TOPOLOGICAL TVERBERG'S THEOREM

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1. Intersections of Convex Hulls: Tverberg's Theorem

We're going to dive into questions about sets of points in Euclidean space and their intersection.

Definition 1.1. Let U be a subset of $\mathbb{R}^d$. U is *convex* if for every $x, y \in U$, the line segment from x to y is entirely contained in U.

The *convex hull* $\mathrm{Conv}(U)$ of U is the intersection of all convex sets containing U.

For sets of finitely many points, there is an explicit formulation of their convex hull.

Lemma 1.2. *For $S \subseteq \mathbb{R}^d$,*

$$\mathrm{Conv}(S) = \left\{ \sum_{i=1}^{n} \alpha_i x_i \mid \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1, \{x_1, \ldots, x_n\} \subseteq S \right\}.$$

In particular, for $X = \{x_1, \ldots, x_n\} \subseteq \mathbb{R}^d$,

$$\mathrm{Conv}(X) = \left\{ \sum_{i=1}^{n} \alpha_i x_i \mid \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1 \right\}.$$

Proof. Homework exercise. $\square$

Here is a starting question:

Question 1.3. Given n points in $\mathbb{R}^d$, when can we partition them into disjoint subsets whose convex hulls intersect?

For example, if we have 4 points in $\mathbb{R}^2$, we can see directly that they can be partitioned into two subsets whose convex hulls intersect. The convex hull of our 4 points is a quadrilateral, a triangle, or a line.

But we'd like to know in more generality. For starters, what if we're not living in the plane? Let's look at a more general result for splitting into two subsets.

Theorem 1.4 (Radon's Theorem)**.** *Let $X = \{x_1, \ldots, x_{d+2}\}$ be a set of $d+2$ points in $\mathbb{R}^d$. Then X can be divided into two disjoint subsets whose convex hulls intersect.*

The proof uses a concept we'll come back to later, called *affine independence*.

Definition 1.5. A finite set of points $x_1, \ldots, x_n$ in $\mathbb{R}^d$ is *affinely dependent* if there exist real numbers $\alpha_1, \ldots, \alpha_n$, not all zero, with

$$\sum_{i=1}^{n} \alpha_i = 0 \text{ and } \sum_{i=1}^{n} \alpha_i x_i = 0.$$

Such an expression is called an *affine dependence*. If no affine dependence exists, then the x_i's are *affinely independent*.

This is a concept that looks a lot like linear independence, and for good reason.

Exercise 1.6. Let $x_1, \dots, x_n \in \mathbb{R}^d$. The following are equivalent.

(1) $x_1, \dots, x_n$ are affinely independent.
(2) The $n-1$ vectors $x_2 - x_1, x_3 - x_1, \dots, x_n - x_1$ are linearly independent.
(3) The n vectors in $\mathbb{R}^{d+1}$ given by $(1, x_1), \dots, (1, x_n)$, are linearly independent.

In particular, this tells us that whenever we have $d+2$ or more vectors in $\mathbb{R}^d$, they are affinely dependent. With that definition under our belt, let's examine Radon's Theorem.

Proof of Radon's Theorem. Consider the set $\{(1, x_1), \dots, (1, x_{d+2})\}$ of $d+2$ points in $\mathbb{R}^{d+1}$. These must be linearly dependent. Thus $\{x_1, \dots, x_{d+2}\}$ is affinely dependent, so there exist $\alpha_1, \dots, \alpha_{d+2} \in \mathbb{R}$, not all 0, so that

$$\sum_{i=1}^{d+2} \alpha_i = 0 \text{ and } \sum_{i=1}^{d+2} \alpha_i x_i = 0.$$

Now define $I_1 = \{i \mid \alpha_i > 0\}$ and $I_2 = \{i \mid \alpha_i \leq 0\}$. Our two sets are then $X_1 = \{x_i \mid i \in I_1\}$ and $X_2 = \{x_i \mid i \in I_2\}$. We will construct a point in both convex hulls. Let $S = \sum_{i \in I_1} \alpha_i = \sum_{j \in I_2} (-\alpha_j)$. Then we take the point

$$x = \sum_{i \in I_1} \frac{\alpha_i}{S} x_i = \sum_{j \in I_2} \left(-\frac{\alpha_j}{S} \right) x_j.$$

The first expression shows that x is in the convex hull of X_1, and the second that x is in the convex hull of X_2. □

Great, so we've gotten a good start. But what about dividing our sets into more than two disjoint pieces, where all convex hulls intersect in some point?

Theorem 1.7 (Tverberg's Theorem). *Let $X = \{x_1, \dots, x_n\}$ be a set of points in $\mathbb{R}^d$, with $n \geq (r-1)(d+1)+1$. Then there is a partition $I_1, \dots, I_r$ of $\{1, 2, \dots, n\}$ such that*

$$\bigcap_{j=1}^{r} \operatorname{Conv}\{x_i \mid i \in I_j\} \neq \emptyset.$$

Note that this is saying that there is a point in the intersection of *all* convex hulls, which is much stronger than saying that any two convex hulls intersect.

This is really great! Now we have a nice generalization of Radon's Lemma telling us about dividing sets of points into arbitrarily many pieces in arbitrarily many dimensions. Our next goal will be proving it. To do so, we'll go over a couple of other theorems that look a lot like Radon's Theorem.

Proposition 1.8 (Caratheodory's Theorem). *Let $S \subseteq \mathbb{R}^d$. If $x \in \operatorname{Conv}(S)$, then $x \in \operatorname{Conv}(X)$ for some $X \subseteq S$, where X is a set of at most $d+1$ isolated points.*

Proof. Let $x \in \operatorname{Conv}(S)$. Then x is a convex combination of finitely many points in P, so

$$x = \sum_{j=1}^{k} \alpha_j x_j,$$

with $\sum_{j=1}^{k}\alpha_j = 1$, $\alpha_j \geq 0$, $x_j \in P$. If $k \leq d+1$, we are done. Otherwise, we will show that we could have represented it as a convex combination of fewer elements. Assume $k \geq d+2$. Then $x_1, \ldots, x_k$ must be affinely dependent, so there exist scalars $\beta_1, \beta_2, \ldots, \beta_k \in \mathbb{R}$, not all 0, such that

$$\sum_{j=1}^{k}\beta_j = 0 \text{ and } \sum_{j=1}^{k}\beta_j x_j = 0.$$

Not all β_j are 0, so at least one must be positive. Then for all $\gamma \in \mathbb{R}$,

$$x = \sum_{j=1}^{k}\alpha_j x_j - \gamma\sum_{j=1}^{k}\beta_j x_j = \sum_{j=1}^{k}(\alpha_j - \gamma\beta_j)x_j.$$

Let γ be given by $\gamma = \min_{1\leq j\leq k}\left\{\frac{\alpha_j}{\beta_j} \mid \beta_j > 0\right\}$, and let $\frac{\alpha_i}{\beta_i}$ be the instance attaining the minimum. Then $\gamma > 0$ and for all j, $\alpha_j - \gamma\beta_j \geq 0$, with $\alpha_i - \gamma\beta_i = 0$ by definition of γ. Thus

$$x = \sum_{j=1}^{k}(\alpha_j - \gamma\beta_j)x_j$$

is a combination of the x_j's where all coefficients are positive, their sum is one, and at least one coefficient is 0. Thus x can be represented as a convex combination of $k-1$ points of P whenever $k \geq d+2$, so we can repeat this process until we represent x as a combination of $d+1$ or fewer points in P. □

Proposition 1.9 (Colorful Caratheodory's Theorem). *Let $S_1, S_2, \ldots, S_{d+1}$ be $d+1$ sets in $\mathbb{R}^d$; think of each set as being a different color. Suppose that $x \in \bigcap_{i=1}^{r}\mathrm{Conv}(S_i)$. Then there are $x_1 \in S_1, \ldots, x_{d+1} \in S_{d+1}$ such that $x \in \mathrm{Conv}\{x_1, \ldots, x_{d+1}\}$.*

So we have $d+1$ colored sets, and a point in the intersection of all of their convex hulls. Colorful Caratheodory tells us that that point is in the convex hull of a set consisting of one point of each color.

Proof. By Caratheodory's Theorem, we can assume that each S_i is finite. Then there are finitely many options for our choices of $x_1, \ldots, x_{d+1}$, so we can assume that we have chosen $x_1, \ldots, x_{d+1}$ so that the minimum distance between x and the convex hull $C = \mathrm{Conv}\{x_1 \ldots, x_{d+1}\}$ is minimized. If $x \in C$, we are done. Assume by contradiction that $x \notin C$, and let $c \in C$ be the point nearest to x. Since C is the convex hull of finitely many points, it is a polytope, and since c is closest to x it must be on a facet of this polytope. Thus c lies in the convex hull of some points in $d-1$ dimensional space, so by Caratheodory's theorem for $d-1$-dimensional space, it can be written as a convex combination of at most d of the points x_i. Thus some point is not used; assume without loss of generality that that point is x_1. Then c will still be in the convex hull of our points if we swap in any other choice of a point in S_1 for our choice of x_1, so x_1 can be replaced without increasing the distance between x and C.

Consider the hyperplane through c orthogonal to $x - c$. It separates x from x_1. But $\mathrm{Conv}(S_1)$ contains x, so there is some $x_1' \in S_1$ on the same side of this hyperplane as x. Then c will no longer be the closest point to x, which contradicts the minimality of c. □

With that under our belt, we can prove Tverberg's Theorem.

Proof of Tverberg's Theorem. Assume that $n = (r-1)(d+1)+1$, where we can throw out extra points if we have some to throw out. Let $y_i \in \mathbb{R}^{d+1}$ be the vector whose first d coordinates are x_i's coordinates and whose last coordinate is 1, so $y_i = (x_i, 1)$.

Meanwhile, we will select r vectors in $\mathbb{R}^{r-1}$. Let $\{v_1, \dots, v_{r-1}\}$ be the standard basis, and let $v_r = -v_1 - v_2 - \cdots - v_{r-1}$. In particular, note that $v_1 + \cdots + v_r = 0$, but this is the only linear relation among the v_j's.

Now for each $1 \leq i \leq n$ and $1 \leq j \leq r$, we will examine the $(r-1)\times(d+1)$-sized matrix $v_j y_i^\top$. Let

$$A_i = \{v_1 y_i^\top, v_2 y_i^\top, \dots, v_r y_i^\top\},$$

for each $1 \leq i \leq n$. Then A_i is a set of matrices of size $(r-1)\times(d+1)$, which we can interpret as a set of vectors in $\mathbb{R}^{(r-1)\cdot(d+1)} = \mathbb{R}^{n-1}$. Since $v_1 + v_2 + \cdots + v_r = 0$,

$$\frac{1}{r} v_1 y_i^\top + \cdots + \frac{1}{r} v_r y_i^\top = 0,$$

so $0 \in \mathrm{Conv}(A_i)$ for every i. Thus we can apply the Colorful Caratheodory theorem for our sets A_i, so there exists $a_i \in A_i$ for all i such that 0 is in the convex hull of $A = \{a_1, \dots, a_n\}$. This means that

$$\sum_{i=1}^{n} \alpha_i a_i = 0$$

for some coefficients $\alpha_i \geq 0$, $\sum \alpha_i = 1$.

For each i, $a_i \in A_i$, so $a_i = v_{\sigma(i)} y_i^\top$ for some index $\sigma(i) \in \{1, \dots, r\}$. This index will be how we define our partition of $\{1, \dots, n\}$ into r subsets. In particular, we define

$$I_k = \{i : \sigma(i) = k\}.$$

Then $\bigcup_{k=1}^r I_k = \{1, \dots, n\}$, and this is a partition of our points; we will show it is the one we are looking for. Let $z_k = \sum_{i \in I_k} \alpha_i y_i$. Then

$$\begin{aligned} \sum_{i=1}^{n} \alpha_i v_{\sigma(i)} y_i^\top &= 0 \\ \Rightarrow \sum_{k=1}^{r} \sum_{j \in I_k} \alpha_j v_{\sigma(j)} y_j^\top &= 0 \\ \Rightarrow \sum_{k=1}^{r} v_k \sum_{j \in I_k} \alpha_j y_j^\top &= 0 \\ \Rightarrow \sum_{k=1}^{r} v_k z_k^\top &= 0. \end{aligned}$$

We chose our v_i's so that the only linear dependence of them is $v_1 + \cdots + v_r = 0$, so every entry of the z_k's must be the same, and thus the z_k's are all equal. Let $z_k = \begin{pmatrix} c_1 \\ \vdots \\ c_{d+1} \end{pmatrix}$, which is independent of k. Let $z^* = \begin{pmatrix} \frac{c_1}{c_{d+1}} \\ \vdots \\ \frac{c_d}{c_{d+1}} \end{pmatrix} \in \mathbb{R}^d$. Since z_k is a convex combination of the vectors $y_i = (x_i, 1)$ contained in each I_k, z^* must be a convex combination of the vectors x_i contained in each I_k. Thus $z^* \in \bigcap_{k=1}^r \mathrm{Conv}\{x_i \mid i \in I_k\}$, so we are done. □

2. "Topological"

2.1. **Maps from simplices.** We've been talking a lot about finitely many points and their convex hulls. But, we could rephrase the entire question in terms of *maps from simplices*. Intuitively, a simplex is an arbitrary-dimensional generalization of a tetrahedron or a triangle.

Definition 2.1. A *standard n-simplex* is the set in $\mathbb{R}^{n+1}$ consisting of the convex hull of the standard basis vectors $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \dots, e_{n+1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$. We will denote it σ^n.

In general, an *n-simplex* in $\mathbb{R}^d$ is the convex hull of n affinely independent vectors in $\mathbb{R}^d$.

So indeed, σ^1 is a line, σ^2 is a triangle, and σ^3 is a tetrahedron.

We won't use it till a bit later, but let's go ahead and define a generalization of a simplex, called a *simplicial complex*.

Definition 2.2. A *simplicial complex* Δ is a nonempty family of simplices such that the following two conditions hold:

(1) Each face of any simplex in Δ is a simplex in Δ.
(2) The intersection $\sigma \cap \rho$ of any two simplices $\sigma, \rho \in \Delta$ is a face of σ and a face of ρ.

The first requirement is just a nice way for us to say that our simplicial complex records all of its faces, and the second one tells us that parts of a simplicial complex have to snap together perfectly along faces.

In an abstract combinatorial setting, we don't always want to know exactly how our simplicial complex is realized in space, just what vertices are connected as faces. This gives the following definition:

Definition 2.3. An *abstract simplicial complex* K on a finite vertex set $V(K)$ is a set of subsets of $V(K)$ such that for all $F \in K$ and $G \subseteq F$ a subset, $G \in K$ as well.

In other words, $V(K)$ is recording the vertices of a simplicial complex, and K is the set of all faces. Any subset of a face must also be a face, and with this data we are by definition not allowing any faces to intersect along anything but faces.

Proposition 2.4. *For any subset S of the standard basis vectors in $\mathbb{R}^{n+1}$, $\mathrm{Conv}(S)$ is a face of σ^{n+1} of dimension $|S| - 1$.*

Proof. Exercise! □

So, any two points of a triangle determine an edge, as do any two points of a tetrahedron, and any three points of a tetrahedron determine a two-dimensional face, and so on.

Recall that the convex hull of finitely many points is the set of convex combinations of those points, or

$$\mathrm{Conv}\{x_1, \dots, x_n\} = \left\{ \sum_{i=1}^n a_i x_i \mid a_i \geq 0, \sum_{i=1}^n a_i = 1 \right\}.$$

So if we have a linear map from one Euclidean space to another, taking convex hulls commutes with that linear map. Since the standard basis vectors are linearly independent, for any set of points $x_1, \ldots, x_n \in \mathbb{R}^d$, there exists a linear map from $\mathbb{R}^n$ taking $e_i \mapsto x_i$. This restricts to a unique linear map $\sigma^{n-1} \to \mathrm{Conv}\{x_1, \ldots, x_n\}$, where the convex hulls of subsets of the points are exactly the images of the faces of our simplex. In this language, the Tverberg theorem becomes:

Theorem 2.5 (Tverberg's Theorem). *Let $f : \sigma^{n-1} \to \mathbb{R}^d$ be a linear map, with $n \geq (r-1)(d+1)+1$. Then there exist r disjoint faces of σ^{n-1} whose images all intersect.*

So then, a natural question to ask is, why are we just looking at linear maps? What if f is any continuous function from $\sigma^{n-1} \to \mathbb{R}^d$?

Theorem 2.6 (Topological Tverberg's Conjecture). *Let $f : \sigma^{n-1} \to \mathbb{R}^d$ be a continuous map, with $n \geq (r-1)(d+1)+1$. Then there exist r disjoint faces of σ^{n-1} whose images all intersect.*

This would be pretty magical. If we go back to our first Radon's Lemma case, with four points in $\mathbb{R}^2$, this would say that no matter how wiggly or curvy or crazy my edges are, if I draw a tetrahedron on the blackboard, two disjoint faces intersect.

The proof in the case when r, the number of sets we're partitioning, is prime, was completed first by Tverberg. We'll spend the next couple days proving it.

Theorem 2.7 (Topological Tverberg - prime case). *Let $d \in \mathbb{N}_{\geq 1}$ and let $p \in \mathbb{N}$ be prime. Let $n \in \mathbb{N}$ with $n \geq (r-1)(d+1)+1$. Then for any continuous function $f : \sigma^{n-1} \to \mathbb{R}^d$, there exist p disjoint faces of σ^{n-1} whose images all intersect.*

3. $\mathbb{Z}_2$-SPACES AND BORSUK ULAM: THE TOPOLOGICAL RADON THEOREM

Let's start with the case of the topological Radon theorem, which will introduce us to the important techniques. Here's the theorem we're working towards:

Theorem 3.1 (Topological Radon). *Let $d \in \mathbb{N}_{\geq 1}$.. Let $n \in \mathbb{N}$ with $n \geq d+2$. Then for any continuous function $f : \sigma^{n-1} \to \mathbb{R}^d$, there exist two disjoint faces of σ^{n-1} whose images intersect.*

In other words, it's the case $p = 2$. To attack it, we'll start by defining $\mathbb{Z}_2$-spaces.

Definition 3.2. A $\mathbb{Z}_2$*-space* is a pair (X, ν), with X a topological space (think $\mathbb{R}^n$, S^n, or the boundary of a simplicial complex) and $\nu : X \to X$ is a continuous function with $\nu^2 = \nu \circ \nu = \mathrm{id}_X$.

The $\mathbb{Z}_2$-action ν is *free* if $\nu(x) \neq x$ for all $x \in X$, i.e. there are no fixed points.

For $\mathbb{Z}_2$ spaces (X, ν) and (Y, ω), a $\mathbb{Z}_2$*-map* $f : (X, \nu) \to (Y, \omega)$ is a continuous map $X \to Y$ that is compatible with the actions, i.e. $f(\nu(x)) = \omega(f(x))$. $\mathbb{Z}_2$ maps are also called *invariant* or *antipodal*.

For example, $\mathbb{R}^n$ and S^n are $\mathbb{Z}_2$ spaces under the map $x \mapsto -x$, which unless otherwise specified will always be the assumed $\mathbb{Z}_2$ action on these spaces.

Definition 3.3. We will say that two $\mathbb{Z}_2$-spaces X and Y are $\mathbb{Z}_2$*-compatible* if there exists a $\mathbb{Z}_2$-map from X to Y, and we will write $X \overset{\mathbb{Z}_2}{\to} Y$. Otherwise, we will write $X \overset{\mathbb{Z}_2}{\not\to} Y$.

Exercise 3.4. Assume that (Y, ω) is a nonfree $\mathbb{Z}_2$ space, i.e. $\omega(y_0) = y_0$ for some y_0. Then for all $\mathbb{Z}_2$ spaces (X, ν), $X \overset{\mathbb{Z}_2}{\to} Y$.

The compatibility relation is reflexive and transitive, but it is not symmetric. For example, $S^n \overset{\mathbb{Z}_2}{\to} S^{n+1}$ via the inclusion map at the equator. However, the reverse does not hold. This is an important result, called the Borsuk-Ulam theorem, that a lot of what we'll talk about this week rests on. So, let's prove it.

3.1. **Borsuk-Ulam.**

Theorem 3.5 (Borsuk-Ulam)**.** *There is no antipodal mapping $S^n \to S^{n-1}$. In other words, with S^n and S^{n-1} given the $\mathbb{Z}_2$ action of mapping $x \mapsto -x$, $S^n \overset{\mathbb{Z}_2}{\not\to} S^{n-1}$.*

This statement that we'll need is equivalent to, and in particular implied by, a different statement of the Borsuk-Ulam theorem, called Tucker's Lemma.

Let $\diamond^{n-1}$ denote the simplicial complex consisting of the boundary of the *crosspolytope*, which is dual to the n-cube. If that is schmancy, you can think of it as follows: the vertex set $V(\diamond^{n-1})$ is the set $\{\pm 1, \pm 2, \ldots, \pm n\}$, and a subset $F \subseteq V(\diamond^{n-1})$ if and only if it never contains both a number and its negative.

We could inflate $\diamond^{n-1}$ like a beach ball, which is a quick, intuitive way of saying that it is homeomorphic to S^{n-1}. The polytope it bounds will be denoted by $\hat{B}^n$. Another way of considering it is examining the distance function $||x||_1 := \sum_i |x_i|$ on $\mathbb{R}^n$. In this distance, $\hat{B}^n$ becomes the unit ball.

Definition 3.6. A triangulation of $\hat{B}^n$ is *special* if it refines the octahedral subdivision.

Theorem 3.7 (Tucker's Lemma)**.** *Let T be a special triangulation of $\hat{B}^n$ that is antipodally symmetric on the boundary $\diamond^{n-1}$. Assume that the vertices of T are assigned labels from $\{\pm 1, \ldots, \pm n\}$ via a labeling map $\lambda : V(T) \to \{\pm 1, \ldots, \pm n\}$. If antipodal vertices of T on $\diamond^{n-1}$ receive labels that are equal but of opposite sign, i.e. if $\lambda(-v) = -\lambda(v)$ for $v \in V(T) \cap \diamond^{n-1}$, then T contains an edge with the same property, i.e. an edge whose vertices have opposite labels.*

So we have two proofs ahead of us. First, that Tucker's Lemma will prove Borsuk-Ulam, and second, a proof of Tucker's Lemma.

Tucker's Lemma implies Borsuk-Ulam. Let $f : S^n \to \mathbb{R}^n$ be any continuous function. We will find a pair of antipodal points x and $-x$ that are mapped to the same point in $\mathbb{R}^n$. Define $g : \hat{B}^n \to \mathbb{R}^n$ via

$$g(x) = f(x, 1 - ||x||_1) - f(-x, ||x||_1 - 1).$$

Then $g(-x) = -g(x)$ whenever $x \in S^{n-1}$, the boundary. Also, we are now searching for a point x with $g(x) = 0$, since this will imply that $(x, 1 - ||x||_1)$ and its antipode are mapped to the same point.

For any special triangulation T of $\hat{B}^n$, let i be the coordinate such that $|g(v)_i|$, the absolute value of the ith coordinate of $g(v)$, is maximal. Then label vertex v with a $+i$ if $g(v)_i > 0$ and a $-i$ if $g(v)_i < 0$. If $g(v) = 0$, then we must have $f(v, 1 - ||v||_1) = f(-v, ||v||_1 - 1)$, so $(v, 1 - ||v||_1)$ and its negative is our pair of antipodal points mapped to the same image under f. If there is no unique maximal coordinate, we choose the smallest index of a maximal coordinate.

This labelling satisfies the requirements of the lemma, since g is antipodal on the boundary. Thus T contains an edge whose vertices have opposite signs. Now we can do it again while requiring that the maximal radius of T approach 0, to get a sequence of these edges decreasing in length whose boundaries always have opposite sign. Let l be any limit point of a sequence of points on these edges. By continuity, since these smaller and smaller edges get closer and closer to l, $g(l) = 0$. But then $f(l, 1 - ||l||_1) = f(-l, ||l||_1 - 1)$, so $(l, 1 - ||l||_1)$ and its antipode are mapped by f to the same point.

Now we know that any continuous map $f : S^n \to \mathbb{R}^n$ takes some point and its antipode to the same image. Assume by contradiction there is a continuous antipodal map $f : S^n \to S^{n-1}$. Then via composition with $S^{n-1} \hookrightarrow \mathbb{R}^n$, we have a continuous map $f : S^n \to \mathbb{R}^n$ where no point is taken to the same image as its antipode, which is a contradiction. □

Proof of Tucker's Lemma. Let T be a special triangulation of $\hat{B}^n$. Note that by subdividing, we can construct arbitrarily fine special triangulations. Let $\lambda : V(T) \to \{\pm 1, \ldots, \pm n\}$ be a labeling antipodal on the boundary.

For a simplex $\sigma \in T$, we write $\lambda(\sigma) = \{\lambda(v) \mid v \text{ is a vertex of } \sigma\}$. We will also define another set $S(\sigma)$. For a vertex x in the interior of σ, $S(\sigma)$ is a set of labels given by

$$S(\sigma) := \{+i \mid x_i > 0, i = 1, \ldots, n\} \cup \{-i \mid x_i < 0, i = 1, \ldots, n\}.$$

This is the set of indices signed by whether their entries in x are positive or negative.

Since T is a special triangulation, all choices of $x \in \sigma$ give the same $S(\sigma)$ for any σ, since each simplex σ is contained in one quadrant of $\hat{B}^n$, where no sign of an entry will change.

Let's call a simplex $\sigma \in T$ *happy* if $S(\sigma) \subseteq \lambda(\sigma)$. In this way, we've decided on sets of labels that we like, and our simplex is happy if all of our favorite labels appear in its labels. SEE PICTURE.

OK, so let's talk about happy simplices for a bit. Let σ be a happy simplex, and let $k = |S(\sigma)|$. Then σ lies in the k-dimensional subspace L_σ spanned by the k coordinate axes x_i such that $\pm i \in S(\sigma)$, so $\dim \sigma \leq k$. But σ has at least k vertex labels, so $\dim \sigma \geq k - 1$. We'll say that σ is *tight* if $\dim \sigma = k - 1$, so that all vertex labels are needed to make σ happy. If $\dim \sigma = k$, by contrast, we will call σ *loose*. For a loose happy simplex σ, either some vertex label is repeated, or there is a label not appearing at all in $S(\sigma)$.

If σ is a boundary happy simplex, then it must be tight (since no vertex is at the origin, so for it to be in the span of k vectors it must be $k-1$-dimensional). Meanwhile, a nonboundary happy simplex may be tight or loose. The simplex containing only the point 0 is happy and loose.

Now define an (undirected) graph Γ as follows. The vertices are all happy simplices, and we connect two happy simplices $\sigma, \tau \in T$ by an edge if

(a) σ and τ are antipodal boundary simplices, or
(b) σ is a facet of τ, i.e. a face of τ of dimension $\dim \tau - 1$, and $\lambda(\sigma) = S(\tau)$. In other words, the labels of σ alone make τ happy.

The simplex $\{0\}$ has degree 1 in Γ, since it connects to the edge of the triangulation that is made happy by the label $\lambda(0)$. We prove that if there is no complementary edge, then any other vertex σ of our graph Γ has degree 2. A finite graph can't contain only one vertex of odd degree, so this will establish Tucker's lemma.

Now we get to split into cases.

1. σ is a tight happy simplex. Then any neighbor τ of σ either equals $-\sigma$ or has σ as a facet. We will further split into cases now:
   (a) σ lies on the boundary $\diamond^{n-1}$. Then $-\sigma$ is one neighbor of σ. Any other neighbor τ has σ as a facet and is made happy by the labels of σ. Thus it must lie in the

coordinate subspace L_σ as above, i.e. the space spanned by the coordinate axes x_i with $\pm i \in S(\sigma)$. The dimension of L_σ is $k = \dim \sigma + 1$. The intersection $L_\sigma \cap \hat{B}^n$ is a k-dimensional polytope triangulated by the simplices of T contained in L_σ. Since σ is a boundary $(k-1)$-simplex, it is the facet of precisely one such simplex.
Thus σ is connected to two other vertices of Γ.

(b) Assume now that σ doesn't lie on the boundary. By the same argument as above, σ is a facet of exactly two simplices made happy by its labels, which are its two neighbors.

2. σ is a loose happy simplex. We again do some subcases!
   (a) If $S(\sigma) = \lambda(\sigma)$ and some label occurs twice on σ. Then σ is adjacent to two of its facets, where each duplicated label is removed once. It cannot be the facet of a larger happy simiplex, for it is too loose.
   (b) There is an extra label $i \in \lambda(\sigma) \setminus S(\sigma)$. Since there is no opposite or complementaray edge, $-i \notin \lambda(\sigma)$. One of the neighbors of σ is the facet of σ not containing the vertex labeled by i. Also, σ is a facet of exactly one loose simplex σ' made happy by the labels of σ; namely, one with $S(\sigma') = \lambda(\sigma) = S(\sigma) \cup \{i\}$. How do we find σ'? Well, if we go from an interior point of σ in the direction of the $x_{|i|}$-axis, in the positive direction for $i > 0$ and negative for $i < 0$, we will find ourselves in σ'.

Thus for each possibility, if we assume we have no complementary edge, there are exactly two neighbors, whcih completes the proof that there must be a complementary edge. $\square$

3.2. $\mathbb{Z}_2$ **indices.** Now that we have the Borsuk-Ulam theorem under our belts, let's get back to $\mathbb{Z}_2$-spaces and something new, $\mathbb{Z}_2$-indices. (I promise, eventually we will truly return to Tverberg). Recall that before we digressed with Borsuk-Ulam, we defined a relation $X \stackrel{\mathbb{Z}_2}{\to} Y$ by saying that $X \stackrel{\mathbb{Z}_2}{\to} Y$ if there exists a $\mathbb{Z}_2$ map $f : X \to Y$, with $f(\nu(x)) = \omega(f(y))$ for (X, ν) and (Y, ω) the $\mathbb{Z}_2$ structures. As hinted at by our extensive discussion of Borsuk-Ulam, spheres play a key role. We'll use them as a yardstick to measure the "size" of $\mathbb{Z}_2$-spaces with respect to our relation $\stackrel{\mathbb{Z}_2}{\to}$.

Definition 3.8. Let (X, ν) be a $\mathbb{Z}_2$-space. The $\mathbb{Z}_2$-*index* of (X, ν) is defined as

$$\operatorname{ind}_{\mathbb{Z}_2}(X) = \min\{n \in \mathbb{N} \mid X \stackrel{\mathbb{Z}_2}{\to} S^n\},$$

where S^n has the standard antipodal $\mathbb{Z}_2$ action.

The index can be a natural number or infinite. For example, $\operatorname{ind}_{\mathbb{Z}_2}(\mathbb{R}) = \infty$, since 0 cannot be mapped to any point in any sphere via a $\mathbb{Z}_2$ map.

Proposition 3.9 (Index Properties). *(i) If $X \stackrel{\mathbb{Z}_2}{\to} Y$, then $\operatorname{ind}_{\mathbb{Z}_2}(X) \leq \operatorname{ind}_{\mathbb{Z}_2}(Y)$. Equivalently, if $\operatorname{ind}_{\mathbb{Z}_2}(X) > \operatorname{ind}_{\mathbb{Z}_2}(Y)$, then $X \stackrel{\mathbb{Z}_2}{\not\to} Y$.*
(ii) $\operatorname{ind}_{\mathbb{Z}_2}(S^n) = n$ for all $n \geq 0$.
(iii) If X is $n-1$-connected, meaning that every continuous map $S^k \to X$ can be extended to a map $B^{k+1} \to X$ for all $k \leq n$, then $\operatorname{ind}_{\mathbb{Z}_2}(X) \geq n$.

(iv) *If K is a simplicial complex that has a free $\mathbb{Z}_2$ space structure and is of dimension n, then* $\text{ind}_{\mathbb{Z}_2}(K) \leq n$.

As a note to the definition of n-connectivity, the n-sphere is $(n-1)$-connected for all n; this is an important fact. Any map $S^k \to S^n$, for $k < n$, must miss some point, so it is a map $S^k \to \mathbb{R}^n$ via composition with the stereographic projection. Thus it can be contracted in $\mathbb{R}^n$ to a map to a point, so the same holds for the map in S^n; and thus we can fill in the map to be a map from $B^{k+1} \to S^n$.

Proof. (i) This follows from the definition. If $X \overset{\mathbb{Z}_2}{\to} Y$ and $Y \overset{\mathbb{Z}_2}{\to} S^n$, then $X \overset{\mathbb{Z}_2}{\to} Y \overset{\mathbb{Z}_2}{\to} S^n$, so $X \overset{\mathbb{Z}_2}{\to} S^n$, which implies that $\text{ind}_{\mathbb{Z}_2}(X) \leq \text{ind}_{\mathbb{Z}_2}(Y)$. The second statement is the contrapositive.

(ii) This is the Borsuk-Ulam Theorem! $S^k \overset{\mathbb{Z}_2}{\to} S^{n-1}$ for $k \leq n-1$, so if $S^n \overset{\mathbb{Z}_2}{\to} S^k$ for $k < n$, then $S^n \overset{\mathbb{Z}_2}{\to} S^k \overset{\mathbb{Z}_2}{\to} S^{n-1}$, contradicting Borsuk-Ulam.

(iii) By part (ii), it suffices to find some $\mathbb{Z}_2$-map $g : S^n \to X$. Then by parts (i) and (ii), $\text{ind}_{\mathbb{Z}_2}(X) \geq \text{ind}_{\mathbb{Z}_2}(S^n) = n$. We will construct a $\mathbb{Z}_2$-map $g_k : S^k \to X$ inductively for $k \leq n$.

If $k = 0$, we pick two antipodes and map one point to one, and the other point to the other. For the inductive step, consider S^{k-1} as an equatorial subset of S^k. Now, via the projection map $\pi : \mathbb{R}^{k+1} \to \mathbb{R}^k$ that deletes the last coordinate, the upper hemisphere $H_k^+ = \{x \in S^k \mid x_{k+1} \geq 0\}$ is homeomorphic to the ball B^k. Since X is $(k-1)$-connected, if a $\mathbb{Z}_2$-map $g_{k-1} : S^{k-1} \to X$ has been constructed, we can extend it to a continuous map $\bar{g}_{k-1} : B^k \to X$. Then via π, we define $g_k : H_k^+$ by $g_k = \bar{g}_{k-1} \circ \pi : H_k^+ \to B^k \to X$. Meanwhile, for $x \in H_k^-$, set $g_k(x) = \nu(g_k(-x))$ to get a map $g_k : S^k \to X$. Since g_k is antipodal on the intersection of the hemispheres (namely, the equator S^{k-1}, this map is well-defined. Since it is continuous on both hemispheres, it is continuous, and by construction it is a $\mathbb{Z}_2$-map. This completes the inductive step, so we are done.

(iv) Because of part (i), we need only prove that $K \overset{\mathbb{Z}_2}{\to} S^n$. Let $K^{\leq k}$ be the k-skeleton of K, i.e. just containing all simplices of dimension k or lower. We will inductively construct $\mathbb{Z}_2$-maps $g_k : K^{\leq k} \to S^n$. As a base case, for $k = 0$, $K^{\leq k}$ is a set of points, some of which may be antipodes; we may pick effectively arbitrary images for these, with antipodes mapped to antipodes.

As for the inductive step, given g_{k-1}, we divide the k-dimensional simplices in K into the orbits under the $\mathbb{Z}_2$-action. Then each class consists of two disjoint simplices F and $\nu(F)$, since the $\mathbb{Z}_2$ action is free on K. Pick one simplex from each class. For these simplices, extend g_{k-1} on the interior of the simplex via the fact that S^n is $(k-1)$-connected. Then define g_k on the interiors of the remaining simplices via the constraint that g_k must be a $\mathbb{Z}_2$-map. This completes the proof.

□

Note that for the last two things, we basically had the same proof twice, where we picked one antipode from each to extend the function. In our first case, this meant looking at a hemisphere, and in the second case, we did it more directly.

4. TOPOLOGICAL JOINS AND DELETED JOINS

The last ingredient that we'll need is the *join* of topological spaces.

Definition 4.1. Let X and Y be topological spaces. The *join* $X * Y$ of X and Y is the space obtained by taking the product $X \times Y \times [0,1]$, identifying $(x,y,0)$ with $(x',y,0)$ for all $x, x' \in X$ and $y \in Y$, and identifying $(x,y,1)$ with $(x,y',1)$ for all $x \in X$ and $y, y' \in Y$.

Another way to think of the join is by taking the set of all lines from any point in X to any point in Y. This makes it very intuitive that it would be associative; the join of htree spaces is the set of all triangles with a corner in each space.

Example 4.2. Draw! The join of two line segments is a tetrahedron.
The join of a point and a shape is a cone over that shape.
The join of two disjoint points and two other disjoint points is a circle.
The join of a circle and two disjoint points is a sphere.
The join of an n-sphere and two disjoint points is an $n+1$-sphere.
For spaces we'll be looking at, the join is associative (this is somewhat tricky), so the join of an n-sphere and an m-sphere is an $n+m+1$-sphere.

A special case of the join is the join of two simplicial complexes.

Definition 4.3. For K and L simplicial complexes, their join $K * L$ is the simplicial complex with vertex set $V(K) \sqcup V(L)$ and simplices $\{F \cup G \mid F \in K, G \in L\}$.

It's clearer to see that this join is associative.
One nice thing about joins is that they play nice with indices.

Lemma 4.4. *If X and Y are $\mathbb{Z}_2$-spaces,*

$$\mathrm{ind}_{\mathbb{Z}_2}(X * Y) \leq \mathrm{ind}_{\mathbb{Z}_2}(X) + \mathrm{ind}_{\mathbb{Z}_2}(Y) + 1.$$

This follows from the fact that $S^n * S^m \cong S^{n+m+1}$. If $X \overset{\mathbb{Z}_2}{\to} S^n$ and $Y \overset{\mathbb{Z}_2}{\to} S^m$, then pointwise we can map $X * Y \to S^n * S^m$, and this will preserve the $\mathbb{Z}_2$-structure. Thus $X * Y \overset{\mathbb{Z}_2}{\to} S^n * S^m \cong S^{n+m+1}$, which proves the lemma.

Now we can define a *deleted join of a simplicial complex*, which we will eventually use to prove the Topological Tverberg Theorem. This is close to the definition of taking the join of a simplicial complex with itself, except that we *delete* the portion that corresponds to the join of each simplex with itself.

Definition 4.5. Let K be a simplicial complex. The *deleted join* K^{*2}_{Δ} of K is a simplicial complex defined as follows.

(1) It has vertex set
$$V(K^{*2}_{\Delta}) = V(K) \times \{1,2\}.$$
or two copies of $V(K)$ labeled by 1 and by 2.

(2) The faces are given by
$$K^{*2}_{\Delta} = \{(F_1 \times \{1\}) \cup (F_2 \times \{2\}) \mid F_1, F_2 \in K, F_1 \cap F_2 = \emptyset\}.$$

The $F_1 \cap F_2 = \emptyset$ requirement is what makes this join deleted.

Example 4.6. (1) The deleted join $(\sigma^0)^{*2}_{\Delta}$ of a single point is two disjoint points.

(2) The deleted join K^{*2}_{Δ} of a simplicial complex consisting of disjoint points is a bipartite graph missing all horizontal lines.
(3) The deleted join of two disjoint edges is a square.

Note that joins have $\mathbb{Z}_2$-actions from swapping the two coordinates of K. So do deleted joins, but because we've deleted this diagonal, we've in fact deleted exactly the portion that would prevent this action from being free. So K^{*2}_{Δ} is a free $\mathbb{Z}_2$-space.

Deleted joins commute nicely with joins.

Lemma 4.7. *For K, L simplicial complexes,*

$$(K * L)^{*2}_{\Delta} = K^{*2}_{\Delta} * L^{*2}_{\Delta}.$$

Proof. The simplices on the left hand side are of the form $(F_1 \sqcup G_1) \sqcup (F_2 \sqcup G_2)$ with $F_1, F_2 \in K$, $G_1, G_2 \in L$, $F_1 \cap F_2 = \emptyset = G_1 \cap G_2$. On the right hand side, we instead have the simplex $(F_1 \sqcup F_2) \sqcup (G_1 \sqcup G_2)$, with the same conditions on the Fs and Gs. □

Corollary 4.8.

$$(\sigma^n)^{*2}_{\Delta} \cong S^n.$$

Proof. $\sigma^n \cong (\sigma^0)^{*(n+1)}$, so

$$((\sigma^0)^{*(n+1)})^{*2}_{\Delta} \cong ((\sigma^0)^{*2}_{\Delta})^{*(n+1)} \cong (S^0)^{*(n+1)} \cong S^n.$$

□

So now that we have all that machinery, let's talk nonembeddability! (Aka, Radon.)

Here's the basic idea. We'll prove that if we have a continuous map f from a simplicial complex K to $\mathbb{R}^d$ which is "bad," (for example, which contradicts Radon's theorem, so all disjoint faces have disjoint images), then we'll have a map $K^{*2}_{\Delta} \xrightarrow{\mathbb{Z}_2} (\mathbb{R}^d)^{*2}_{\Delta}$. From our discussion of indices, this will us that $\text{ind}_{\mathbb{Z}_2}(K^{*2}_{\Delta}) \leq \text{ind}_{\mathbb{Z}_2}(\mathbb{R}^d)^{*2}_{\Delta}$. If we compute indices and show that this inequality doesn't hold, we get exactly what we want. OK, let's do it. We'll always be considering a simplicial complex K, and maps from $K \to \mathbb{R}^d$.

Definition 4.9. A map $f : K \to \mathbb{R}^d$ is *bad* if $f(x_1) \neq f(x_2)$ whenever the simplices containing x_1 and x_2 in their interiors are disjoint.

OK, so a bad map is bad. We want to prove they don't exist if the dimension of K is high enough. Our first trick is to consider the map $f * f : K^{*2}_{\Delta} \to (\mathbb{R}^d)^{*2}_{\Delta}$ rather than the map f itself, where

$$f * f(tx, (1-t)y) = (tf(x), (1-t)f(y)).$$

Now the images of some disjoint pair $(tx_1, (1-t)x_2)$ under $f * f$ have the form

$$(tf(x_1), (1-t)f(x_2)) \in (\mathbb{R}^d)^{*2}.$$

Since $f(x_1) \neq f(x_2)$ for $x_1, x_2 \in K$ with disjoint simplices, all such points are contained in

$$\{(ty_1, (1-t)y_2) \mid t \in [0,1], y_1, y_2 \in \mathbb{R}^d, y_1 \neq y_2\},$$

which in turn is contained in the "deleted join" of $\mathbb{R}^d$, namely

$$(\mathbb{R}^d)^{*2}_{\Delta} := (\mathbb{R}^d)^{*2} \setminus \{(y/2, y/2) \mid y \in \mathbb{R}^d\}.$$

This is very analogous to our other deleted joins, in that we just took out the diagonal identifications.

This set has a $\mathbb{Z}_2$ action on it, from swapping the coordinates y_1 and y_2! Note that we're still recording the coordinate t. And since there is no diagonal, this is a free $\mathbb{Z}_2$ space as well.

Both $\mathbb{Z}_2$ actions are just swapping coordinates, so we in fact have $f^{*2} : K^{*2}_\Delta \xrightarrow{\mathbb{Z}_2} (\mathbb{R}^d)^{*2}_\Delta$. This tells us that

$$\mathrm{ind}_{\mathbb{Z}_2}(K^{*2}_\Delta) \leq \mathrm{ind}_{\mathbb{Z}_2}((\mathbb{R}^d)^{*2}_\Delta).$$

OK, so this is starting to feel like we're almost there. We wanted a contradiction from this bad f, and now as long as we show that the two indices above are in the reverse order, we're good! To that end, let's compute the index of $(\mathbb{R}^d)^{*2}_\Delta$.

Lemma 4.10. *There is a $\mathbb{Z}_2$-map $g : (\mathbb{R}^d)^{*2}_\Delta \to S^d$, so $\mathrm{ind}_{\mathbb{Z}_2}(\mathbb{R}^d)^{*2}_\Delta \leq d$.*

Proof. Instead of $\mathbb{R}^d$, we'll consider a d-dimensional open ball B which is homeomorphic, so that we'll have a bounded set. We'll consider them as subsets of $\mathbb{R}^{2d+2}$, with $\varphi_1, \varphi_2 : \mathbb{R}^d \to \mathbb{R}^{d+2}$ given by

$$\varphi_1(x) = (1, x_1, \ldots, x_d, 0, \ldots, 0), \varphi_2(y) = (0, \ldots, 0, 1, y_1, \ldots, y_d).$$

These are two copies of $\mathbb{R}^d$ in $\mathbb{R}^{2d+2}$, so we will consider a copy of the open ball B in each. Then define $h : B^{*2}_\Delta \to (\mathbb{R}^{d+1})^2$ via

$$h(tx, (1-t)y) \mapsto t\varphi_1(x) + (1-t)\varphi_2(y),$$

so B^{*2}_Δ lives inside $\mathbb{R}^{d+2}$ in this way. This is a $\mathbb{Z}^2$ map where the action is given by swapping coordinates. Crucially, the map avoids the diagonal, since t is recorded by the first coordinate and x is never equal to y in the deleted join. Then we can compose with a map from $(\mathbb{R}^{d+1})^2 \setminus D$, for D the diagonal, to $S^d \subseteq \mathbb{R}^{d+1}$ given by

$$(x, y) \mapsto \frac{x-y}{||x-y||}.$$

This is also a $\mathbb{Z}_2$ map, so the composition is, and we are done. □

This completes the proof of the following theorem.

Theorem 4.11. *Let K be a simplicial complex. If*

$$\mathrm{ind}_{\mathbb{Z}_2}(K^{*2}_\Delta) > d,$$

then for every continuous mapping $f : K \to \mathbb{R}^d$, the image of some two disjoint faces of K intersect.

In other words, if the index of K^{*2}_Δ is too big, K can admit no bad maps. Finally, we are ready for the topological Radon theorem.

Theorem 4.12 (Topological Radon)**.** *Let $d \in \mathbb{N}_{\geq 1}$.. Let $n \in \mathbb{N}$ with $n \geq d+2$. Then for any continuous function $f : \sigma^{n-1} \to \mathbb{R}^d$, there exist two disjoint faces of σ^{n-1} whose images intersect.*

Proof. By the above theorem, it suffices to show that

$$\mathrm{ind}_{\mathbb{Z}_2}(\sigma^{n-1})^{*2}_\Delta > d.$$

By Corollary 4.8, $(\sigma^{n-1})^{*2}_\Delta \cong S^{n-1}$, and $\mathrm{ind}_{\mathbb{Z}_2}(S^{n-1}) = n-1 \geq d+2-1 = d+1 > d$. So we're done! □

Wow, we did something! (insert spiel recapping intuition). Let's do it again, but a little harder this time, for the topological Tverberg theorem.

5. $\mathbb{Z}_p$-SPACES AND $\mathbb{Z}_p$-ACTIONS: PRIME TOPOLOGICAL TVERBERG

OK, so the Topological Radon theorem was the case $p = 2$, and we were working with $\mathbb{Z}_2$. Now, we want to work with arbitrary p, so we'll work with $\mathbb{Z}_p$. Many of the definitions are a bit more complicated, but the results will follow the same outline. From now on, we will let p be a positive prime number.

Definition 5.1. A $\mathbb{Z}_p$*-space* is a pair (X, ν), with X a topological space and $\nu : X \to X$ a continuous function with $\nu^p = \nu \circ \cdots \circ \nu = \mathrm{id}_X$.

Just like with $\mathbb{Z}_2$, the $\mathbb{Z}_p$ action is *free* if there are no fixed points, and a $\mathbb{Z}_p$-map between spaces (X, ν) and (Y, ω) is a map $f : X \to Y$ satisfying $f(\nu(x)) = \omega(f(x))$ for all x.

For $\mathbb{Z}_2$-spaces, we had a nice example of spheres where we could map any point to its antipode. For $\mathbb{Z}_p$-spaces, we can't do quite the same thing as nicely - how do we have a $\mathbb{Z}_3$ action on S^2 with no fixed points, for example? Well, here's what we can do. One way of writing the n-sphere, as we saw, was $(S^0)^{*(n+1)}$, with the $\mathbb{Z}_2$ action acting componentwise. Note that S^0 is just two points, so we could have written $(\mathbb{Z}_2)^{*(n+1)}$ instead of $(S^0)^{*(n+1)}$. But hey, this means that we could write $(\mathbb{Z}_p)^{*(n+1)}$ if we want to get a $\mathbb{Z}_p$-action!

So, these are going to be our generalizations of "spheres" with their antipodal actions: instead, we have $(\mathbb{Z}_p)^{*(n+1)}$ with a coordinatewise rotation action. Notably, these spaces are also $(n-1)$-connected, so many proofs about them go similarly as for spheres.

Definition 5.2. Let X be a $\mathbb{Z}_p$ space. Then $\mathbb{Z}_p$-index of X is defined as

$$\mathrm{ind}_{\mathbb{Z}_p}(X) = \min\{n : X \overset{\mathbb{Z}_p}{\to} (\mathbb{Z}_p)^{*(n+1)}\}.$$

But before, we had this nice result that index wasn't silly, i.e. that $S^n \overset{\mathbb{Z}_2}{\not\to} S^{n-1}$, which was Borsuk-Ulam. Let's generalize again!

Theorem 5.3 (A Bigger Borsuk-Ulam)**.** *There is no $\mathbb{Z}_p$-map of an $(\mathbb{Z}_p)^{*(n+1)}$ into $(\mathbb{Z}_p)^{*n}$, i.e. $(\mathbb{Z}_p)^{*(n+1)} \overset{\mathbb{Z}_p}{\not\to} (\mathbb{Z}_p)^{*n}$.*

So for $p = 2$, this is exactly Borsuk-Ulam. Parts of the proof look a lot like the proof of Borsuk-Ulam, but it uses a bit more algebraic topology, so for the purposes of our course we will leave it as a black box.

Just like for the $\mathbb{Z}_2$-index, we have a bunch of nice properties.

Proposition 5.4 (More indexy index properties)**.**

(i) If $X \overset{\mathbb{Z}_p}{\to} Y$, then $\mathrm{ind}_{\mathbb{Z}_p}(X) \le \mathrm{ind}_{\mathbb{Z}_p}(Y)$. Equivalently, if $\mathrm{ind}_{\mathbb{Z}_p}(X) > \mathrm{ind}_{\mathbb{Z}_p}(Y)$, then $X \overset{\mathbb{Z}_p}{\not\to} Y$.
(ii) $\mathrm{ind}_{\mathbb{Z}_p}((\mathbb{Z}_p)^{*n}) = n$ for all $n \ge 0$.
(iii) $\mathrm{ind}_{\mathbb{Z}_p}(X * Y) \le \mathrm{ind}_{\mathbb{Z}_p}(X) + \mathrm{ind}_{\mathbb{Z}_p}(Y) + 1$.
(iv) If X is $n-1$-connected*, meaning that every continuous map $S^k \to X$ can be extended to a map $B^{k+1} \to X$ for all $k \le n$, then $\mathrm{ind}_{\mathbb{Z}_p}(X) \ge n$.*

(v) *If K is a simplicial complex that has a free $\mathbb{Z}_p$ space structure and is of dimension n, then* $\mathrm{ind}_{\mathbb{Z}_p}(K) \leq n$.

We will not leave this as a black box, but the proof will be relegated to the homework, since most of it is the same proof as before, but where you erase 2 and write p.

So now we again want to eventually argue that if we have a bad map, we must have an index contradiction. But from what space? Last time, we looked at induced maps on the deleted join. In this case, we will again need a generalization.

Definition 5.5. Let $n \geq k \geq 2$ be integers. The *n-fold pairwise deleted join* of a simplicial complex K is given by

$$K^{*n}_{\Delta(2)} = \{F_1 \sqcup F_2 \sqcup \cdots \sqcup F_n \mid (F_1, \dots, F_n) \text{ are pairwise disjoint.}\}$$

For any topological space X, the *n-fold middle deleted join* of X is given by

$$X^{*n}_{\Delta} = X^{*n} \setminus \{(\tfrac{1}{n}x, \dots, \tfrac{1}{n}x) \mid x \in X\}.$$

One way of thinking about this is that before, we were mostly looking at taking just one deleted joins. But now with a tuple of a deleted join, we need to worry about which diagonal we're removing. For the middle deleted join, we're removing only the strict diagonal of the space, where we take the deleted join and remove only the true diagonal, where all coordinates are the same. For the pairwise deleted join, we're removing anything that has any overlap, so we're removing as much as possible rather than as little as possible. For each n, the cyclic group $\mathbb{Z}_n$ acts on any kind of n-fold deleted join by permuting the coordinates, and so does the group of *all* permutations S_n.

Remark 5.6. The action of S_n (and also $\mathbb{Z}_n$) on a pairwise deleted join is always free. But the action of S_n on the middle deleted join is not free for $n \geq 3$, and the action of $\mathbb{Z}_n$ is free if and only if n is prime. If n is prime, then we can check that the generator has no fixed point. But for example if $n = 6$, with ν a cyclic shift, then for X a topological space with $x \neq y \in X$, the element $(1/6x, 1/6y, 1/6x, 1/6y, 1/6x, 1/6y) \in X^{*6}_{\Delta}$ is a fixed point of ν^2, so the action is not free.

This property is where we are crucially using primality. This is why the proof won't work for arbitrary n.

Our variations on deleted joins also act nice with joins:

Lemma 5.7. *For K, L simplicial complexes,*

$$(K * L)^{*2}_{\Delta(2)} = K^{*2}_{\Delta(2)} * L^{*2}_{\Delta(2)}.$$

The proof, as before, is a computation of simplices. (Make it homework!)

Corollary 5.8.

$$\mathrm{ind}_{\mathbb{Z}_p}((\sigma^n)^{*p}_{\Delta(2)}) = n.$$

Proof. In this case, we have

$$(\sigma^n)^{*p}_{\Delta(2)} \cong ((\sigma^0)^{*(n+1)})^{*p}_{\Delta(2)} \cong ((\sigma^0)^{*p}_{\Delta(2)})^{*(n+1)} \cong (\mathbb{Z}_p)^{*(n+1)},$$

so since the latter has index n, so does the first thing. $\square$

So now let's proceed with our nonembeddability! We want to say that if we have a "bad" map from a simplicial complex K to $\mathbb{R}^d$, then we'll have a map $K^{*p}_{\Delta(2)} \xrightarrow{\mathbb{Z}_p} (\mathbb{R}^d)^{*p}_{\Delta}$. This induces an inequality on indices, which we will then show doesn't hold.

We'll start with the index computation for $\mathbb{R}^d$.

Proposition 5.9. *Let $d \geq 1$, and let p be an odd prime. Then*

$$\operatorname{ind}_{\mathbb{Z}_p}((\mathbb{R}^d)^{*p}_{\Delta}) \leq (d+1)(p-1) - 1.$$

Hey hey, this is great! If we remember from day 1, that's the number we wanted! Awesome.

Proof. First, we will construct a $\mathbb{Z}_p$ map $h : (\mathbb{R}^d)^{*p}_{\Delta} \to Y = \{(x_1, \ldots, x_{p(d+1)}) \in (\mathbb{R}^{d+1})^p \mid x_i\text{'s not all equal.}\}$.

As before, we consider the deleted join of a bounded set, say B^d, rather than $\mathbb{R}^d$. Place p copies of B^d into $(\mathbb{R}^{d+1})^p$ using hte embeddings $\psi_1, \ldots, \psi_p$, where $\psi_i(x)$ has $(1, x_1, \ldots, x_d)$ in the ith block of coordinates and 0s elsewhere. Then the map h is given by

$$h(t_1 x_1, \ldots, t_p x_p) = t_1\psi_1(x_1) + \cdots + t_p\psi_p(x_p).$$

This is a $\mathbb{Z}_p$-map, and it misses the diagonal because it comes from the deleted join.

Now we construct a $\mathbb{Z}_p$-map $g : Y \to S^{(d+1)(p-1)-1}$. Note that $S^{(d+1)(p-1)-1} \cong (S^1)^{*(d+1)(p-1)/2}$, since p is odd and thus $(p-1)/2$ is an integer. There is a $\mathbb{Z}_p$-action on the circle S^1 by rotating by $2\pi/p$, so this induces the $\mathbb{Z}_p$-action that we'll use on the sphere.

Interpret $\mathbb{R}^{(d+1)p}$, which is Y with the diagonal added back, as the space of matrices with $d+1$ rows and p columns. The $\mathbb{Z}_p$-action now cyclically shifts the columns, and elements of Y are all matrices that don't have all columns equal. Consider the orthogonal projection g_1 of $\mathbb{R}^{(d+1)p}$ on the $(d+1)(p-1)$-dimensional subspace L perpendicular to the diagonal. In coordinates, L consists of all $(d+1)\times p$ matrices with zero row sums, and g_1 maps a matrix $X = (x_{ij})$ to the matrix

$$g_1(X) = \left(x_{ij} - \frac{1}{p}\sum_{k=1}^{p} x_{ik} \right)_{ij},$$

so the average of all columns is subtracted from each column, to make the average of all columns 0. Then $g_1(X)$ is the zero matrix if and only if all columns are equal, so g_1 provides a surjective $\mathbb{Z}_p$-map from Y to $L \setminus \{0\}$.

Now we have a punctured Euclidean space, which is going to become our sphere. Let $g(X) = \frac{g_1(X)}{||g_1(X)||}$. The range of g is the unit sphere in L, which is a $(d+1)(p-1)-1$-sphere. This completes our $\mathbb{Z}_p$-map.

The last step is to provide a $\mathbb{Z}_p$-map from $S^{(d+1)(p-1)-1} \to (\mathbb{Z}_p)^{*(d+1)(p-1)-1}$. This is a homework exercise. □

Now we finally get to the topological Tverberg in the prime case. Let's recall the statement.

Theorem 5.10 (Prime case: Topological Tverberg)**.** *Let p be a prime, let $d \geq 1$ be arbitrary, and let $N = (d+1)(p-1)$. For every continuous map $f : \sigma^N \to \mathbb{R}^d$, there exist p disjoint faces*

$F_1, \ldots, F_p \subseteq \sigma^N$ *whose images under* f *intersect, i.e.*

$$f(F_1) \cap f(F_2) \cap \cdots \cap f(F_p) \neq \emptyset.$$

Proof. Assume by contradiction that there exists f such that no p disjoint faces have intersecting images. Consider the p-fold join f^{*p} as a map from the p-fold pairwise deleted join. Then we have

$$f^{*p} : (\sigma^N)^{*p}_{\Delta(2)} \to (\mathbb{R}^d)^{*p}_{\Delta}.$$

Why do the points land in the p-fold middle deleted join? Well, if they did not, we would have some point of the domain, meaning it is a list of points in p disjoint faces of σ^N, all of which map to the same point in $\mathbb{R}^d$, since it all maps to the diagonal. But we've assumed precisely that this does not happen.

Note that this map is automatically a $\mathbb{Z}_p$-map; the action of $\mathbb{Z}_p$, of permuting coordinates, is the same on either side. Thus

$$\operatorname{ind}_{\mathbb{Z}_p}(\sigma^N)^{*p}_{\Delta(2)} \leq \operatorname{ind}_{\mathbb{Z}_p}(\mathbb{R}^d)^{*p}_{\Delta}.$$

But by the corollary above, $\operatorname{ind}_{\mathbb{Z}_p}(\sigma^N)^{*p}_{\Delta(2)} = N$, and by the proposition above, we have $(\mathbb{R}^d)^{*p}_{\Delta} \leq (d+1)(p-1) - 1 = N - 1$. Thus this inequality gives us

$$N \leq N - 1,$$

which is a contradiction, so we are done. □

6. When the Topological Tverberg's conjecture is false

So now we have shown (booyah!) the topological Tverberg conjecture in the prime case. It would be very reasonable to expect this to be true in general. People did! Let's talk history for a bit:

- In 1921, Johann Radon published Radon's theorem, the non-topological version, with proof.
- In 1965, Helge Tverberg proved Tverberg's theorem, the non-topological version.
- In 1981, Imre Bárány, András Szűcs, and Senya B. Shlosman presented the proof we just did of the topological Tverberg theorem in the prime case.
- In 1987, in an unpublished preprint, Murad Özaydin extended the proof (very nontrivially!) to the prime power case. This extension hinges on the fact that there are finite fields of prime power order. If that's meaningless, don't worry about it; this extension has new ideas and is interesting, but it requires a bit more background and we have finite time, so we won't go into it here.
- For 28 years, the theory was expanded in many ways, with things like colored versions, partitions, and certain dimension questions. But, nobody could prove the non-prime power case.
- In 2015, Florian Frick proved that the Topological Tverberg conjecture is *false* whenever the number r of disjoint faces desired is not a prime power.

So that was a shock.

A lot of the techniques he used were very powerful, so today will be a day when we leave out some proofs.

The first ingredient is *deleted products*, which are very similar to the deleted joins we've been discussing.

Definition 6.1. For a simplicial complex K, the *r-fold pairwise deleted product* of K is the space
$$K^{\times r}_{\Delta(2)} = \{(x_1, \dots, x_r) \in \sigma_1 \times \cdots \sigma_r \mid \sigma_i \subseteq K \text{ a face}, \sigma_i \cap \sigma_j = \emptyset \forall i \neq j\}.$$

So, it's like the deleted join, but instead of deleting the diagonal (in the broad sense) from the join of two spaces, which is this space of line segments, we're just deleting the diagonal from the product.

Let $W_r = \{(x_1, \dots, x_r) \in \mathbb{R}^r \mid \sum_i x_i = 0\}$, which is a skew copy of $\mathbb{R}^{r-1}$ in $\mathbb{R}^r$, with the symmetric group S_r of permutations acting on W_r via permuting coordinates. Let $W_r^{\oplus rk}$ denote the vector space of dimension $(r-1)^{rk}$, where one basis is the rk-tuples of basis elements of W_r. In this way, $W_r^{\oplus rk}$ has a coordinatewise S_r action coming from the permutation on W_r. Then the following theorem is true.

Theorem 6.2 (Mabillard, Wagner). *Let $r \geq 2$, $k \geq 3$, and let K be a simplicial complex of dimension $(r-1)k$. Then the following are equivalent.*

(i) There exists an S_r-map $K^{\times r}_{\Delta(2)} \to S(W_r^{\oplus rk})$, where the latter is the unit sphere inside $W_r^{\oplus rk}$.
(ii) There exists a continuous map $f : K \to \mathbb{R}^{rk}$ such that for any r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of K, $f(\sigma_1) \cap \cdots \cap f(\sigma_r) = \emptyset$.

So this is sort of intuitive; so far we've talked about something that looks like the correspondence from (ii) to (i), where we assume that we have such a "bad" map $f : K \to \mathbb{R}^d$ for some d and prove that there exists some S_r-map that we don't want. But that's not exactly what's above, and more importantly we definitely haven't had the correspondence going in the other direction, so we haven't had anything so far where the *existence* of S_r-maps would tell us anything about the existence of "bad" maps. But now we do. OK, so let's look at the existence of maps.

First, there's a helpful lemma of Özaydin, which tells us the following.

Lemma 6.3. *In order to check that there exist S_r-maps, it suffices to check that there exist G-maps for G a subgroup of S_r of prime power order.*

Proposition 6.4. *Let $r \geq 6$ be an integer that is not a prime power, and $k \geq 3$ an integer. For any N, there exists a continuous map $f : \Delta_N \to \mathbb{R}^{rk}$ such that for any r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of Δ_N, with* $\dim \sigma_i \leq (r-1)k$, *we have*
$$f(\sigma_1) \cap \cdots \cap f(\sigma_n) = \emptyset.$$

Proof. Let K be the $((r-1)k)$-dimensional skeleton of the simplex Δ_N on $N+1$ vertices. We need only construct f on K, since all pairwise disjoint faces under consideration will be in this skeleton. By Mabillard and Wagner's theorem, we want to show that there exists a S_r-map $K^{\times r}_{\Delta(2)} \to S(W_r^{\oplus rk})$. The free S_r space $K^{\times r}_{\Delta(2)}$ has dimension at most $d = r(r-1)k$, since k is $(r-1)k$-dimensional. Also, the unit sphere $S(W_r^{\oplus rk})$ is a $(d-1)$-sphere and thus $(d-2)$-connected. By the lemma, we need only check the existence of G-maps for subgroups G of prime power order. Let G be such a subgroup, of order a power of a prime p. Since r is not a prime power, any p-group action on $S(W_r^{\oplus rk})$ will have fixed points (this is a strong statement we haven't proven), so we can map the whole group to a chosen fixed point. □

This is aaaaalllmost a counterexample, but let's go through the fix that makes it a full counterexample.

Theorem 6.5. *Let $r \geq 6$ be an integer that is not a prime power, with $k \geq 3$ an integer. Let $d = rk + 1$, and let $N = (r-1)(d+1)$. There exists a continuous map $F : \Delta_N \to \mathbb{R}^d$ such that for any r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of Δ_N,*

$$F(\sigma_1) \cap \cdots \cap F(\sigma_r) = \emptyset.$$

Proof. Let $f : \Delta_N \to \mathbb{R}^{rk}$ be a continuous map as in the proposition, so for any r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of Δ_N with $\dim \sigma_i \leq (r-1)k$, we have $f(\sigma_1) \cap \cdots \cap f(\sigma_r) = \emptyset$.

Define $F : \Delta_N \to \mathbb{R}^{rk+1} = \mathbb{R}^d$ via $x \mapsto (f(x), \mathrm{dist}(x, \Delta_N^{((r-1)k)}))$, with $\Delta_N^{((r-1)k)}$ the $(r-1)k$-skeleton of Δ_N, so the first coordinate is f and the second is the distance from x to the $(r-1)k$-skeleton. This will be our F. Why does it work? Well, suppose we had r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of Δ_N such that we have points $x_i \in \sigma_i$, for $1 \leq i \leq r$, with $F(x_1) = \cdots = F(x_r)$. We can assume without loss of generality, by taking smaller faces if necessary, that x_i is in the interior of σ_i, not on the boundary. Since all x_i have the same final coordinate, they all have the same distance to the $(r-1)k$-skeleton of Δ_N.

We will now show that some σ_i must have dimension $(r-1)k$ or less. Suppose by contradiction that all σ_i had dimension at least $(r-1)k+1$. Since the σ_i are all disjoint, they would have between them at least $r((r-1)k+2)$ vertices, with r faces total and at least $(r-1)k+2$ vertices per face. But $r((r-1)k+2) = (r-1)rk + 2r - 2 + 2 = (r-1)(rk+2) + 2 > N+1$ vertices, and we don't have that many vertices to go around!

Thus some face σ_i must have dimension $(r-1)k$ or less. But then if $x_i \in \sigma_i$, the point x_i is in the $(r-1)k$-skeleton, so its distance from the skeleton is 0. But then all points are in the $(r-1)k$-skeleton, and thus all simplices σ_j are in the $(r-1)k$-skeleton. But then they all satisfy the dimension constraint, so they contradict our assumption on f. □

So there's a counterexample! It's not small, though. The smallest case for $r = 6$ is that this shows that if $k = 3$, then $d = 6 * 3 + 1 = 19$, and $N = (r-1)(d+1) = 100$. So, this constructs a continuous map $\Delta_{100} \to \mathbb{R}^{19}$ such that any six pairwise disjoint faces have images that do not intersect in a common point.