfrak{S}(\mathcal{D}) = \prod_p \frac{1-\nu_p(\mathcal{D})}{(1-1/p)^k}$ is the singular series. Both of these have series representations; we'll work with $\mathfrak{S}_0$, and we will use the expression

$$\mathfrak{S}_0(\mathcal{D}) = \sum_{1<q_1,\ldots,q_k<\infty} \left(\prod_{i=1}^{k} \frac{\mu(q_i)}{\phi(q_i)}\right) \sum_{\substack{a_1,\ldots,a_k \\ 1\le a_i\le q_i \\ (a_i,q_i)=1 \\ \sum a_i/q_i\in\mathbb{Z}}} e\left(\sum_{i=1}^{k} \frac{a_i d_i}{q_i}\right)$$

The crucial result on the way to a result about moments of primes (dependent on Hardy-Littlewood) is a result about averaging $\mathfrak{S}_0(\mathcal{D})$ over bounded sets $\mathcal{D}$. We define

$$R_k(h) = \sum_{\substack{d_1,\ldots,d_k \\ 1\le d_i\le h \\ d_i \text{ distinct}}} \mathfrak{S}_0(\mathcal{D})$$

Montgomery and Soundararajan prove the following.

Theorem 3.2. *For $h > 1$,*

$$R_k(h) = \mu_k(-h\log h + Ah)^{k/2} + O_k(h^{k/2-1/(7k)+\varepsilon})$$

for any nonnegative integer k, where $A = 2 - \gamma - \log 2\pi$.

Leaving out many details, we'll present a sketch of the argument here. The first goal is to restrict the outer sum in $\mathfrak{S}_0(\mathcal{D})$ to only terms with only small prime factors. Let $Q = \prod_{p\le h^{k+1}} p$. Using a lemma of Hardy and Littlewood, one can argue that

$$R_k(h) = \sum_{\substack{d_1,\ldots,d_k \\ 1\le d_i\le h \\ d_i \text{ distinct}}} \sum_{1<q_1,\ldots,q_k|Q} \left(\prod_{i=1}^{k} \frac{\mu(q_i)}{\phi(q_i)}\right) \sum_{\substack{a_1,\ldots,a_k \\ 1\le a_i\le q_i \\ (a_i,q_i)=1 \\ \sum a_i/q_i\in\mathbb{Z}}} e\left(\sum_{i=1}^{k} \frac{d_i a_i}{q_i}\right) + O(1),$$

where the only change here is that we require $q_i|Q$, and in return gain an $O(1)$ error.

Now in fact, the main term would be exactly $V_k(Q;h)$ if only we were not requiring that the d_i's be distinct, so the main obstruction is finding something that we can work with that doesn't have this distinctness condition. After quite a bit of work, Montgomery and Soundararajan arrive at

$$R_k(h) = \sum_{0\le j\le k/2} \binom{k}{2j} \frac{(2j)!}{j!2^j} \left(-h \sum_{\substack{d|Q \\ d>1}} \frac{\mu(d)^2}{\phi(d)}\right)^j V_{k-2j}(Q;h) + O(h^{(k-1)/2+\varepsilon}).$$

Now, if k is odd, then so is $k-2j$, so every term in this expression will be small. If k is even, the main term is

$$\sum_{0\le j\le k/2}\binom{k}{2j}\frac{(2j)!}{j!2^j}\left(-h\sum_{1<d|Q}\frac{\mu(d)^2}{\phi(d)}\right)^j\mu_{k-2j}V_2(q;h)^{k/2-j}$$
$$=\mu_k\left(V_2(Q;h)-h\sum_{1<d|Q}\frac{\mu(d)^2}{\phi(d)}\right)^{k/2},$$

where this is not an obvious step, but is derived from the binomial theorem. The last step is the following lemma concerning the size of $V_2(Q;h)$:

Lemma 3.3.

$$V_2(Q;h)-h\sum_{1<d|Q}\frac{\mu(d)^2}{\phi(d)}=-h\log h+(2-\gamma-\log 2\pi)h+O(h^{1/2+\varepsilon}).$$

This precisely gives us that for k even, we have

$$R_k(h)=\mu_k(-h\log h+(2-\gamma-\log 2\pi)h)^{k/2}+O(h^{(k-1)/2+\varepsilon}).$$

Now, there's still a question of connecting this to a true result about the distribution of primes, rather than a result about sums of singular series. We'll briefly see how this ties in. We have $(\log N)^{1+\delta}\le h\le N^{1-\delta}$ and want to evaluate

$$\frac{1}{N}\sum_{n\le N}(\psi(n+h)-\psi(n)-h)^r=\sum_{d_1,\dots,d_r\le h}\frac{1}{N}\sum_{n\le N}\Lambda_0(n+d_1)\cdots\Lambda_0(n+d_r).$$

After a little combinatorics, this can be written as

$$\sum_{k=1}^{r}\sum_{\substack{m_1,\dots,m_k\ge 1\\ \sum m_i=r}}\binom{r}{m_1,\dots,m_k}\frac{1}{k!}\sum_{\substack{h_1,\dots,h_k\le h\\ h_j\text{ distinct}}}\frac{1}{N}\sum_{n=1}^{N}\prod_{i=1}^{k}\Lambda_0(n+h_i)^{m_i}.$$

There are two different scenarios here: either $m_i=1$ or $m_i>1$. Let $\mathcal{I}\subseteq\{1,\dots,k\}$ be the subset of indices such that $m_i=1$ precisely for $i\in\mathcal{I}$. If $i\notin\mathcal{I}$, we think of $\Lambda_0(n+h_i)$ as being essentially $(\log N)^{m_i-1}\Lambda(n+h_i)$, since both quantities have about the same expected value, namely $(\log N)^{m_i-1}$. This is unlike the case when $m_i=1$, where the expected value of $\Lambda(n+h_i)$ is 1, but that of Λ_0 is 0. So the inner sum above is close to

$$\frac{(\log N)^{r-k}}{N}\sum_{n=1}^{N}\prod_{i\in\mathcal{I}}\Lambda_0(n+h_i)\prod_{\substack{1\le i\le k\\ i\notin\mathcal{I}}}(\Lambda_0(n+h_i)+1)$$
$$=\frac{(\log N)^{r-k}}{N}\sum_{\mathcal{I}\subseteq\mathcal{J}\subseteq\{1,\dots,k\}}\sum_{n=1}^{N}\prod_{j\in\mathcal{J}}\Lambda_0(n+h_j)$$

At this point we're at a place where invoking our effective version of Hardy-Littlewood for this product over $j\in\mathcal{J}$ is helpful; it is then summed over the h_i's, which further allows us to invoke our estimate for $R_k(h)$. This ultimately yields the following theorem:

Theorem 3.4. *Let* $E_k(x;\mathcal{D})$ *be given by*

$$\sum_{n\le x}\prod_{i=1}^{k}\Lambda(n+d_i)=\mathfrak{S}(\mathcal{D})x+E_k(x;\mathcal{D}),$$

and suppose that $E_k(x;\mathcal{D})\ll N^{1/2+\varepsilon}$ *uniformly for* $1\le k\le K, 0\le x\le N$, *and distinct* d_i *with* $1\le d_i\le H$. *Then*

$$\begin{aligned}M_K(N;H)&=\sum_{n=1}^{N}(\psi(n+H)-\psi(n)-H)^K\\&=\mu_K H^{K/2}\int_1^N(\log x/H+B)^{K/2}\mathrm{d}x\\&\quad+O\left(N(\log N)^{K/2}H^{K/2}\left(\frac{H}{\log N}\right)^{-1/(8K)}+H^K N^{1/2+\varepsilon}\right)\end{aligned}$$

uniformly for $\log N\le H\le N^{1/K}$, *where* $B=1-\gamma-\log 2\pi$.

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