Asymmetric graph removal

Yuval Wigderson

Tel Aviv University

RS&A 2023

Joint with Lior Gishboliner and Asaf Shapira

The graph removal lemma

The graph removal lemma

Theorem (triangle removal lemma)

Let G have n vertices.
If G is ε-far from triangle-free, then G contains $\geq \delta n^3$ triangles, where $\delta = \delta(\varepsilon) > 0$ depends only on ε.

The graph removal lemma

G is ε-far from $\mathscr{P}$ if one must add/delete $\geq \varepsilon n^2$ edges to satisfy $\mathscr{P}$.

Theorem (triangle removal lemma)

Let G have n vertices.
If G is ε-far from triangle-free, then G contains $\geq \delta n^3$ triangles, where $\delta = \delta(\varepsilon) > 0$ depends only on ε.

The graph removal lemma

G is ε-far from $\mathscr{P}$ if one must add/delete $\geq \varepsilon n^2$ edges to satisfy $\mathscr{P}$.

Theorem (triangle removal lemma)

Let G have n vertices.
If G is ε-far from triangle-free, then G contains $\geq \delta n^3$ triangles, where $\delta = \delta(\varepsilon) > 0$ depends only on ε.

Despite the simple statement, all known proofs are **hard**.

The graph removal lemma

G is ε-far from $\mathscr{P}$ if one must add/delete $\geq \varepsilon n^2$ edges to satisfy $\mathscr{P}$.

Theorem (triangle removal lemma)

Let G have n vertices.
If G is ε-far from triangle-free, then G contains $\geq \delta n^3$ triangles,
where $\delta = \delta(\varepsilon) > 0$ depends only on ε.

Despite the simple statement, all known proofs are **hard**. Why?

The graph removal lemma

G is ε-far from $\mathscr{P}$ if one must add/delete $\geq \varepsilon n^2$ edges to satisfy $\mathscr{P}$.

Theorem (triangle removal lemma)

Let G have n vertices.
If G is ε-far from triangle-free, then G contains $\geq \delta n^3$ triangles, where $\delta = \delta(\varepsilon) > 0$ depends only on ε.

Despite the simple statement, all known proofs are **hard**. Why?

- Many **hard** applications in graph theory, number theory, discrete geometry, theoretical computer science, ...

The graph removal lemma

G is ε-far from $\mathscr{P}$ if one must add/delete $\geq \varepsilon n^2$ edges to satisfy $\mathscr{P}$.

Theorem (triangle removal lemma)

Let G have n vertices.
If G is ε-far from triangle-free, then G contains $\geq \delta n^3$ triangles, where $\delta = \delta(\varepsilon) > 0$ depends only on ε.

Despite the simple statement, all known proofs are **hard**. Why?

- Many **hard** applications in graph theory, number theory, discrete geometry, theoretical computer science, …
- A "simple" proof would presumably give polynomial bounds.

The graph removal lemma

G is ε-far from $\mathscr{P}$ if one must add/delete $\geq \varepsilon n^2$ edges to satisfy $\mathscr{P}$.

Theorem (triangle removal lemma)

Let G have n vertices.
If G is ε-far from triangle-free, then G contains $\geq \delta n^3$ triangles, where $\delta = \delta(\varepsilon) > 0$ depends only on ε.

Despite the simple statement, all known proofs are **hard**. Why?

- Many **hard** applications in graph theory, number theory, discrete geometry, theoretical computer science, ...
- A "simple" proof would presumably give polynomial bounds.

Theorem (Ruzsa–Szemerédi 1978)

One cannot take $\delta(\varepsilon) = \mathrm{poly}(\varepsilon)$.

The graph removal lemma

G is ε-far from $\mathscr{P}$ if one must add/delete $\geq \varepsilon n^2$ edges to satisfy $\mathscr{P}$.

Theorem (triangle removal lemma)

Let G have n vertices.
If G is ε-far from triangle-free, then G contains $\geq \delta n^3$ triangles, where $\delta = \delta(\varepsilon) > 0$ depends only on ε.

Despite the simple statement, all known proofs are **hard**. Why?

- Many **hard** applications in graph theory, number theory, discrete geometry, theoretical computer science, ...
- A "simple" proof would presumably give polynomial bounds.

Theorem (Ruzsa–Szemerédi 1978)

One cannot take $\delta(\varepsilon) = \mathrm{poly}(\varepsilon)$.

In the other direction, only tower-type bounds are known.

The graph removal lemma

G is ε-far from $\mathscr{P}$ if one must add/delete $\geq \varepsilon n^2$ edges to satisfy $\mathscr{P}$.

Theorem (graph removal lemma)

Let G have n vertices and H have h vertices.
If G is ε-far from H-free, then G contains $\geq \delta n^h$ copies of H,
where $\delta = \delta(\varepsilon, H) > 0$ depends only on ε and H.

Despite the simple statement, all known proofs are **hard**. Why?

- Many **hard** applications in graph theory, number theory, discrete geometry, theoretical computer science, ...
- A "simple" proof would presumably give polynomial bounds.

Theorem (Ruzsa–Szemerédi 1978)

One cannot take $\delta(\varepsilon) = \text{poly}(\varepsilon)$.

In the other direction, only tower-type bounds are known.

The graph removal lemma

G is ε-far from $\mathscr{P}$ if one must add/delete $\geq \varepsilon n^2$ edges to satisfy $\mathscr{P}$.

Theorem (graph removal lemma)

Let G have n vertices and H have h vertices.
If G is ε-far from H-free, then G contains $\geq \delta n^h$ copies of H,
where $\delta = \delta(\varepsilon, H) > 0$ depends only on ε and H.

Despite the simple statement, all known proofs are **hard**. Why?

- Many **hard** applications in graph theory, number theory, discrete geometry, theoretical computer science, …
- A "simple" proof would presumably give polynomial bounds.

Theorem (Ruzsa–Szemerédi 1978, Alon 2002)

One cannot take $\delta(\varepsilon, H) = \mathrm{poly}(\varepsilon)$. *(unless H is bipartite)*

In the other direction, only tower-type bounds are known.

The asymmetric removal lemma

Theorem (graph removal lemma)

Let G have n vertices and H have h vertices.
If G is ε-far from H-free, then G contains $\geq \delta n^h$ copies of H, where $\delta = \delta(\varepsilon, H) > 0$ depends only on ε and H.

The asymmetric removal lemma

Theorem (asymmetric removal lemma)

Let G have n vertices and H have h vertices. Suppose $\chi(H) = 3$.
If G is ε-far from triangle-free, then G contains $\geq \delta^* n^h$ copies of H, where $\delta^* = \delta^*(\varepsilon, H) > 0$ depends only on ε and H.

The asymmetric removal lemma

Theorem (asymmetric removal lemma)

Let G have n vertices and H have h vertices. Suppose $\chi(H) = 3$.
If G is ε-far from triangle-free, then G contains $\geq \delta^* n^h$ copies of H, where $\delta^* = \delta^*(\varepsilon, H) > 0$ depends only on ε and H.

Can we expect better bounds on $\delta^*(\varepsilon, H)$?

The asymmetric removal lemma

Theorem (asymmetric removal lemma)

Let G have n vertices and H have h vertices. Suppose $\chi(H) = 3$.
If G is ε-far from triangle-free, then G contains $\geq \delta^* n^h$ copies of H, where $\delta^* = \delta^*(\varepsilon, H) > 0$ depends only on ε and H.

Can we expect better bounds on $\delta^*(\varepsilon, H)$?

Not in general: If $K_3 \subseteq H$, then $\delta^*(\varepsilon, H) \approx \delta(\varepsilon, K_3)$.

The asymmetric removal lemma

Theorem (asymmetric removal lemma)

Let G have n vertices and H have h vertices. Suppose $\chi(H) = 3$.
If G is ε-far from triangle-free, then G contains $\geq \delta^* n^h$ copies of H, where $\delta^* = \delta^*(\varepsilon, H) > 0$ depends only on ε and H.

Can we expect better bounds on $\delta^*(\varepsilon, H)$?

Not in general: If $K_3 \subseteq H$, then $\delta^*(\varepsilon, H) \approx \delta(\varepsilon, K_3)$.

Theorem (Csaba 2021)

We can take $\delta^*(\varepsilon, C_5)$ to be $2^{-\operatorname{poly}(1/\varepsilon)}$.

The asymmetric removal lemma

Theorem (asymmetric removal lemma)

Let G have n vertices and H have h vertices. Suppose $\chi(H) = 3$. If G is ε-far from triangle-free, then G contains $\geq \delta^* n^h$ copies of H, where $\delta^* = \delta^*(\varepsilon, H) > 0$ depends only on ε and H.

Can we expect better bounds on $\delta^*(\varepsilon, H)$?

Not in general: If $K_3 \subseteq H$, then $\delta^*(\varepsilon, H) \approx \delta(\varepsilon, K_3)$.

Theorem (Csaba 2021)

We can take $\delta^*(\varepsilon, C_5)$ to be $2^{-\operatorname{poly}(1/\varepsilon)}$.

This is **much** better than the tower-type bounds known for the symmetric removal lemma.

The asymmetric removal lemma

Theorem (asymmetric removal lemma)

Let G have n vertices and H have h vertices. Suppose $\chi(H) = 3$.
If G is ε-far from triangle-free, then G contains $\geq \delta^* n^h$ copies of H, where $\delta^* = \delta^*(\varepsilon, H) > 0$ depends only on ε and H.

Can we expect better bounds on $\delta^*(\varepsilon, H)$?

Not in general: If $K_3 \subseteq H$, then $\delta^*(\varepsilon, H) \approx \delta(\varepsilon, K_3)$.

Theorem (Csaba 2021)

We can take $\delta^*(\varepsilon, C_5)$ to be $2^{-\operatorname{poly}(1/\varepsilon)}$.

This is **much** better than the tower-type bounds known for the symmetric removal lemma. The proof uses a weak regularity lemma.

The asymmetric removal lemma

Theorem (asymmetric removal lemma)

Let G have n vertices and H have h vertices. Suppose $\chi(H) = 3$. If G is ε-far from triangle-free, then G contains $\geq \delta^* n^h$ copies of H, where $\delta^* = \delta^*(\varepsilon, H) > 0$ depends only on ε and H.

Can we expect better bounds on $\delta^*(\varepsilon, H)$?

Not in general: If $K_3 \subseteq H$, then $\delta^*(\varepsilon, H) \approx \delta(\varepsilon, K_3)$.

Theorem (Csaba 2021)

We can take $\delta^*(\varepsilon, C_5)$ to be $2^{-\operatorname{poly}(1/\varepsilon)}$.

This is **much** better than the tower-type bounds known for the symmetric removal lemma. The proof uses a weak regularity lemma.

Theorem (Gishboliner–Shapira–W. 2022)

$\delta^*(\varepsilon, C_5) = \operatorname{poly}(\varepsilon)$.

Odd cycles

Theorem (Gishboliner–Shapira–W. 2022)

$\delta^*(\varepsilon, C_5) = \text{poly}(\varepsilon)$.

Odd cycles

Theorem (Gishboliner–Shapira–W. 2022)

$\delta^*(\varepsilon, C_5) = \mathrm{poly}(\varepsilon)$. *In fact,* $\Omega(\varepsilon^{10}) \leq \delta^*(\varepsilon, C_5) \leq O(\varepsilon^5)$.

Odd cycles

Theorem (Gishboliner–Shapira–W. 2022)

$\delta^*(\varepsilon, C_5) = \text{poly}(\varepsilon)$. *In fact,* $\Omega(\varepsilon^{10}) \le \delta^*(\varepsilon, C_5) \le O(\varepsilon^5)$.

If G is ε-far from triangle-free, then G contains $\Omega(\varepsilon^{10} n^5)$ copies of C_5.

Odd cycles

Theorem (Gishboliner–Shapira–W. 2022)

$\delta^*(\varepsilon, C_5) = \text{poly}(\varepsilon)$. *In fact,* $\Omega(\varepsilon^{10}) \le \delta^*(\varepsilon, C_5) \le O(\varepsilon^5)$.

If G is ε-far from triangle-free, then G contains $\Omega(\varepsilon^{10} n^5)$ copies of C_5.

The upper bound comes from the Erdős–Rényi random graph $G_{n,\varepsilon}$.

Odd cycles

Theorem (Gishboliner–Shapira–W. 2022)

$\delta^*(\varepsilon, C_5) = \text{poly}(\varepsilon)$. *In fact,* $\Omega(\varepsilon^{10}) \leq \delta^*(\varepsilon, C_5) \leq O(\varepsilon^5)$.

If G is ε-far from triangle-free, then G contains $\Omega(\varepsilon^{10} n^5)$ copies of C_5.
The upper bound comes from the Erdős–Rényi random graph $G_{n,\varepsilon}$.

Theorem (Gishboliner–Shapira–W. 2022)

Fix odd integers $3 \leq k < \ell$.
If G *is* ε*-far from* C_k*-free, then* G *contains* $\Omega(\varepsilon^{2\ell} n^{\ell})$ *copies of* C_ℓ.

Odd cycles

Theorem (Gishboliner–Shapira–W. 2022)

$\delta^*(\varepsilon, C_5) = \mathrm{poly}(\varepsilon)$. *In fact,* $\Omega(\varepsilon^{10}) \le \delta^*(\varepsilon, C_5) \le O(\varepsilon^5)$.

If G is ε-far from triangle-free, then G contains $\Omega(\varepsilon^{10} n^5)$ copies of C_5.

The upper bound comes from the Erdős–Rényi random graph $G_{n,\varepsilon}$.

Theorem (Gishboliner–Shapira–W. 2022)

Fix odd integers $3 \le k < \ell$.
If G is ε-far from C_k-free, then G contains $\Omega(\varepsilon^{2\ell} n^{\ell})$ copies of C_ℓ.

The asymmetry is key! If $k = \ell$, no polynomial bound can hold.

Odd cycles

Theorem (Gishboliner–Shapira–W. 2022)

$\delta^*(\varepsilon, C_5) = \text{poly}(\varepsilon)$. *In fact,* $\Omega(\varepsilon^{10}) \leq \delta^*(\varepsilon, C_5) \leq O(\varepsilon^5)$.

If G is ε-far from triangle-free, then G contains $\Omega(\varepsilon^{10} n^5)$ copies of C_5.
The upper bound comes from the Erdős–Rényi random graph $G_{n,\varepsilon}$.

Theorem (Gishboliner–Shapira–W. 2022)

Fix odd integers $3 \leq k < \ell$.
If G *is* ε*-far from* C_k*-free, then* G *contains* $\Omega(\varepsilon^{2\ell} n^\ell)$ *copies of* C_ℓ.

The asymmetry is key! If $k = \ell$, no polynomial bound can hold.
The bound $\varepsilon^{2\ell}$ is tight up to the factor of 2 in the exponent.

Odd cycles

Theorem (Gishboliner–Shapira–W. 2022)

$\delta^*(\varepsilon, C_5) = \text{poly}(\varepsilon)$. *In fact,* $\Omega(\varepsilon^{10}) \le \delta^*(\varepsilon, C_5) \le O(\varepsilon^5)$.

If G is ε-far from triangle-free, then G contains $\Omega(\varepsilon^{10} n^5)$ copies of C_5.

The upper bound comes from the Erdős–Rényi random graph $G_{n,\varepsilon}$.

Theorem (Gishboliner–Shapira–W. 2022)

Fix odd integers $3 \le k < \ell$.
If G *is* ε*-far from* C_k*-free, then* G *contains* $\Omega(\varepsilon^{2\ell} n^{\ell})$ *copies of* C_ℓ.

The asymmetry is key! If $k = \ell$, no polynomial bound can hold.

The bound $\varepsilon^{2\ell}$ is tight up to the factor of 2 in the exponent.

The proof uses elementary (but subtle!) averaging arguments.

Odd cycles

Theorem (Gishboliner–Shapira–W. 2022)

$\delta^*(\varepsilon, C_5) = \text{poly}(\varepsilon)$. *In fact,* $\Omega(\varepsilon^{10}) \leq \delta^*(\varepsilon, C_5) \leq O(\varepsilon^5)$.

If G is ε-far from triangle-free, then G contains $\Omega(\varepsilon^{10} n^5)$ copies of C_5.

The upper bound comes from the Erdős–Rényi random graph $G_{n,\varepsilon}$.

Theorem (Gishboliner–Shapira–W. 2022)

Fix odd integers $3 \leq k < \ell$.
If G *is* ε*-far from* C_k*-free, then* G *contains* $\Omega(\varepsilon^{2\ell} n^{\ell})$ *copies of* C_ℓ.

The asymmetry is key! If $k = \ell$, no polynomial bound can hold.

The bound $\varepsilon^{2\ell}$ is tight up to the factor of 2 in the exponent.

The proof uses elementary (but subtle!) averaging arguments.

This result has applications in property testing.

K_3-abundant graphs

K_3-abundant graphs

Theorem (asymmetric removal lemma)

Let G have n vertices and H have h vertices. Suppose $\chi(H) = 3$. If G is ε-far from triangle-free, then G contains $\geq \delta^* n^h$ copies of H, where $\delta^* = \delta^*(\varepsilon, H) > 0$ depends only on ε and H.

K_3-abundant graphs

Theorem (asymmetric removal lemma)

Let G have n vertices and H have h vertices. Suppose $\chi(H) = 3$. If G is ε-far from triangle-free, then G contains $\geq \delta^* n^h$ copies of H, where $\delta^* = \delta^*(\varepsilon, H) > 0$ depends only on ε and H.

If $K_3 \subseteq H$, then $\delta^*(\varepsilon, H) \neq \text{poly}(\varepsilon)$.

K_3-abundant graphs

Theorem (asymmetric removal lemma)

Let G have n vertices and H have h vertices. Suppose $\chi(H) = 3$. If G is ε-far from triangle-free, then G contains $\geq \delta^* n^h$ copies of H, where $\delta^* = \delta^*(\varepsilon, H) > 0$ depends only on ε and H.

If $K_3 \subseteq H$, then $\delta^*(\varepsilon, H) \neq \text{poly}(\varepsilon)$.

If H is C_ℓ, $\ell \geq 5$ odd, then $\delta^*(\varepsilon, H) = \text{poly}(\varepsilon)$.

K_3-abundant graphs

Theorem (asymmetric removal lemma)

Let G have n vertices and H have h vertices. Suppose $\chi(H) = 3$. If G is ε-far from triangle-free, then G contains $\geq \delta^* n^h$ copies of H, where $\delta^* = \delta^*(\varepsilon, H) > 0$ depends only on ε and H.

If $K_3 \subseteq H$, then $\delta^*(\varepsilon, H) \neq \text{poly}(\varepsilon)$.

If H is homomorphic to C_ℓ, $\ell \geq 5$ odd, then $\delta^*(\varepsilon, H) = \text{poly}(\varepsilon)$.

K_3-abundant graphs

Theorem (asymmetric removal lemma)

Let G have n vertices and H have h vertices. Suppose $\chi(H) = 3$*.*
If G is ε*-far from triangle-free, then G contains* $\geq \delta^* n^h$ *copies of H, where* $\delta^* = \delta^*(\varepsilon, H) > 0$ *depends only on* ε *and H.*

If $K_3 \subseteq H$, then $\delta^*(\varepsilon, H) \neq \text{poly}(\varepsilon)$.

If H is homomorphic to C_ℓ, $\ell \geq 5$ odd, then $\delta^*(\varepsilon, H) = \text{poly}(\varepsilon)$.

Definition

A tripartite graph H is K_3-abundant if $\delta^*(\varepsilon, H) = \text{poly}(\varepsilon)$.

K_3-abundant graphs

Theorem (asymmetric removal lemma)

Let G have n vertices and H have h vertices. Suppose $\chi(H) = 3$. If G is ε-far from triangle-free, then G contains $\geq \delta^* n^h$ copies of H, where $\delta^* = \delta^*(\varepsilon, H) > 0$ depends only on ε and H.

If $K_3 \subseteq H$, then $\delta^*(\varepsilon, H) \neq \text{poly}(\varepsilon)$.

If H is homomorphic to C_ℓ, $\ell \geq 5$ odd, then $\delta^*(\varepsilon, H) = \text{poly}(\varepsilon)$.

Definition

A tripartite graph H is K_3-abundant if $\delta^*(\varepsilon, H) = \text{poly}(\varepsilon)$.

Question: Is every triangle-free tripartite graph K_3-abundant?

K_3-abundant graphs

Theorem (asymmetric removal lemma)

Let G have n vertices and H have h vertices. Suppose $\chi(H) = 3$. If G is ε-far from triangle-free, then G contains $\geq \delta^* n^h$ copies of H, where $\delta^* = \delta^*(\varepsilon, H) > 0$ depends only on ε and H.

If $K_3 \subseteq H$, then $\delta^*(\varepsilon, H) \neq \text{poly}(\varepsilon)$.

If H is homomorphic to C_ℓ, $\ell \geq 5$ odd, then $\delta^*(\varepsilon, H) = \text{poly}(\varepsilon)$.

Definition

A tripartite graph H is K_3-abundant if $\delta^*(\varepsilon, H) = \text{poly}(\varepsilon)$.

Question: Is every triangle-free tripartite graph K_3-abundant? **No!**

Theorem (Gishboliner–Shapira–W. 2022)

There exist triangle-free tripartite H which are not K_3-abundant*

K_3-abundant graphs

Theorem (asymmetric removal lemma)

Let G have n vertices and H have h vertices. Suppose $\chi(H) = 3$.
If G is ε-far from triangle-free, then G contains $\geq \delta^* n^h$ copies of H, where $\delta^* = \delta^*(\varepsilon, H) > 0$ depends only on ε and H.

If $K_3 \subseteq H$, then $\delta^*(\varepsilon, H) \neq \text{poly}(\varepsilon)$.

If H is homomorphic to C_ℓ, $\ell \geq 5$ odd, then $\delta^*(\varepsilon, H) = \text{poly}(\varepsilon)$.

Definition

A tripartite graph H is K_3-abundant if $\delta^*(\varepsilon, H) = \text{poly}(\varepsilon)$.

Question: Is every triangle-free tripartite graph K_3-abundant? **No!**

Theorem (Gishboliner–Shapira–W. 2022)

There exist triangle-free tripartite H which are not K_3-abundant*
*(*assuming Ruzsa's genus conjecture in additive number theory).*

Proof sketch: The Ruzsa–Szemerédi construction

Proof sketch: The Ruzsa–Szemerédi construction

Theorem (Gishboliner–Shapira–W. 2022)

There exist triangle-free tripartite H which are not * *K_3-abundant.*

Proof sketch: The Ruzsa–Szemerédi construction

Theorem (Ruzsa–Szemerédi 1978)

K_3 is not K_3-abundant.

Proof sketch: The Ruzsa–Szemerédi construction

Theorem (Ruzsa–Szemerédi 1978)

K_3 is not K_3-abundant.

Want: G which is ε-far from K_3-free, but has $\leq \varepsilon^{\omega(1)} n^3$ copies of K_3.

Proof sketch: The Ruzsa–Szemerédi construction

Theorem (Ruzsa–Szemerédi 1978)

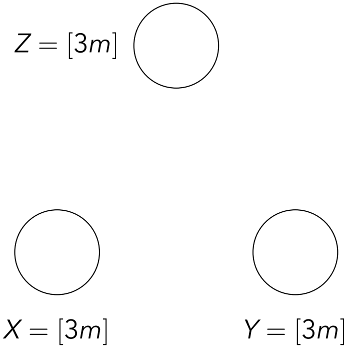
K_3 is not K_3-abundant.

Want: G which is ε-far from K_3-free, but has $\leq \varepsilon^{\omega(1)} n^3$ copies of K_3.
Let $R \subseteq [m]$ with $|R| \geq \varepsilon m$. We define the graph $G = \mathrm{RS}(m, R)$:

Theorem (Ruzsa–Szemerédi 1978)

K_3 is not K_3-abundant.

Want: G which is ε-far from K_3-free, but has $\leq \varepsilon^{\omega(1)} n^3$ copies of K_3.
Let $R \subseteq [m]$ with $|R| \geq \varepsilon m$. We define the graph $G = \mathrm{RS}(m, R)$:



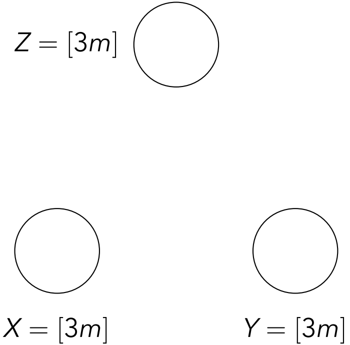
Proof sketch: The Ruzsa–Szemerédi construction

Theorem (Ruzsa–Szemerédi 1978)

K_3 is not K_3-abundant.

Want: G which is ε-far from K_3-free, but has $\leq \varepsilon^{\omega(1)} n^3$ copies of K_3.
Let $R \subseteq [m]$ with $|R| \geq \varepsilon m$. We define the graph $G = \mathrm{RS}(m, R)$:

G has $n = \Theta(m)$ vertices.

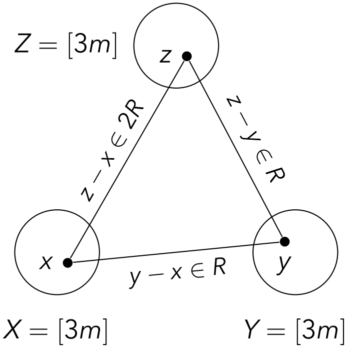


Proof sketch: The Ruzsa–Szemerédi construction

Theorem (Ruzsa–Szemerédi 1978)

K_3 is not K_3-abundant.

Want: G which is ε-far from K_3-free, but has $\leq \varepsilon^{\omega(1)} n^3$ copies of K_3.
Let $R \subseteq [m]$ with $|R| \geq \varepsilon m$. We define the graph $G = \mathrm{RS}(m, R)$:



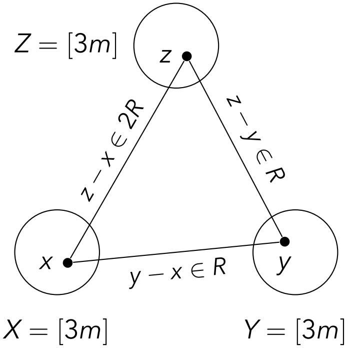
G has $n = \Theta(m)$ vertices.

Proof sketch: The Ruzsa–Szemerédi construction

Theorem (Ruzsa–Szemerédi 1978)

K_3 is not K_3-abundant.

Want: G which is ε-far from K_3-free, but has $\leq \varepsilon^{\omega(1)} n^3$ copies of K_3.
Let $R \subseteq [m]$ with $|R| \geq \varepsilon m$. We define the graph $G = \mathrm{RS}(m, R)$:



G has $n = \Theta(m)$ vertices.

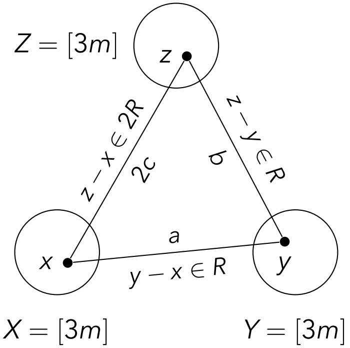
Suppose x, y, z form a triangle.

Proof sketch: The Ruzsa–Szemerédi construction

Theorem (Ruzsa–Szemerédi 1978)

K_3 is not K_3-abundant.

Want: G which is ε-far from K_3-free, but has $\leq \varepsilon^{\omega(1)} n^3$ copies of K_3.
Let $R \subseteq [m]$ with $|R| \geq \varepsilon m$. We define the graph $G = \mathrm{RS}(m, R)$:



G has $n = \Theta(m)$ vertices.

Suppose x, y, z form a triangle. Let

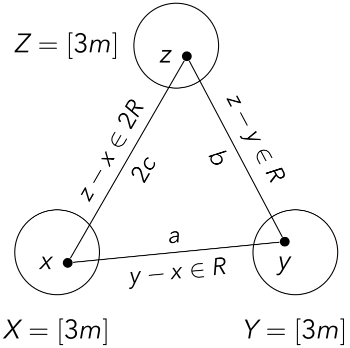
$$a = y - x; \quad b = z - y; \quad c = \frac{z-x}{2}.$$

Proof sketch: The Ruzsa–Szemerédi construction

Theorem (Ruzsa–Szemerédi 1978)

K_3 is not K_3-abundant.

Want: G which is ε-far from K_3-free, but has $\leq \varepsilon^{\omega(1)} n^3$ copies of K_3.
Let $R \subseteq [m]$ with $|R| \geq \varepsilon m$. We define the graph $G = \mathrm{RS}(m, R)$:



G has $n = \Theta(m)$ vertices.

Suppose x, y, z form a triangle. Let

$$a = y - x; \quad b = z - y; \quad c = \frac{z - x}{2}.$$

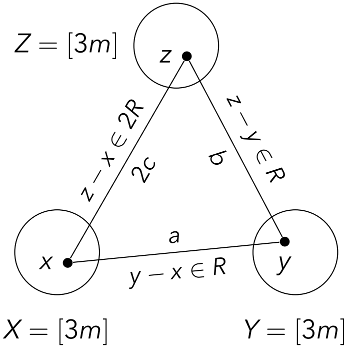
Then $a + b = 2c$.

Proof sketch: The Ruzsa–Szemerédi construction

Theorem (Ruzsa–Szemerédi 1978)

K_3 is not K_3-abundant.

Want: G which is ε-far from K_3-free, but has $\leq \varepsilon^{\omega(1)} n^3$ copies of K_3.
Let $R \subseteq [m]$ with $|R| \geq \varepsilon m$. We define the graph $G = \mathrm{RS}(m, R)$:



G has $n = \Theta(m)$ vertices.

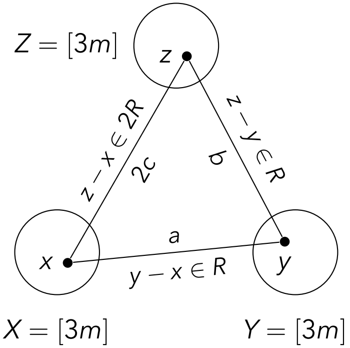
Triangles $\longleftrightarrow$ solutions in R to $a + b = 2c$ (plus choice of basepoint).

Proof sketch: The Ruzsa–Szemerédi construction

Theorem (Ruzsa–Szemerédi 1978)

K_3 is not K_3-abundant.

Want: G which is ε-far from K_3-free, but has $\leq \varepsilon^{\omega(1)} n^3$ copies of K_3.
Let $R \subseteq [m]$ with $|R| \geq \varepsilon m$. We define the graph $G = \mathrm{RS}(m, R)$:



G has $n = \Theta(m)$ vertices.

Triangles $\longleftrightarrow$ solutions in R to $a + b = 2c$ (plus choice of basepoint).

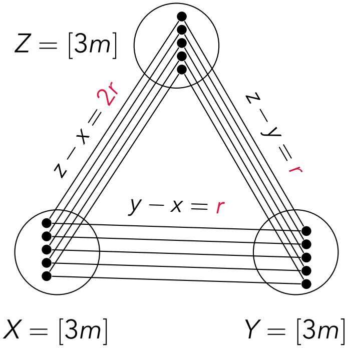
Each $r \in R$ yields a trivial solution

Proof sketch: The Ruzsa–Szemerédi construction

Theorem (Ruzsa–Szemerédi 1978)

K_3 is not K_3-abundant.

Want: G which is ε-far from K_3-free, but has $\leq \varepsilon^{\omega(1)} n^3$ copies of K_3.
Let $R \subseteq [m]$ with $|R| \geq \varepsilon m$. We define the graph $G = \mathrm{RS}(m, R)$:



G has $n = \Theta(m)$ vertices.

Triangles $\longleftrightarrow$ solutions in R to $a + b = 2c$ (plus choice of basepoint).

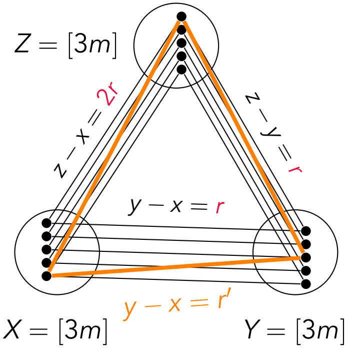
Each $r \in R$ yields a trivial solution $\implies m$ vertex-disjoint triangles.

Proof sketch: The Ruzsa–Szemerédi construction

Theorem (Ruzsa–Szemerédi 1978)

K_3 is not K_3-abundant.

Want: G which is ε-far from K_3-free, but has $\leq \varepsilon^{\omega(1)} n^3$ copies of K_3.
Let $R \subseteq [m]$ with $|R| \geq \varepsilon m$. We define the graph $G = \mathrm{RS}(m, R)$:



G has $n = \Theta(m)$ vertices.

Triangles $\longleftrightarrow$ solutions in R to $a + b = 2c$ (plus choice of basepoint).

Each $r \in R$ yields a trivial solution
$\Longrightarrow m$ vertex-disjoint triangles.

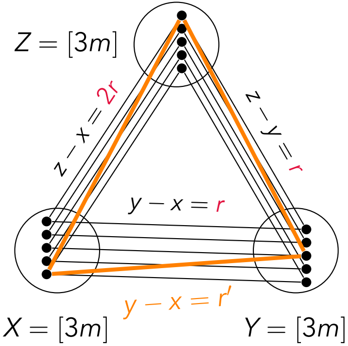
For $r \neq r'$, they are edge-disjoint

Proof sketch: The Ruzsa–Szemerédi construction

Theorem (Ruzsa–Szemerédi 1978)

K_3 is not K_3-abundant.

Want: G which is ε-far from K_3-free, but has $\leq \varepsilon^{\omega(1)} n^3$ copies of K_3.
Let $R \subseteq [m]$ with $|R| \geq \varepsilon m$. We define the graph $G = \mathrm{RS}(m, R)$:



G has $n = \Theta(m)$ vertices.

Triangles $\longleftrightarrow$ solutions in R to $a + b = 2c$ (plus choice of basepoint).

Each $r \in R$ yields a trivial solution
$\Longrightarrow$ m vertex-disjoint triangles.

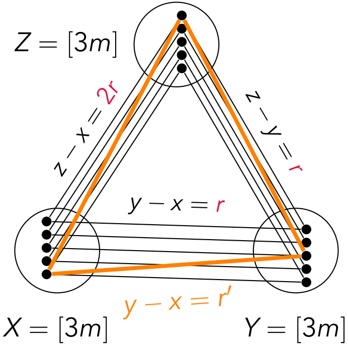
For $r \neq r'$, they are edge-disjoint
$\Longrightarrow$ $m|R|$ edge-disjoint triangles

Proof sketch: The Ruzsa–Szemerédi construction

Theorem (Ruzsa–Szemerédi 1978)

K_3 is not K_3-abundant.

Want: G which is ε-far from K_3-free, but has $\leq \varepsilon^{\omega(1)} n^3$ copies of K_3.
Let $R \subseteq [m]$ with $|R| \geq \varepsilon m$. We define the graph $G = \mathrm{RS}(m, R)$:



G has $n = \Theta(m)$ vertices.

Triangles $\longleftrightarrow$ solutions in R to $a + b = 2c$ (plus choice of basepoint).

Each $r \in R$ yields a trivial solution $\Longrightarrow m$ vertex-disjoint triangles.

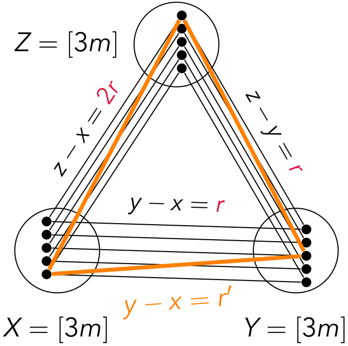
For $r \neq r'$, they are edge-disjoint $\Longrightarrow m|R|$ edge-disjoint triangles $\Longrightarrow G$ is $\Theta(\varepsilon)$-far from triangle-free.

Proof sketch: The Ruzsa–Szemerédi construction

Theorem (Ruzsa–Szemerédi 1978)

K_3 is not K_3-abundant.

Want: G which is ε-far from K_3-free, but has $\leq \varepsilon^{\omega(1)} n^3$ copies of K_3.
Let $R \subseteq [m]$ with $|R| \geq \varepsilon m$. We define the graph $G = \mathrm{RS}(m, R)$:



G has $n = \Theta(m)$ vertices.

Triangles $\longleftrightarrow$ solutions in R to $a + b = 2c$ (plus choice of basepoint).

Each $r \in R$ yields a trivial solution
$\Longrightarrow m$ vertex-disjoint triangles.

For $r \neq r'$, they are edge-disjoint
$\Longrightarrow m|R|$ edge-disjoint triangles
$\Longrightarrow G$ is $\Theta(\varepsilon)$-far from triangle-free.

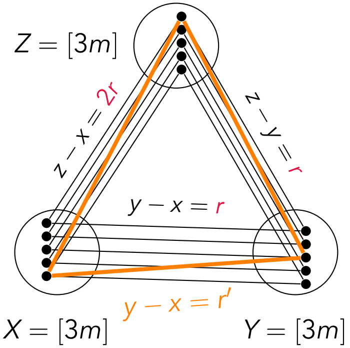
If R has no non-trivial solutions,
$\#\{K_3 \text{ in } G\} = m|R| \leq m^2 = \Theta\left(\frac{1}{m}\right) \cdot n^3$.

Proof sketch: The Ruzsa–Szemerédi construction

Theorem (Ruzsa–Szemerédi 1978)

K_3 is not K_3-abundant.

Want: G which is ε-far from K_3-free, but has $\leq \varepsilon^{\omega(1)} n^3$ copies of K_3.
Let $R \subseteq [m]$ with $|R| \geq \varepsilon m$. We define the graph $G = \mathrm{RS}(m, R)$:



G has $n = \Theta(m)$ vertices.

Triangles $\longleftrightarrow$ solutions in R to $a + b = 2c$ (plus choice of basepoint).

Each $r \in R$ yields a trivial solution
$\Longrightarrow m$ vertex-disjoint triangles.

For $r \neq r'$, they are edge-disjoint
$\Longrightarrow m|R|$ edge-disjoint triangles
$\Longrightarrow G$ is $\Theta(\varepsilon)$-far from triangle-free.

If R has no non-trivial solutions,
$\#\{K_3 \text{ in } G\} = m|R| \leq m^2 = \Theta\left(\frac{1}{m}\right) \cdot n^3$.

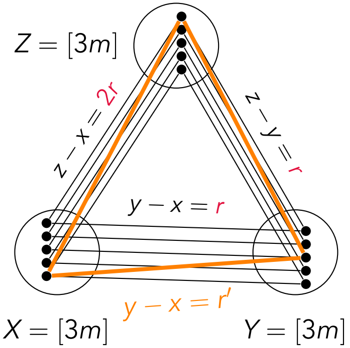
Behrend (1946): There exists such $R \subseteq [m]$ with $m \geq (1/\varepsilon)^{\omega(1)}$.

Proof sketch: The Ruzsa–Szemerédi construction

Theorem (Gishboliner–Shapira–W. 2022)

There exist triangle-free tripartite H which are not K_3-abundant.*

Want: G which is ε-far from K_3-free, but has $\leq \varepsilon^{\omega(1)} n^3$ copies of K_3.
Let $R \subseteq [m]$ with $|R| \geq \varepsilon m$. We define the graph $G = \mathrm{RS}(m, R)$:



G has $n = \Theta(m)$ vertices.

Triangles $\longleftrightarrow$ solutions in R to $a + b = 2c$ (plus choice of basepoint).

Each $r \in R$ yields a trivial solution
$\Longrightarrow m$ vertex-disjoint triangles.

For $r \neq r'$, they are edge-disjoint
$\Longrightarrow m|R|$ edge-disjoint triangles
$\Longrightarrow G$ is $\Theta(\varepsilon)$-far from triangle-free.

If R has no non-trivial solutions,
$\#\{K_3 \text{ in } G\} = m|R| \leq m^2 = \Theta\left(\frac{1}{m}\right) \cdot n^3$.

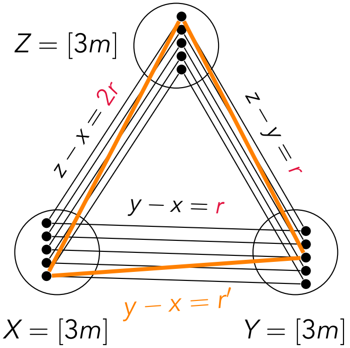
Behrend (1946): There exists such $R \subseteq [m]$ with $m \geq (1/\varepsilon)^{\omega(1)}$.

Proof sketch: The Ruzsa–Szemerédi construction

Theorem (Gishboliner–Shapira–W. 2022)

*There exist triangle-free tripartite H which are not** K_3*-abundant.*

Want: G which is ε-far from K_3-free, but has $\le \varepsilon^{\omega(1)} n^h$ copies of H.
Let $R \subseteq [m]$ with $|R| \ge \varepsilon m$. We define the graph $G = \mathrm{RS}(m, R)$:



G has $n = \Theta(m)$ vertices.

Triangles $\longleftrightarrow$ solutions in R to $a + b = 2c$ (plus choice of basepoint).

Each $r \in R$ yields a trivial solution
$\Longrightarrow m$ vertex-disjoint triangles.

For $r \ne r'$, they are edge-disjoint
$\Longrightarrow m|R|$ edge-disjoint triangles
$\Longrightarrow G$ is $\Theta(\varepsilon)$-far from triangle-free.

If R has no non-trivial solutions,
$\#\{K_3 \text{ in } G\} = m|R| \le m^2 = \Theta\left(\frac{1}{m}\right) \cdot n^3$.

Behrend (1946): There exists such $R \subseteq [m]$ with $m \ge (1/\varepsilon)^{\omega(1)}$.

Proof sketch: The Ruzsa–Szemerédi construction

Theorem (Gishboliner–Shapira–W. 2022)

There exist triangle-free tripartite H which are not K_3-abundant.*

Want: G which is ε-far from K_3-free, but has $\leq \varepsilon^{\omega(1)} n^h$ copies of H.

Let $R \subseteq [m]$ with $|R| \geq \varepsilon m$. We define the graph $G = \mathrm{RS}(m, R)$:

G has $n = \Theta(m)$ vertices.

Triangles $\longleftrightarrow$ solutions in R to $a + b = 2c$ (plus choice of basepoint).

Each $r \in R$ yields a trivial solution $\implies m$ vertex-disjoint triangles.

For $r \neq r'$, they are edge-disjoint $\implies m|R|$ edge-disjoint triangles $\implies G$ is $\Theta(\varepsilon)$-far from triangle-free.

If R has no non-trivial solutions, $\#\{K_3 \text{ in } G\} = m|R| \leq m^2 = \Theta\left(\frac{1}{m}\right) \cdot n^3$.

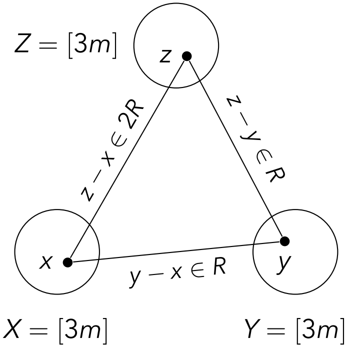
Behrend (1946): There exists such $R \subseteq [m]$ with $m \geq (1/\varepsilon)^{\omega(1)}$.

Proof sketch: The Ruzsa–Szemerédi construction

Theorem (Gishboliner–Shapira–W. 2022)

There exist triangle-free tripartite H which are not K_3-abundant.*

Want: G which is ε-far from K_3-free, but has $\leq \varepsilon^{\omega(1)} n^h$ copies of H.
Let $R \subseteq [m]$ with $|R| \geq \varepsilon m$. We define the graph $G = \mathrm{RS}(m, R)$:



G has $n = \Theta(m)$ vertices.

Triangles $\longleftrightarrow$ solutions in R to $a + b = 2c$ (plus choice of basepoint).

Each $r \in R$ yields a trivial solution $\Longrightarrow$ m vertex-disjoint triangles.

For $r \neq r'$, they are edge-disjoint $\Longrightarrow$ $m|R|$ edge-disjoint triangles $\Longrightarrow$ G is $\Theta(\varepsilon)$-far from triangle-free.

If R has no non-trivial solutions, $\#\{K_3 \text{ in } G\} = m|R| \leq m^2 = \Theta\left(\frac{1}{m}\right) \cdot n^3$.

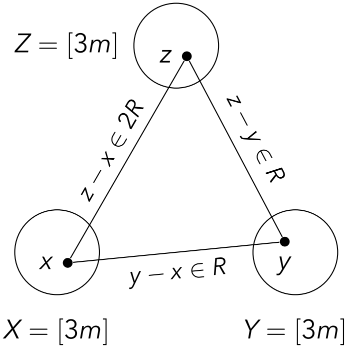
Behrend (1946): There exists such $R \subseteq [m]$ with $m \geq (1/\varepsilon)^{\omega(1)}$.

Proof sketch: The Ruzsa–Szemerédi construction

Theorem (Gishboliner–Shapira–W. 2022)

There exist triangle-free tripartite H which are not * K_3*-abundant.*

Want: G which is ε-far from K_3-free, but has $\leq \varepsilon^{\omega(1)} n^h$ copies of H.
Let $R \subseteq [m]$ with $|R| \geq \varepsilon m$. We define the graph $G = \mathrm{RS}(m, R)$:



G is $\Theta(\varepsilon)$-far from triangle-free.

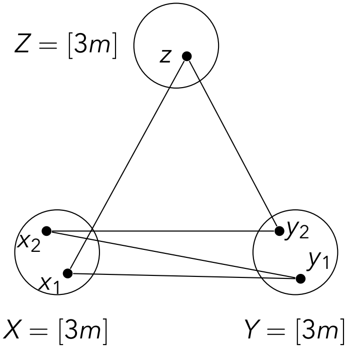
Behrend (1946): There exists such $R \subseteq [m]$ with $m \geq (1/\varepsilon)^{\omega(1)}$.

Proof sketch: The Ruzsa–Szemerédi construction

Theorem (Gishboliner–Shapira–W. 2022)

There exist triangle-free tripartite H which are not K_3-abundant.*

Want: G which is ε-far from K_3-free, but has $\leq \varepsilon^{\omega(1)} n^h$ copies of H.
Let $R \subseteq [m]$ with $|R| \geq \varepsilon m$. We define the graph $G = \mathrm{RS}(m, R)$:



G is $\Theta(\varepsilon)$-far from triangle-free.

Suppose we have a C_5 in G.

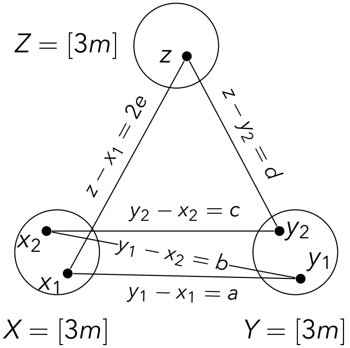
Behrend (1946): There exists such $R \subseteq [m]$ with $m \geq (1/\varepsilon)^{\omega(1)}$.

Proof sketch: The Ruzsa–Szemerédi construction

Theorem (Gishboliner–Shapira–W. 2022)

*There exist triangle-free tripartite H which are not** K_3*-abundant.*

Want: G which is ε-far from K_3-free, but has $\leq \varepsilon^{\omega(1)} n^h$ copies of H.
Let $R \subseteq [m]$ with $|R| \geq \varepsilon m$. We define the graph $G = \mathrm{RS}(m, R)$:



G is $\Theta(\varepsilon)$-far from triangle-free.

Suppose we have a C_5 in G.

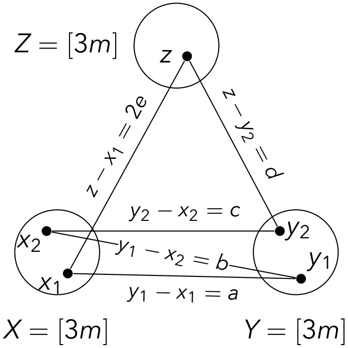
Behrend (1946): There exists such $R \subseteq [m]$ with $m \geq (1/\varepsilon)^{\omega(1)}$.

Proof sketch: The Ruzsa–Szemerédi construction

Theorem (Gishboliner–Shapira–W. 2022)

There exist triangle-free tripartite H which are not K_3-abundant.*

Want: G which is ε-far from K_3-free, but has $\leq \varepsilon^{\omega(1)} n^h$ copies of H.
Let $R \subseteq [m]$ with $|R| \geq \varepsilon m$. We define the graph $G = \mathrm{RS}(m, R)$:



G is $\Theta(\varepsilon)$-far from triangle-free.

Suppose we have a C_5 in G.

$$a - b + c + d - 2e = 0$$

Behrend (1946): There exists such $R \subseteq [m]$ with $m \geq (1/\varepsilon)^{\omega(1)}$.

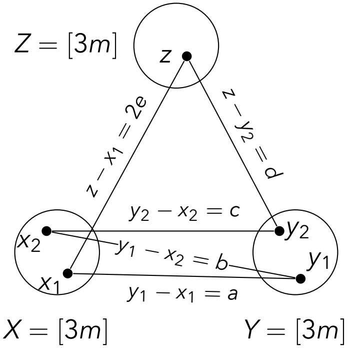
Proof sketch: The Ruzsa–Szemerédi construction

Theorem (Gishboliner–Shapira–W. 2022)

There exist triangle-free tripartite H which are not K_3-abundant.*

Want: G which is ε-far from K_3-free, but has $\le \varepsilon^{\omega(1)} n^h$ copies of H.

Let $R \subseteq [m]$ with $|R| \ge \varepsilon m$. We define the graph $G = \mathrm{RS}(m, R)$:



G is $\Theta(\varepsilon)$-far from triangle-free.

Suppose we have a C_5 in G.

$$a - b + c + d - 2e = 0$$

For any H, copies of H in $G \longleftrightarrow$ solutions in R of E.

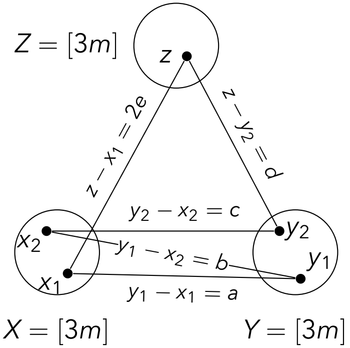
Behrend (1946): There exists such $R \subseteq [m]$ with $m \ge (1/\varepsilon)^{\omega(1)}$.

Proof sketch: The Ruzsa–Szemerédi construction

Theorem (Gishboliner–Shapira–W. 2022)

There exist triangle-free tripartite H which are not K_3-abundant.*

Want: G which is ε-far from K_3-free, but has $\le \varepsilon^{\omega(1)} n^h$ copies of H.
Let $R \subseteq [m]$ with $|R| \ge \varepsilon m$. We define the graph $G = \mathrm{RS}(m, R)$:



G is $\Theta(\varepsilon)$-far from triangle-free.

Suppose we have a C_5 in G.

$$a - b + c + d - 2e = 0$$

For any H, copies of H in G $\longleftrightarrow$ solutions in R of E.

E = system of linear equations:
variables = edges of H,
equations = cycles of H.

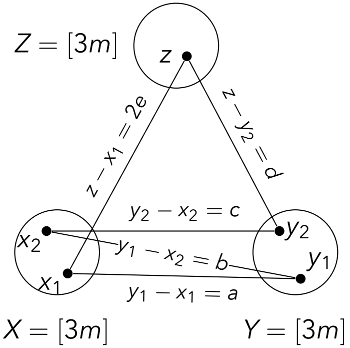
Behrend (1946): There exists such $R \subseteq [m]$ with $m \ge (1/\varepsilon)^{\omega(1)}$.

Proof sketch: The Ruzsa–Szemerédi construction

Theorem (Gishboliner–Shapira–W. 2022)

*There exist triangle-free tripartite H which are not** K_3*-abundant.*

Want: G which is ε-far from K_3-free, but has $\leq \varepsilon^{\omega(1)} n^h$ copies of H.
Let $R \subseteq [m]$ with $|R| \geq \varepsilon m$. We define the graph $G = \mathrm{RS}(m, R)$:



G is $\Theta(\varepsilon)$-far from triangle-free.

Suppose we have a C_5 in G.

$$a - b + c + d - 2e = 0$$

For any H, copies of H in $G \longleftrightarrow$ solutions in R of E.

E = system of linear equations:
variables = edges of H,
equations = cycles of H.

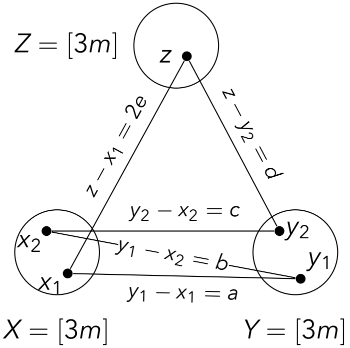
Want: R has no non-trivial solutions.

Behrend (1946): There exists such $R \subseteq [m]$ with $m \geq (1/\varepsilon)^{\omega(1)}$.

Theorem (Gishboliner–Shapira–W. 2022)

There exist triangle-free tripartite H which are not K_3-abundant.*

Want: G which is ε-far from K_3-free, but has $\leq \varepsilon^{\omega(1)} n^h$ copies of H.
Let $R \subseteq [m]$ with $|R| \geq \varepsilon m$. We define the graph $G = \mathrm{RS}(m, R)$:



G is $\Theta(\varepsilon)$-far from triangle-free.

Suppose we have a C_5 in G.

$$a - b + c + d - 2e = 0$$

For any H, copies of H in G $\longleftrightarrow$ solutions in R of E.

E = system of linear equations:
variables = edges of H,
equations = cycles of H.

Want: R has no non-trivial solutions.

Ruzsa (1993): For which E is there such $R \subseteq [m]$ with $m \geq (1/\varepsilon)^{\omega(1)}$?

Ruzsa's genus conjecture

Ruzsa (1993): For which E is there such $R \subseteq [m]$ with $m \geq (1/\varepsilon)^{\omega(1)}$?

Ruzsa's genus conjecture

Ruzsa (1993): For which E is there such $R \subseteq [m]$ with $m \geq (1/\varepsilon)^{\omega(1)}$?
(i.e. $|R| \geq \varepsilon m$ and R has no non-trivial solutions to E)

Ruzsa's genus conjecture

Ruzsa (1993): For which E is there such $R \subseteq [m]$ with $m \geq (1/\varepsilon)^{\omega(1)}$?
(i.e. $|R| \geq \varepsilon m$ and R has no non-trivial solutions to E)

Theorem/Conjecture (Ruzsa 1993)

There exists such R if and only if E has genus one.

Ruzsa's genus conjecture

Ruzsa (1993): For which E is there such $R \subseteq [m]$ with $m \geq (1/\varepsilon)^{\omega(1)}$?
(i.e. $|R| \geq \varepsilon m$ and R has no non-trivial solutions to E)

Theorem/Conjecture (Ruzsa 1993)

There exists such R if and only if E has genus one.

Genus one = a simple linear-algebraic condition on the coefficients.

Ruzsa's genus conjecture

Ruzsa (1993): For which E is there such $R \subseteq [m]$ with $m \geq (1/\varepsilon)^{\omega(1)}$?
(i.e. $|R| \geq \varepsilon m$ and R has no non-trivial solutions to E)

Theorem/Conjecture (Ruzsa 1993)

There exists such R if and only if E has genus one.

Genus one = a simple linear-algebraic condition on the coefficients.

It remains to construct H such that its E has genus one.

Ruzsa's genus conjecture

Ruzsa (1993): For which E is there such $R \subseteq [m]$ with $m \geq (1/\varepsilon)^{\omega(1)}$?
(i.e. $|R| \geq \varepsilon m$ and R has no non-trivial solutions to E)

Theorem/Conjecture (Ruzsa 1993)

There exists such R if and only if E has genus one.

Genus one = a simple linear-algebraic condition on the coefficients.

It remains to construct H such that its E has genus one.

Let H be a random triangle-free tripartite graph.

Ruzsa's genus conjecture

Ruzsa (1993): For which E is there such $R \subseteq [m]$ with $m \geq (1/\varepsilon)^{\omega(1)}$?
(i.e. $|R| \geq \varepsilon m$ and R has no non-trivial solutions to E)

Theorem/Conjecture (Ruzsa 1993)

There exists such R if and only if E has genus one.

Genus one = a simple linear-algebraic condition on the coefficients.

It remains to construct H such that its E has genus one.

Let H be a random triangle-free tripartite graph.

Lemma (Gishboliner–Shapira–W. 2022)

For such H, its set E of equations has genus one w.h.p.

Ruzsa's genus conjecture

Ruzsa (1993): For which E is there such $R \subseteq [m]$ with $m \geq (1/\varepsilon)^{\omega(1)}$?
(i.e. $|R| \geq \varepsilon m$ and R has no non-trivial solutions to E)

Theorem/Conjecture (Ruzsa 1993)

There exists such R if and only if E has genus one.

Genus one = a simple linear-algebraic condition on the coefficients.

It remains to construct H such that its E has genus one.

Let H be a (pseudo)random triangle-free tripartite graph.

Lemma (Gishboliner–Shapira–W. 2022)

For such H, its set E of equations has genus one w.h.p.

Ruzsa's genus conjecture

Ruzsa (1993): For which E is there such $R \subseteq [m]$ with $m \geq (1/\varepsilon)^{\omega(1)}$?
(i.e. $|R| \geq \varepsilon m$ and R has no non-trivial solutions to E)

Theorem/Conjecture (Ruzsa 1993)

There exists such R if and only if E has genus one.

Genus one = a simple linear-algebraic condition on the coefficients.

It remains to construct H such that its E has genus one.

Let H be a (pseudo)random triangle-free tripartite graph.

Lemma (Gishboliner–Shapira–W. 2022)

For such H, its set E of equations has genus one w.h.p.

The proof uses global, Ramsey-esque arguments, plus some structural information arising from the pseudorandomness.

Conclusion and open problems

Theorem (asymmetric removal lemma)

Let G have n vertices and H have h vertices. Suppose $\chi(H) = 3$.
If G is ε-far from triangle-free, then G contains $\geq \delta^* n^h$ copies of H, where $\delta^* = \delta^*(\varepsilon, H) > 0$ depends only on ε and H.

Conclusion and open problems

Theorem (asymmetric removal lemma)

Let G have n vertices and H have h vertices. Suppose $\chi(H) = 3$. If G is ε-far from triangle-free, then G contains $\geq \delta^* n^h$ copies of H, where $\delta^* = \delta^*(\varepsilon, H) > 0$ depends only on ε and H.

Theorem (Gishboliner–Shapira–W. 2022)

If $H = C_\ell$ for odd $\ell \geq 5$, we have $\delta^*(\varepsilon, H) = \mathrm{poly}(\varepsilon)$.

Conclusion and open problems

Theorem (asymmetric removal lemma)

Let G have n vertices and H have h vertices. Suppose $\chi(H) = 3$.
If G is ε-far from triangle-free, then G contains $\geq \delta^* n^h$ copies of H, where $\delta^* = \delta^*(\varepsilon, H) > 0$ depends only on ε and H.

Theorem (Gishboliner–Shapira–W. 2022)

If $H = C_\ell$ for odd $\ell \geq 5$, we have $\delta^*(\varepsilon, H) = \mathrm{poly}(\varepsilon)$.
There exist * *triangle-free H such that $\delta^*(\varepsilon, H) \neq \mathrm{poly}(\varepsilon)$.*

Conclusion and open problems

Theorem (asymmetric removal lemma)

Let G have n vertices and H have h vertices. Suppose $\chi(H) = 3$.
If G is ε-far from triangle-free, then G contains $\geq \delta^* n^h$ copies of H, where $\delta^* = \delta^*(\varepsilon, H) > 0$ depends only on ε and H.

Theorem (Gishboliner–Shapira–W. 2022)

If $H = C_\ell$ for odd $\ell \geq 5$, we have $\delta^*(\varepsilon, H) = \mathrm{poly}(\varepsilon)$.
There exist * *triangle-free H such that $\delta^*(\varepsilon, H) \neq \mathrm{poly}(\varepsilon)$.*

Open problems:

Conclusion and open problems

Theorem (asymmetric removal lemma)

Let G have n vertices and H have h vertices. Suppose $\chi(H) = 3$.
If G is ε-far from triangle-free, then G contains $\geq \delta^* n^h$ copies of H, where $\delta^* = \delta^*(\varepsilon, H) > 0$ depends only on ε and H.

Theorem (Gishboliner–Shapira–W. 2022)

If $H = C_\ell$ for odd $\ell \geq 5$, we have $\delta^*(\varepsilon, H) = \mathrm{poly}(\varepsilon)$.
There exist * *triangle-free H such that $\delta^*(\varepsilon, H) \neq \mathrm{poly}(\varepsilon)$.*

Open problems:

1. Delete the *.

Conclusion and open problems

Theorem (asymmetric removal lemma)

Let G have n vertices and H have h vertices. Suppose $\chi(H) = 3$.
If G is ε-far from triangle-free, then G contains $\geq \delta^* n^h$ copies of H, where $\delta^* = \delta^*(\varepsilon, H) > 0$ depends only on ε and H.

Theorem (Gishboliner–Shapira–W. 2022)

If $H = C_\ell$ for odd $\ell \geq 5$, we have $\delta^*(\varepsilon, H) = \mathrm{poly}(\varepsilon)$.
There exist * *triangle-free H such that $\delta^*(\varepsilon, H) \neq \mathrm{poly}(\varepsilon)$.*

Open problems:

1. Delete the *.
   - Prove Ruzsa's genus conjecture, at least in some special cases.

Conclusion and open problems

Theorem (asymmetric removal lemma)

*Let G have n vertices and H have h vertices. Suppose $\chi(H) = 3$.
If G is ε-far from triangle-free, then G contains $\geq \delta^* n^h$ copies of H, where $\delta^* = \delta^*(\varepsilon, H) > 0$ depends only on ε and H.*

Theorem (Gishboliner–Shapira–W. 2022)

*If $H = C_\ell$ for odd $\ell \geq 5$, we have $\delta^*(\varepsilon, H) = \text{poly}(\varepsilon)$.
There exist* * *triangle-free H such that $\delta^*(\varepsilon, H) \neq \text{poly}(\varepsilon)$.*

Open problems:

1. Delete the *.
   - Prove Ruzsa's genus conjecture, at least in some special cases.
   - Find another construction, beyond RS(m, R), for such problems.

Conclusion and open problems

Theorem (asymmetric removal lemma)

Let G have n vertices and H have h vertices. Suppose $\chi(H) = 3$.
If G is ε-far from triangle-free, then G contains $\geq \delta^* n^h$ copies of H, where $\delta^* = \delta^*(\varepsilon, H) > 0$ depends only on ε and H.

Theorem (Gishboliner–Shapira–W. 2022)

If $H = C_\ell$ for odd $\ell \geq 5$, we have $\delta^*(\varepsilon, H) = \mathrm{poly}(\varepsilon)$.
There exist * *triangle-free H such that $\delta^*(\varepsilon, H) \neq \mathrm{poly}(\varepsilon)$.*

Open problems:

1. Delete the *.
    - Prove Ruzsa's genus conjecture, at least in some special cases.
    - Find another construction, beyond $\mathrm{RS}(m, R)$, for such problems.
2. Is the Petersen graph K_3-abundant? Which graphs are?

Conclusion and open problems

Theorem (asymmetric removal lemma)

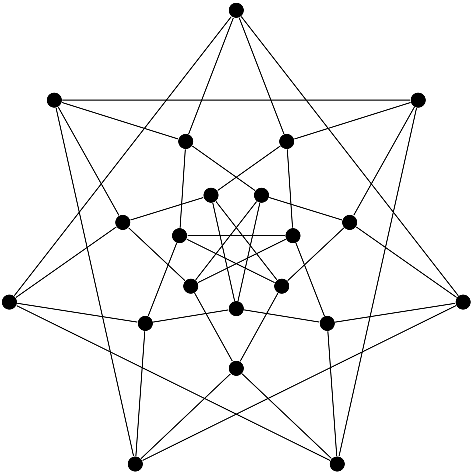
Let G have n vertices and H have h vertices. Suppose $\chi(H) = 3$. If G is ε-far from triangle-free, then G contains $\geq \delta^* n^h$ copies of H, where $\delta^* = \delta^*(\varepsilon, H) > 0$ depends only on ε and H.

Theorem (Gishboliner–Shapira–W. 2022)

If $H = C_\ell$ for odd $\ell \geq 5$, we have $\delta^*(\varepsilon, H) = \mathrm{poly}(\varepsilon)$.
There exist * *triangle-free H such that $\delta^*(\varepsilon, H) \neq \mathrm{poly}(\varepsilon)$.*

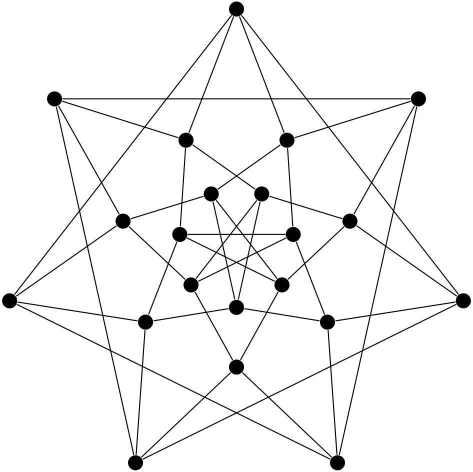
Open problems:

1. Delete the *.
   - Prove Ruzsa's genus conjecture, at least in some special cases.
   - Find another construction, beyond $\mathrm{RS}(m, R)$, for such problems.
2. Is the Petersen graph K_3-abundant? Which graphs are?
3. Are there any K_4-abundant graphs? The first open case is the Brinkmann graph.



The Brinkmann graph

Thank you!



The Brinkmann graph

Genus-one systems of equations

Genus-one systems of equations

Write E in matrix form.

Genus-one systems of equations

Write E in matrix form.

$$E_1 = \begin{cases} a - b + c - d = 0 \\ a + 2b - 2c - d = 0 \end{cases} \longrightarrow \quad M_1 = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 2 & -2 & -1 \end{pmatrix}$$

$$E_2 = \begin{cases} a - b + c - d = 0 \\ 2a + b - 2c - d = 0 \end{cases} \longrightarrow \quad M_2 = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 2 & 1 & -2 & -1 \end{pmatrix}$$

Genus-one systems of equations

Write E in matrix form.

$$E_1 = \begin{cases} a - b + c - d = 0 \\ a + 2b - 2c - d = 0 \end{cases} \longrightarrow \quad M_1 = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 2 & -2 & -1 \end{pmatrix}$$

$$E_2 = \begin{cases} a - b + c - d = 0 \\ 2a + b - 2c - d = 0 \end{cases} \longrightarrow \quad M_2 = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 2 & 1 & -2 & -1 \end{pmatrix}$$

Definition (Ruzsa 1993)

E has genus one if no proper subset of the columns sums to zero.

Genus-one systems of equations

Write E in matrix form.

$$E_1 = \begin{cases} a - b + c - d = 0 \\ a + 2b - 2c - d = 0 \end{cases} \longrightarrow M_1 = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 2 & -2 & -1 \end{pmatrix}$$

$$E_2 = \begin{cases} a - b + c - d = 0 \\ 2a + b - 2c - d = 0 \end{cases} \longrightarrow M_2 = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 2 & 1 & -2 & -1 \end{pmatrix}$$

Definition (Ruzsa 1993)

E has genus one if no proper subset of the columns sums to zero.

Genus-one systems of equations

Write E in matrix form.

$$E_1 = \begin{cases} a - b + c - d = 0 \\ a + 2b - 2c - d = 0 \end{cases} \longrightarrow \quad M_1 = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 2 & -2 & -1 \end{pmatrix}$$

$$E_2 = \begin{cases} a - b + c - d = 0 \\ 2a + b - 2c - d = 0 \end{cases} \longrightarrow \quad M_2 = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 2 & 1 & -2 & -1 \end{pmatrix}$$

Definition (Ruzsa 1993)

E has genus one if no proper subset of the columns sums to zero.

Let $R \subseteq [m]$ with $|R| \geq \varepsilon m$ have no non-trivial solutions to E.

Genus-one systems of equations

Write E in matrix form.

$$E_1 = \begin{cases} a - b + c - d = 0 \\ a + 2b - 2c - d = 0 \end{cases} \longrightarrow M_1 = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 2 & -2 & -1 \end{pmatrix}$$

$$E_2 = \begin{cases} a - b + c - d = 0 \\ 2a + b - 2c - d = 0 \end{cases} \longrightarrow M_2 = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 2 & 1 & -2 & -1 \end{pmatrix}$$

Definition (Ruzsa 1993)

E has genus one if no proper subset of the columns sums to zero.

Let $R \subseteq [m]$ with $|R| \geq \varepsilon m$ have no non-trivial solutions to E.

Theorem (Ruzsa 1993)

If E does not have genus one, then $m \leq O((1/\varepsilon)^2)$.

Conjecture (Ruzsa 1993)

If E does have genus one, there is such R with $m \geq (1/\varepsilon)^{\omega(1)}$.