

The Bar-Natan Theory Splits

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Abstract

We show that over the binary field \mathbb{F}_2 , the Bar-Natan perturbation of Khovanov homology splits as the direct sum of its two reduced theories, which we also prove are isomorphic. This extends Shumakovitch’s analogous result for ordinary Khovanov homology, without the perturbation.

1 Introduction

In [5], Khovanov introduced what he called the “categorification of the Jones polynomial”, and which has since become known as Khovanov homology. Khovanov’s idea was to construct a link invariant as the homology of a certain chain complex associated to the link, in such a way that the graded Euler characteristic of this homology theory is the link’s Jones polynomial. In [1], Bar-Natan calculated the Khovanov homology of many knots, and found that it is a strictly stronger invariant than the Jones polynomial—there are pairs of knots whose Jones polynomials are equal but whose Khovanov homologies are non-isomorphic.

However, it is misleading to think of Khovanov homology as a simple extension of the Jones polynomial, as its theory turns out to be very rich. It has been developed and extended in many ways, including the anti-commutative “odd” version defined by Osváth, Rasmussen, and Szábo in [10], the higher-dimensional analogue introduced by Szábo in [12], the $\mathfrak{sl}(n)$ -homology defined by Khovanov and Rozansky in [4, 8], the functoriality properties developed by Jacobsson and others in [3] and further papers, and its extensions to tangle invariants defined by Khovanov in [6] and by Bar-Natan in [2]. This latter paper, [2], is also important because it is where Bar-Natan introduced a “perturbation” to Khovanov homology (similar to Lee’s perturbation [9]) which has become a major area of study.

In this paper, we begin by introducing Khovanov homology and the Bar-Natan perturbation. Following Khovanov’s [7], the two reduced Bar-Natan theories are introduced, and we finish by extending a result of Shumakovitch in [11] to show that over the binary field \mathbb{F}_2 , the Bar-Natan theory splits as the direct sum of the two reduced theories.

The proof technique used in this paper and in [11] cannot be obviously extended to work over other rings, as many of the simplifications used rely very heavily on the fact that we work in characteristic 2. In fact, it is known that Khovanov homology does not split over \mathbb{Z} , as the example of the trefoil knot shows, which means that Shumakovitch’s result cannot immediately be extended to characteristic zero. Moreover, since the Bar-Natan theory reduces to ordinary Khovanov homology, the same is true of this paper’s results. However, it was shown in [10] that the analogue of Shumakovitch’s result is true in odd Khovanov homology; i.e. odd Khovanov homology splits as the direct sum of two isomorphic reduced theories even over \mathbb{Z} . Since \mathbb{F}_2 Khovanov homology is a reduction both of ordinary and of odd Khovanov homologies, it seems possible that our results could be extended to the odd regime, assuming a suitable analogue of the Bar-Natan perturbation could be defined there.

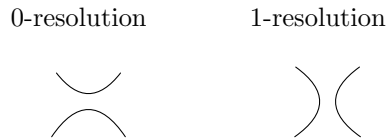
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2 Khovanov Homology

Let \mathcal{L} be a link, and \mathbb{F}_2 be the field with two elements¹. In this section, we summarize the definition of the Khovanov homology of \mathcal{L} , as defined in [5]. For this, we begin by fixing an oriented link diagram L of \mathcal{L} : this is a collection of oriented arcs in the plane, with n double points. We write $n = n_+ + n_-$, where n_+ (resp. n_-) is the number of positive (resp. negative) crossings in L . We order these double points arbitrarily, thus identifying them with the set $\{1, 2, \dots, n\}$. Each double point is of the form



We define two “resolutions” of such a crossing:



So for each vertex α of the hypercube $\{0, 1\}^n$, we can resolve each crossing of L in the obvious way: if the j th coordinate of α is 0, then we use the 0-resolution for crossing j , and we use the 1-resolution otherwise. Note that any such “full resolution” can have no crossing points, so it must be a collection of disjoint circles embedded in the plane. We call this collection of circles $D_\alpha(L)$. We further associate to each vertex α a vector space $V_\alpha(L)$ of dimension 2^k , where k is the number of circles in D_α . We pick a basis of $V_\alpha(L)$ and identify it with the collection of labellings of the circles in $D_\alpha(L)$ by two symbols, which we call $\mathbf{1}$ and \mathbf{x} . We can do this formally by letting W be the 2-dimensional vector space spanned by $\mathbf{1}$ and \mathbf{x} , and then declaring $V_\alpha(L) := W^{\otimes k}$. Moreover, we introduce a grading to elements of $V_\alpha(L)$ by declaring $gr(\mathbf{1}) = 1, gr(\mathbf{x}) = -1$ and by extending grading additively to the tensor product $V_\alpha(L)$. Thus, in this particular basis, a basis element of $V_\alpha(L)$ is a labelling of the k circles in $D_\alpha(L)$, and its grading is the number of $\mathbf{1}$'s in this labelling minus the number of \mathbf{x} 's.

2.1 Chain Groups

The chain groups in the chain complex defining Khovanov homology are direct sums of these V_α 's:

Definition 1 (Khovanov, [5]). The Khovanov chain complex of L has, as its i th chain group,

$$C_i := \bigoplus_{\alpha \in \{0,1\}^n: w(\alpha) = i + n_-} V_\alpha(L)$$

where $w(\alpha)$ is the Hamming weight of α , namely the number of 1's in α . Each vector space C_i is endowed with a *quantum grading* defined by

$$q(v) = gr(v) + i + n_+ - n_-$$

Since there are no hypercube vertices α with $w(\alpha) < 0$ or $w(\alpha) > n$, we see that all chain groups C_i with $i < -n_-$ or $i > n_+$ will be the 0 group.

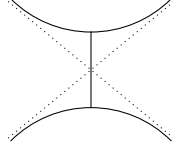
The Khovanov homology of \mathcal{L} will be defined as the homology² of this chain complex, so we need to define a differential on this complex.

¹Everything below also works over any field, and indeed over \mathbb{Z} . However, since our main result deals with the \mathbb{F}_2 version of Khovanov homology, we won't bother with the other versions.

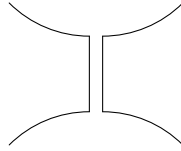
²Under the above definition, the differential will be a map $C_i \rightarrow C_{i+1}$, so Khovanov homology will, strictly speaking, be a cohomology theory.

2.2 The Differential

Defining a differential $d_i : C_i \rightarrow C_{i+1}$ will be done in several steps, beginning by defining a map for each edge of the hypercube. Note that an edge of the hypercube connects two vertices α and β which necessarily differ in exactly one coordinate, say the j th. Without loss of generality, α has a 0 in the j th coordinate, while β will have a 1 there. This means that the only difference between $D_\alpha(L)$ and $D_\beta(L)$ is in the resolution of the j th crossing, where D_α uses a 0-resolution, while D_β uses a 1-resolution. In order to mark how changing a 0-resolution to a 1-resolution will affect the diagram, we place a little arc in each 0-resolution, as follows:



As shown, we place this arc so that it overlaps with the original crossing point. Then, going from a 0-resolution to a 1-resolution simply involves performing surgery along this arc: we translate the arc a bit to the left and a bit to the right and treat these new segments as part of the 1-resolution.



We now consider the two possible cases:

Case 1: The two arcs in the 0-resolution of the j th crossing belong to two distinct circles of $D_\alpha(L)$. In this case, moving from a 0-resolution to a 1-resolution turns these two circles into one, while all other circles remain unchanged. This situation is called a *merge*.

Case 2: The two arcs in the 0-resolution of the j th crossing belong to the same circle. In this case, going from a 0-resolution to a 1-resolution will turn this single circle into two distinct circles, while keeping all other circles unchanged. This situation is called a *split*.

Using the above notation, we see that the following are a merge and a split, respectively:



Given such an edge between α and β , the goal is to define a map $d_{\alpha,\beta} : V_\alpha(L) \rightarrow V_\beta(L)$. Of course, it suffices to define $d_{\alpha,\beta}$ on a basis of $V_\alpha(L)$. Recall that in the chosen basis, a basis vector for $V_\alpha(L)$ is simply a labelling of the circles in $D_\alpha(L)$, and similarly for $V_\beta(L)$. Given such a basis vector, one needs to define its image under $d_{\alpha,\beta}$. By the above, we know that most of the circles in $D_\alpha(L)$ are unchanged by the re-resolution of the j th crossing, so for these circles, one can simply define the labelling of the corresponding circle in $D_\beta(L)$ to be its original labelling in the given basis vector.

Now, if the re-resolution is a split, we have taken into account the original labelling of all but one circle (namely the splitting circle), and we have defined the labellings of all but two target circles (namely the two daughter circles). So all that remains is defining a map $\Delta : W \rightarrow W \otimes W$. This is done by declaring

$$\Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{x} + \mathbf{x} \otimes \mathbf{1} \quad \Delta(\mathbf{x}) = \mathbf{x} \otimes \mathbf{x}$$

Similarly, in the case of a merge, one must define a map $m : W \otimes W \rightarrow W$, which is done by declaring

$$\begin{aligned} m(\mathbf{1} \otimes \mathbf{1}) &= \mathbf{1} & m(\mathbf{x} \otimes \mathbf{x}) &= 0 \\ m(\mathbf{1} \otimes \mathbf{x}) &= \mathbf{x} & m(\mathbf{x} \otimes \mathbf{1}) &= \mathbf{x} \end{aligned}$$

Then, in addition to the above identity actions on the circles not involved in the re-resolution, we have defined a map $d_{\alpha,\beta} : V_\alpha(L) \rightarrow V_\beta(L)$. For convenience, we also define $d_{\alpha,\beta}$ to be the 0 map in case there is no edge between α and β . These maps satisfy a commutativity relation, namely

Lemma 1 (Khovanov, [5]). *For each 2-dimensional face in the hypercube, the maps on the edges commute. More formally, a 2-dimensional face has vertices $\{\alpha, \beta_1, \beta_2, \gamma\}$, where β_1 and β_2 are gotten from α by changing one 0 coordinate to 1, and γ is similarly gotten from both β_1 and β_2 . Then for all such faces,*

$$d_{\beta_1,\gamma} \circ d_{\alpha,\beta_1} = d_{\beta_2,\gamma} \circ d_{\alpha,\beta_2}$$

The differential is then defined as follows.

Definition 2 (Khovanov, [5]). A map $d_i : C_i \rightarrow C_{i+1}$ is defined by

$$d_i := \bigoplus_{\alpha,\beta : w(\alpha)=i+n_-, w(\beta)=i+1-n_-} d_{\alpha,\beta}$$

By Lemma 1, the maps $d_{\alpha,\beta}$ commute on each 2-dimensional face, which is the same as anticommuting since we are working over \mathbb{F}_2 . Therefore, $d_{i+1} \circ d_i = 0$ for all i , which means that d is a differential³.

Definition 3 (Khovanov, [5]). The Khovanov chain complex $\mathcal{C}(L)$ is the chain complex (C_*, d_*) , with chain groups C_i and differential d_i for all i . By Lemma 1 and the above remarks, $\mathcal{C}(L)$ is indeed a chain complex.

The Khovanov homology $Kh_i(L)$ is defined as the i th homology group of $\mathcal{C}(L)$. Since the maps Δ and m both decrease the grading gr by 1, we can see that they will act homogeneously on the quantum grading, which means that q will descend to a grading on the homology $Kh_i(L)$. Thus, $Kh_i(L)$ is a graded vector space.

It is clear that the Khovanov chain complex itself is highly dependent on which link diagram L of the link \mathcal{L} we pick; for instance, the number of chain groups in $\mathcal{C}(L)$ is precisely the number of crossings in L , and there are link diagrams of the same link with arbitrarily many crossings. Nonetheless:

Theorem 1 (Khovanov, [5]). *Khovanov homology is a link invariant. More formally, if L and L' are two link diagrams of the same link \mathcal{L} , then $Kh_i(L) \cong Kh_i(L')$ for all i , where \cong denotes isomorphism of graded vector spaces.*

2.3 The Bar-Natan Perturbation

Now, we define the Bar-Natan perturbation to Khovanov homology. We begin with a purely algebraic formality:

Definition 4. Let H be a formal variable, and let V be an \mathbb{F}_2 -vector space. We extend V to $V[H]$, which is a module over the polynomial ring $\mathbb{F}_2[H]$, as follows. The elements of $V[H]$ are formal linear combinations

$$\sum_{i \geq 0} H^i v_i$$

where $v_i \in V$, and only finitely many summands are nonzero. Addition is defined componentwise, and the action of $\mathbb{F}_2[H]$ is given, naturally, by

$$\lambda \left(\sum_{i \geq 0} H^i v_i \right) = \sum_{i \geq 0} H^i \lambda v_i \quad H \left(\sum_{i \geq 0} H^i v_i \right) = \sum_{i \geq 0} H^{i+1} v_i$$

for $\lambda \in \mathbb{F}_2$.

³For the general Khovanov homology (not over \mathbb{F}_2), it is necessary to add some minus signs to this definition.

In [2], Bar-Natan introduced the map $\Delta' : W \rightarrow W \otimes W$ defined by

$$\Delta'(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1}$$

and $m' : W \otimes W \rightarrow W$ defined by

$$m'(\mathbf{x} \otimes \mathbf{x}) = \mathbf{x}$$

with all other basis vectors sent to 0 in either case⁴. From these, he formed edge maps $h_{\alpha,\beta}$, which are defined as Δ' if the edge α, β is a split, and m' if this edge is a merge. Finally, in analogy with Definition 2, he defined a homomorphism $h_i : C_i \rightarrow C_{i+1}$ by

$$h_i := \bigoplus_{\alpha, \beta: w(\alpha)=i+n_-, w(\beta)=i+1-n_-} h_{\alpha,\beta}$$

Definition 5 (Bar-Natan, [2]). Let H be a formal parameter of degree -2 . The Bar-Natan perturbation to the Khovanov chain complex is the chain complex $\mathcal{C}_{BN}(L) = (C_*[H], d_* + Hh_*)$, though for notational convenience we will often write $d + Hh$ rather than $d_* + Hh_*$. In analogy with Lemma 1, it is straightforward to check that the edge maps commute on all 2-dimensional faces of $\{0, 1\}^n$, so $\mathcal{C}_{BN}(L)$ is indeed a chain complex. Its i th homology group, which is an $\mathbb{F}_2[H]$ -module, is denoted by $BN_i(L)$. Moreover, since H has degree -2 , the differential $d + Hh$ preserves quantum grading, meaning that this grading descends to homology and $BN_i(L)$ can be viewed as a graded $\mathbb{F}_2[H]$ -module.

He also proved the following theorem:

Theorem 2 (Bar-Natan, [2]). *The Bar-Natan homology is a link invariant. More formally, if L and L' are two link diagrams of the same link \mathcal{L} , then $BN_i(L) \cong BN_i(L')$ for all i , where \cong denotes isomorphism of graded $\mathbb{F}_2[H]$ -modules.*

3 The Bar-Natan Theory Splits

We pick a basepoint on L and maintain its position in each complete resolution. Let \mathcal{C}_x and \mathcal{C}_1 denote the subgroups of \mathcal{C}_{BN} in which the circle with the basepoint is labelled \mathbf{x} and $\mathbf{1}$, respectively. Since both d and h map \mathcal{C}_x into itself, we see that \mathcal{C}_x is actually a subcomplex of \mathcal{C}_{BN} , while the quotient $\mathcal{C}_{BN}/\mathcal{C}_x$ is naturally isomorphic (as a group) to \mathcal{C}_1 . Therefore, we can consider \mathcal{C}_1 to be a complex as well, and its differential is given by composing $d + Hh$ with the projection $\mathcal{C}_{BN} \rightarrow \mathcal{C}_{BN}/\mathcal{C}_x \cong \mathcal{C}_1$.

Thus, we have a short exact sequence of chain complexes:

$$0 \longrightarrow \mathcal{C}_x \longrightarrow \mathcal{C}_{BN} \longrightarrow \mathcal{C}_1 \longrightarrow 0$$

This induces a long exact sequence on homology:

$$\dots \xrightarrow{\partial} H(\mathcal{C}_x) \longrightarrow H(\mathcal{C}_{BN}) \longrightarrow H(\mathcal{C}_1) \xrightarrow{\partial} H(\mathcal{C}_x) \longrightarrow \dots$$

The homologies $H(\mathcal{C}_x)$ and $H(\mathcal{C}_1)$ are called the reduced Bar-Natan theories. The connecting map ∂ is surprisingly easy to understand in this context, since it happens to be induced from a chain map $f : \mathcal{C}_1 \rightarrow \mathcal{C}_x$. f acts by applying the differential and then discarding all terms in which the basepoint circle is not labelled \mathbf{x} ; i.e. $f = \pi_x \circ (d + Hh)$, where π_x is a projection onto \mathcal{C}_x . This definition shows that f commutes with the differentials on \mathcal{C}_1 and \mathcal{C}_x , so it is indeed a chain map. Since the Bar-Natan perturbation h never turns a circle labelled $\mathbf{1}$ into a circle labelled \mathbf{x} , it is clear that this connecting map is actually the same as the connecting map gotten in the standard Khovanov case. In particular, f preserves quantum grading. f increases homological grading by 1, but it will be more convenient to have it preserve both homological and quantum grading, so we formally make its codomain be $\mathcal{C}_x[-1]$, a homologically shifted copy of \mathcal{C}_x ; for

⁴Over fields of characteristic $\neq 2$, it is necessary to add a minus sign to the definition of Δ' .

notational simplicity, we usually omit this formal distinction. With this caveat in mind, f preserves both gradings.

In [11], Shumakovitch proved that Khovanov homology splits as the direct sum of these two reduced theories, and we wish to extend this result to the case of the Bar-Natan perturbation. More precisely, we prove the following theorem:

Theorem 3. $H(\mathcal{C}_{BN}) = H(\mathcal{C}_x) \oplus H(\mathcal{C}_1)$ when considered as modules over $\mathbb{F}_2[H]$.

The rest of this section is devoted to proving this theorem. We will do this by proving that the map f is nullhomotopic; this suffices since \mathcal{C}_{BN} is the mapping cone of $f : \mathcal{C}_1 \rightarrow \mathcal{C}_x$, so if f is nullhomotopic we can conclude that \mathcal{C}_{BN} is chain homotopy equivalent to $\mathcal{C}_x \oplus \mathcal{C}_1$ over $\mathbb{F}_2[H]$. Thus, we will get that $H(\mathcal{C}_{BN}) = H(\mathcal{C}_x) \oplus H(\mathcal{C}_1)$ over $\mathbb{F}_2[H]$. A nullhomotopy for f is simply a map K so that $f = [K, d + Hh]$, where $[\varphi, \psi]$ denotes the commutator $\varphi \circ \psi + \psi \circ \varphi$.⁵ K need not be homogeneous, but we can write $K = K_0 + HK_1 + H^2K_2 + \dots$, where each K_i is a map that fixes homological grading but raises quantum grading by $2i$. Thus, our goal is now to construct this family of functions.

The Khovanov Case: Constructing K_0

The following construction is due to Shumakovitch in [11, Theorem 3.2.A].

For a complete resolution D in which the basepoint circle is labelled $\mathbf{1}$, we define

$$K_0(D) := \sum_{\substack{a \\ \circlearrowleft^x}} \circlearrowleft^{\mathbf{1}} \quad \circlearrowleft^x$$

The sum runs over all circles a labelled \mathbf{x} in D . For each such circle, it adds a copy of D in which a has been relabelled to $\mathbf{1}$ and the basepoint circle has been relabelled to \mathbf{x} . If this sum is empty or if the basepoint circle is labelled \mathbf{x} , then K_0 is identically 0.⁶ This map clearly preserves homological and quantum grading. We can also check directly that $f = [K_0, d]$, which is the statement that the ordinary Khovanov theory splits over \mathbb{F}_2 . To see this, we simply check that this equation holds for each 1-dimensional configuration.

We begin with those 1-dimensional configurations involving the basepoint circle. Starting with split maps, we first recall that

$$d \left(\circlearrowleft^{\mathbf{1}} \right) = \begin{bmatrix} \circlearrowleft^{\mathbf{x}} \\ \circlearrowleft^{\mathbf{1}} \end{bmatrix} + \begin{bmatrix} \circlearrowleft^{\mathbf{1}} \\ \circlearrowleft^{\mathbf{x}} \end{bmatrix}$$

from which we immediately see that

$$f \left(\circlearrowleft^{\mathbf{1}} \right) = \circlearrowleft^{\mathbf{1}} \circlearrowleft^{\mathbf{x}}$$

We can also immediately calculate from this and from the definition of K_0 that

$$\begin{aligned} K_0 \left(d \left(\circlearrowleft^{\mathbf{1}} \right) \right) &= K_0 \left(\begin{bmatrix} \circlearrowleft^{\mathbf{x}} \\ \circlearrowleft^{\mathbf{1}} \end{bmatrix} \right) = \begin{bmatrix} \circlearrowleft^{\mathbf{1}} \\ \circlearrowleft^{\mathbf{x}} \end{bmatrix} + \left[\sum_{\substack{a \\ \circlearrowleft^x}} \circlearrowleft^{\mathbf{1}} \quad \circlearrowleft^{\mathbf{x}} \right] \\ d \left(K_0 \left(\circlearrowleft^{\mathbf{1}} \right) \right) &= d \left(\sum_{\substack{a \\ \circlearrowleft^x}} \circlearrowleft^{\mathbf{1}} \quad \circlearrowleft^{\mathbf{x}} \right) = \sum_{\substack{a \\ \circlearrowleft^x}} \circlearrowleft^{\mathbf{1}} \quad \circlearrowleft^{\mathbf{x}} \end{aligned}$$

⁵Since we are working over \mathbb{F}_2 , the commutator and anticommutator are the same.

⁶This last condition is another way of saying that K_0 is a map $\mathcal{C}_1 \rightarrow \mathcal{C}_x$. For if the basepoint circle is labelled \mathbf{x} , then the configuration does not lie in the domain of K_0 , so we declare that K_0 evaluates to 0. This formal extension is necessary because we must consider expressions of the form $K_0 \circ d$, and d may make the basepoint circle be labelled \mathbf{x} .

We see that the $\sum_{\textcircled{a}^x}$ terms will precisely cancel each other out, giving us

$$f\left(\textcircled{\bullet}^1\right) = \textcircled{\bullet}^1 = K_0\left(d\left(\textcircled{\bullet}^1\right)\right) + d\left(K_0\left(\textcircled{\bullet}^1\right)\right)$$

Next, we consider the other split map,

$$d\left(\textcircled{\bullet}^x\right) = \textcircled{\bullet}^x$$

Since the basepoint circle is labelled x , both f and K_0 evaluate to 0 on this configuration. Moreover, since $d\left(\textcircled{\bullet}^x\right)$ has the basepoint labelled x , we see that $K_0\left(d\left(\textcircled{\bullet}^x\right)\right) = 0$ as well. Thus, $f = [K_0, d]$ on this configuration too. In fact, as this example indicates, we can immediately dispense with all diagrams where the basepoint circle is labelled x , as both f and K_0 will evaluate to 0 on such diagrams.

Now we turn to merge maps, which are as straightforward. By the previous comment, the $\textcircled{\bullet}^x - \textcircled{\bullet}^x$ and $\textcircled{\bullet}^x - \textcircled{\bullet}^1$ merges need not be considered, since the basepoint circle is labelled x . Next, we recall that

$$d\left(\textcircled{\bullet}^1 - \textcircled{\bullet}^1\right) = \textcircled{\bullet}^1$$

and therefore

$$f\left(\textcircled{\bullet}^1 - \textcircled{\bullet}^1\right) = 0$$

We then calculate that

$$K_0\left(d\left(\textcircled{\bullet}^1 - \textcircled{\bullet}^1\right)\right) = K_0\left(\textcircled{\bullet}^1\right) = \sum_{\textcircled{a}^x} \textcircled{a}^1 \quad \textcircled{\bullet}^x$$

$$d\left(K_0\left(\textcircled{\bullet}^1 - \textcircled{\bullet}^1\right)\right) = d\left(\sum_{\textcircled{a}^x} \textcircled{a}^1 \quad \textcircled{\bullet}^x - \textcircled{\bullet}^1\right) = \sum_{\textcircled{a}^x} \textcircled{a}^1 \quad \textcircled{\bullet}^x$$

which shows that $f = [K_0, d]$ for this configuration.

Finally, we check the $\textcircled{\bullet}^1 - \textcircled{\bullet}^x$ merge. We have

$$d\left(\textcircled{\bullet}^1 - \textcircled{\bullet}^x\right) = \textcircled{\bullet}^x$$

and therefore

$$f\left(\textcircled{\bullet}^1 - \textcircled{\bullet}^x\right) = \textcircled{\bullet}^x$$

as well. We also see that

$$K_0\left(d\left(\textcircled{\bullet}^1 - \textcircled{\bullet}^x\right)\right) = K_0\left(\textcircled{\bullet}^x\right) = 0$$

$$d\left(K_0\left(\textcircled{\bullet}^1 - \textcircled{\bullet}^x\right)\right) = d\left(\left[\textcircled{\bullet}^x - \textcircled{\bullet}^1\right] + \sum_{\textcircled{a}^x} \textcircled{a}^1 \quad \textcircled{\bullet}^x - \textcircled{\bullet}^x\right) = \textcircled{\bullet}^x$$

Here, we can again see that $f = [K_0, d]$ for this configuration.

Now, we must check that $f = [K_0, d]$ also for 1-dimensional configurations that don't involve the basepoint circle. Note that on all of these configurations, f will be 0, since surgery along any edge not connected to the basepoint circle cannot turn the basepoint circle from a $\mathbf{1}$ to an \mathbf{x} . Therefore, we wish to prove that $[K_0, d] = 0$. Moreover, we can assume that the basepoint circle is labelled $\mathbf{1}$, for otherwise K_0 will automatically evaluate to 0. Starting from the splits, we have

$$K_0 \left(d \left(\bigoplus^{\mathbf{1}} \right) \right) = K_0 \left(\left[\begin{array}{c} \bigcirc^{\mathbf{x}} \\ \bigcirc^{\mathbf{1}} \end{array} \right] + \left[\begin{array}{c} \bigcirc^{\mathbf{1}} \\ \bigcirc^{\mathbf{x}} \end{array} \right] \right) = \left[\sum_{\bigcirc^{\mathbf{x}}} \bigcirc^{\mathbf{1}} \bigcirc^{\mathbf{x}} \bigcirc^{\bullet} \right] + \left[\sum_{\bigcirc^{\mathbf{x}}} \bigcirc^{\mathbf{1}} \bigcirc^{\mathbf{x}} \bigcirc^{\bullet} \right]$$

$$d \left(K_0 \left(\bigoplus^{\mathbf{1}} \right) \right) = d \left(\sum_{\bigcirc^{\mathbf{x}}} \bigcirc^{\mathbf{1}} \bigoplus^{\mathbf{1}} \bigcirc^{\mathbf{x}} \right) = \left[\sum_{\bigcirc^{\mathbf{x}}} \bigcirc^{\mathbf{1}} \bigcirc^{\mathbf{x}} \bigcirc^{\bullet} \right] + \left[\sum_{\bigcirc^{\mathbf{x}}} \bigcirc^{\mathbf{1}} \bigcirc^{\mathbf{x}} \bigcirc^{\bullet} \right]$$

Note that in calculating $K_0 \circ d$, we also get two terms in which both daughter circles are labelled $\mathbf{1}$, but these two terms cancel each other out since we are working over \mathbb{F}_2 .

Next, we consider the \mathbf{x} split:

$$K_0 \left(d \left(\bigoplus^{\mathbf{x}} \right) \right) = K_0 \left(\left[\begin{array}{c} \bigcirc^{\mathbf{x}} \\ \bigcirc^{\mathbf{x}} \end{array} \right] \right) = \left[\sum_{\bigcirc^{\mathbf{x}}} \bigcirc^{\mathbf{1}} \bigcirc^{\mathbf{x}} \bigcirc^{\bullet} \right] + \left[\begin{array}{c} \bigcirc^{\mathbf{x}} \\ \bigcirc^{\mathbf{1}} \end{array} \bigcirc^{\bullet} \right] + \left[\begin{array}{c} \bigcirc^{\mathbf{1}} \\ \bigcirc^{\mathbf{x}} \end{array} \bigcirc^{\bullet} \right]$$

$$d \left(K_0 \left(\bigoplus^{\mathbf{x}} \right) \right) = d \left(\left[\sum_{\bigcirc^{\mathbf{x}}} \bigcirc^{\mathbf{1}} \bigoplus^{\mathbf{x}} \bigcirc^{\bullet} \right] + \left[\bigoplus^{\mathbf{1}} \bigcirc^{\bullet} \right] \right)$$

$$= \left[\sum_{\bigcirc^{\mathbf{x}}} \bigcirc^{\mathbf{1}} \bigcirc^{\mathbf{x}} \bigcirc^{\bullet} \right] + \left[\begin{array}{c} \bigcirc^{\mathbf{x}} \\ \bigcirc^{\mathbf{1}} \end{array} \bigcirc^{\bullet} \right] + \left[\begin{array}{c} \bigcirc^{\mathbf{1}} \\ \bigcirc^{\mathbf{x}} \end{array} \bigcirc^{\bullet} \right]$$

and we see that these are indeed equal.

Next, we turn to the merges. We have

$$K_0 \left(d \left(\bigcirc^{\mathbf{1}} \text{---} \bigcirc^{\mathbf{1}} \right) \right) = K_0 \left(\bigcirc^{\mathbf{1}} \right) = \sum_{\bigcirc^{\mathbf{x}}} \bigcirc^{\mathbf{1}} \bigcirc^{\mathbf{1}} \bigcirc^{\bullet}$$

$$d \left(K_0 \left(\bigcirc^{\mathbf{1}} \text{---} \bigcirc^{\mathbf{1}} \right) \right) = d \left(\sum_{\bigcirc^{\mathbf{x}}} \bigcirc^{\mathbf{1}} \bigcirc^{\mathbf{1}} \text{---} \bigcirc^{\mathbf{1}} \bigcirc^{\bullet} \right) = \sum_{\bigcirc^{\mathbf{x}}} \bigcirc^{\mathbf{1}} \bigcirc^{\mathbf{1}} \bigcirc^{\bullet}$$

$$K_0 \left(d \left(\bigcirc^{\mathbf{x}} \text{---} \bigcirc^{\mathbf{1}} \right) \right) = K_0 \left(\bigcirc^{\mathbf{x}} \right) = \left[\sum_{\bigcirc^{\mathbf{x}}} \bigcirc^{\mathbf{1}} \bigcirc^{\mathbf{x}} \bigcirc^{\bullet} \right] + \left[\bigcirc^{\mathbf{1}} \bigcirc^{\mathbf{x}} \right]$$

$$\begin{aligned}
d\left(K_0\left(\bigcirc^x - \bigcirc^1\right)\right) &= d\left(\left[\sum_{\bigcirc^x} \bigcirc^1 \bigcirc^x - \bigcirc^1 \bigcirc^x \right] + \left[\bigcirc^1 - \bigcirc^1 \bigcirc^x\right]\right) \\
&= \left[\sum_{\bigcirc^x} \bigcirc^1 \bigcirc^x \bigcirc^x\right] + \left[\bigcirc^1 \bigcirc^x\right]
\end{aligned}$$

$$K_0\left(d\left(\bigcirc^x - \bigcirc^x\right)\right) = K_0(0) = 0$$

$$\begin{aligned}
d\left(K_0\left(\bigcirc^x - \bigcirc^x\right)\right) &= d\left(\left[\sum_{\bigcirc^x} \bigcirc^1 \bigcirc^x - \bigcirc^x \bigcirc^x\right] + \left[\bigcirc^x - \bigcirc^1 \bigcirc^x\right] + \left[\bigcirc^1 - \bigcirc^x \bigcirc^x\right]\right) \\
&= 0
\end{aligned}$$

Thus, since each pair sums to 0, we see that $f = [K_0, d]$.

The Bar-Natan Case: The Higher K_i 's

In order to see that the Bar-Natan theory splits as well, we will construct the homotopies K_i for all $i > 0$. Recall that K_i must preserve homological grading but raise quantum grading by $2i$, so there is a natural guess for a definition of K_i :

$$K_i(D) := \sum_{\bigcirc^x \cdots \bigcirc^x} \bigcirc^1 \cdots \bigcirc^1 \bigcirc^x$$

This sum runs over all $(i+1)$ -tuples of circles labelled \mathbf{x} . As usual, if the basepoint circle is labelled \mathbf{x} in D or if this sum is empty, then $K_i(D) = 0$. Note that when $i = 0$, this actually agrees with our above definition of K_0 . Since we intend to define $K = K_0 + H K_1 + H^2 K_2 + \cdots$ and we intend to have $f = [d + Hh, K]$, we can see that it is sufficient to have

$$[K_i, h] + [K_{i+1}, d] = 0$$

satisfied for all $i > 0$. We will begin by proving that this holds for all 1-dimensional configurations involving the basepoint.

We can immediately see that $[K_i, h]$ has only two nonzero terms involving the basepoint, namely:

$$\begin{aligned}
h\left(K_i\left(\bigcirc^1 - \bigcirc^x\right)\right) &= h\left(\left[\sum_{\bigcirc^x \cdots \bigcirc^x} \bigcirc^1 \cdots \bigcirc^1 \bigcirc^x - \bigcirc^1\right] + \right. \\
&\quad \left. + \left[\sum_{\bigcirc^x \cdots \bigcirc^x} \bigcirc^1 \cdots \bigcirc^1 \bigcirc^x - \bigcirc^x - \bigcirc^x\right]\right) \\
&= \sum_{\bigcirc^x \cdots \bigcirc^x} \bigcirc^1 \cdots \bigcirc^1 \bigcirc^x \tag{1}
\end{aligned}$$

$$K_i \left(h \left(\begin{array}{c} \textcircled{1} \\ \bullet \\ \textcircled{1} \end{array} \right) \right) = K_i \left(\begin{array}{c} \textcircled{1} \\ \bullet \\ \textcircled{1} \end{array} \right) = \sum_{\textcircled{a_1}^x \dots \textcircled{a_{i+1}}^x} \textcircled{a_1}^1 \dots \textcircled{a_{i+1}}^1 \begin{array}{c} \textcircled{1} \\ \bullet \\ \textcircled{x} \end{array} \quad (2)$$

We need to check that $[K_{i+1}, d]$ has the same nonzero terms containing the basepoint, and we'll do this by first checking that $[K_{i+1}, d]$ agrees with $[K_i, h]$ on these configurations, and then that it has no other nonzero basepoint terms. First, we note that $K_{i+1} \left(d \left(\begin{array}{c} \textcircled{1} \\ \bullet \\ \textcircled{x} \end{array} \right) \right) = 0$, since $d \left(\begin{array}{c} \textcircled{1} \\ \bullet \\ \textcircled{x} \end{array} \right)$ has the basepoint circle labelled x . On the other hand,

$$\begin{aligned} d \left(K_{i+1} \left(\begin{array}{c} \textcircled{1} \\ \bullet \\ \textcircled{x} \end{array} \right) \right) &= d \left(\left[\sum_{\textcircled{a_1}^x \dots \textcircled{a_{i+1}}^x} \textcircled{a_1}^1 \dots \textcircled{a_{i+1}}^1 \begin{array}{c} \textcircled{x} \\ \bullet \\ \textcircled{1} \end{array} \right] + \right. \\ &\quad \left. + \left[\sum_{\textcircled{a_1}^x \dots \textcircled{a_{i+2}}^x} \textcircled{a_1}^1 \dots \textcircled{a_{i+2}}^1 \begin{array}{c} \textcircled{x} \\ \bullet \\ \textcircled{x} \end{array} \right] \right) \\ &= \sum_{\textcircled{a_1}^x \dots \textcircled{a_{i+1}}^x} \textcircled{a_1}^1 \dots \textcircled{a_{i+1}}^1 \begin{array}{c} \textcircled{x} \\ \bullet \\ \textcircled{1} \end{array} \end{aligned}$$

which agrees with (1).

For the split, we have

$$\begin{aligned} K_{i+1} \left(d \left(\begin{array}{c} \textcircled{1} \\ \bullet \\ \textcircled{1} \end{array} \right) \right) &= K_{i+1} \left(\left[\begin{array}{c} \textcircled{1} \\ \bullet \\ \textcircled{x} \end{array} \right] + \left[\begin{array}{c} \textcircled{x} \\ \bullet \\ \textcircled{1} \end{array} \right] \right) = K_{i+1} \left(\begin{array}{c} \textcircled{x} \\ \bullet \\ \textcircled{1} \end{array} \right) \\ &= \left[\sum_{\textcircled{a_1}^x \dots \textcircled{a_{i+1}}^x} \textcircled{a_1}^1 \dots \textcircled{a_{i+1}}^1 \begin{array}{c} \textcircled{1} \\ \bullet \\ \textcircled{x} \end{array} \right] + \left[\sum_{\textcircled{a_1}^x \dots \textcircled{a_{i+2}}^x} \textcircled{a_1}^1 \dots \textcircled{a_{i+2}}^1 \begin{array}{c} \textcircled{x} \\ \bullet \\ \textcircled{x} \end{array} \right] \end{aligned}$$

$$d \left(K_{i+1} \left(\begin{array}{c} \textcircled{1} \\ \bullet \\ \textcircled{1} \end{array} \right) \right) = d \left(\sum_{\textcircled{a_1}^x \dots \textcircled{a_{i+2}}^x} \textcircled{a_1}^1 \dots \textcircled{a_{i+2}}^1 \begin{array}{c} \textcircled{x} \\ \bullet \\ \textcircled{1} \end{array} \right) = \sum_{\textcircled{a_1}^x \dots \textcircled{a_{i+2}}^x} \textcircled{a_1}^1 \dots \textcircled{a_{i+2}}^1 \begin{array}{c} \textcircled{x} \\ \bullet \\ \textcircled{x} \end{array}$$

And indeed, their sum precisely equals (2).

Now, we check the other configurations involving the basepoint. As before, we need not consider diagrams where the basepoint circle is labelled x , as both f and K_{i+1} will vanish on such configurations. All that remains, therefore, is the $\mathbf{1}, \mathbf{1}$ merge. For this, we have

$$K_{i+1} \left(d \left(\begin{array}{c} \textcircled{1} \\ \bullet \\ \textcircled{1} \end{array} \right) \right) = K_{i+1} \left(\begin{array}{c} \textcircled{1} \\ \bullet \\ \textcircled{1} \end{array} \right) = \sum_{\textcircled{a_1}^x \dots \textcircled{a_{i+2}}^x} \textcircled{a_1}^1 \dots \textcircled{a_{i+2}}^1 \begin{array}{c} \textcircled{x} \\ \bullet \\ \textcircled{1} \end{array}$$

$$\begin{aligned}
d\left(K_{i+1}\left(\begin{array}{c} \textcircled{1} \\ \bullet \text{---} \textcircled{1} \end{array}\right)\right) &= d\left(\sum_{\textcircled{a_1}^x \dots \textcircled{a_{i+2}}^x} \textcircled{a_1}^1 \dots \textcircled{a_{i+2}}^1 \begin{array}{c} \textcircled{x} \\ \bullet \text{---} \textcircled{1} \end{array}\right) \\
&= \sum_{\textcircled{a_1}^x \dots \textcircled{a_{i+2}}^x} \textcircled{a_1}^1 \dots \textcircled{a_{i+2}}^1 \begin{array}{c} \textcircled{x} \\ \bullet \end{array}
\end{aligned}$$

So we indeed see that $[K_{i+1}, d]$ has no other nonzero terms involving the basepoint.

Next, we need to check that $[K_i, h] = [K_{i+1}, d]$ on all 1-dimensional configurations not involving the basepoint. We begin by calculating $[K_i, h]$ on the splits:

$$K_i\left(h\left(\begin{array}{c} \textcircled{1} \\ \ominus \end{array}\right)\right) = K_i\left(\begin{array}{c} \textcircled{1} \\ \textcircled{1} \end{array}\right) = \sum_{\textcircled{a_1}^x \dots \textcircled{a_{i+1}}^x} \textcircled{a_1}^1 \dots \textcircled{a_{i+1}}^1 \begin{array}{c} \textcircled{1} \\ \textcircled{1} \\ \bullet \end{array} \textcircled{x}$$

$$\begin{aligned}
h\left(K_i\left(\begin{array}{c} \textcircled{1} \\ \ominus \end{array}\right)\right) &= h\left(\sum_{\textcircled{a_1}^x \dots \textcircled{a_{i+1}}^x} \textcircled{a_1}^1 \dots \textcircled{a_{i+1}}^1 \begin{array}{c} \textcircled{1} \\ \ominus \\ \bullet \end{array} \textcircled{x}\right) \\
&= \sum_{\textcircled{a_1}^x \dots \textcircled{a_{i+1}}^x} \textcircled{a_1}^1 \dots \textcircled{a_{i+1}}^1 \begin{array}{c} \textcircled{1} \\ \textcircled{1} \\ \bullet \end{array} \textcircled{x}
\end{aligned}$$

$$K_i\left(h\left(\begin{array}{c} \textcircled{x} \\ \ominus \end{array}\right)\right) = K_i(0) = 0$$

$$\begin{aligned}
h\left(K_i\left(\begin{array}{c} \textcircled{x} \\ \ominus \end{array}\right)\right) &= h\left(\left[\sum_{\textcircled{a_1}^x \dots \textcircled{a_{i+1}}^x} \textcircled{a_1}^1 \dots \textcircled{a_{i+1}}^1 \begin{array}{c} \textcircled{x} \\ \ominus \\ \bullet \end{array} \textcircled{x}\right] + \right. \\
&\quad \left. + \left[\sum_{\textcircled{a_1}^x \dots \textcircled{a_i}^x} \textcircled{a_1}^1 \dots \textcircled{a_i}^1 \begin{array}{c} \textcircled{1} \\ \ominus \\ \bullet \end{array} \textcircled{x}\right]\right) \\
&= \sum_{\textcircled{a_1}^x \dots \textcircled{a_i}^x} \textcircled{a_1}^1 \dots \textcircled{a_i}^1 \begin{array}{c} \textcircled{1} \\ \textcircled{1} \\ \bullet \end{array} \textcircled{x} \tag{3}
\end{aligned}$$

Thus, the only nonzero term we need to worry about here is (3), which will be precisely cancelled soon.

Next, we do the same for $[K_{i+1}, d]$:

$$\begin{aligned}
K_{i+1} \left(d \left(\bigoplus^1 \right) \right) &= K_{i+1} \left(\left[\begin{array}{c} \bigcirc^1 \\ \bigcirc^x \end{array} \right] + \left[\begin{array}{c} \bigcirc^x \\ \bigcirc^1 \end{array} \right] \right) \\
&= \left(\left[\sum_{\bigcirc^x a_1 \dots \bigcirc^x a_{i+2}} \bigcirc^1 a_1 \dots \bigcirc^1 a_{i+2} \begin{array}{c} \bigcirc^1 \\ \bigcirc^x \end{array} \bullet^x \right] \right. \\
&\quad \left. + \left[\sum_{\bigcirc^x a_1 \dots \bigcirc^x a_{i+1}} \bigcirc^1 a_1 \dots \bigcirc^1 a_{i+1} \begin{array}{c} \bigcirc^1 \\ \bigcirc^1 \end{array} \bullet^x \right] \right) \\
&\quad + \left(\left[\sum_{\bigcirc^x a_1 \dots \bigcirc^x a_{i+2}} \bigcirc^1 a_1 \dots \bigcirc^1 a_{i+2} \begin{array}{c} \bigcirc^x \\ \bigcirc^1 \end{array} \bullet^x \right] \right. \\
&\quad \left. + \left[\sum_{\bigcirc^x a_1 \dots \bigcirc^x a_{i+1}} \bigcirc^1 a_1 \dots \bigcirc^1 a_{i+1} \begin{array}{c} \bigcirc^1 \\ \bigcirc^1 \end{array} \bullet^x \right] \right) \\
&= \sum_{\bigcirc^x a_1 \dots \bigcirc^x a_{i+2}} \bigcirc^1 a_1 \dots \bigcirc^1 a_{i+2} \left(\left[\begin{array}{c} \bigcirc^1 \\ \bigcirc^x \end{array} \right] + \left[\begin{array}{c} \bigcirc^x \\ \bigcirc^1 \end{array} \right] \right) \bullet^x
\end{aligned}$$

$$\begin{aligned}
d \left(K_{i+1} \left(\bigoplus^1 \right) \right) &= d \left(\sum_{\bigcirc^x a_1 \dots \bigcirc^x a_{i+2}} \bigcirc^1 a_1 \dots \bigcirc^1 a_{i+2} \bigoplus^1 \bullet^x \right) \\
&= \sum_{\bigcirc^x a_1 \dots \bigcirc^x a_{i+2}} \bigcirc^1 a_1 \dots \bigcirc^1 a_{i+2} \left(\left[\begin{array}{c} \bigcirc^1 \\ \bigcirc^x \end{array} \right] + \left[\begin{array}{c} \bigcirc^x \\ \bigcirc^1 \end{array} \right] \right) \bullet^x
\end{aligned}$$

$$\begin{aligned}
K_{i+1} \left(d \left(\bigoplus^x \right) \right) &= K_{i+1} \left(\begin{array}{c} \bigcirc^x \\ \bigcirc^x \end{array} \right) \\
&= \left[\sum_{\bigcirc^x a_1 \dots \bigcirc^x a_{i+2}} \bigcirc^1 a_1 \dots \bigcirc^1 a_{i+2} \begin{array}{c} \bigcirc^x \\ \bigcirc^x \end{array} \bullet^x \right] \\
&\quad + \left[\sum_{\bigcirc^x a_1 \dots \bigcirc^x a_{i+1}} \bigcirc^1 a_1 \dots \bigcirc^1 a_{i+1} \left(\left[\begin{array}{c} \bigcirc^1 \\ \bigcirc^x \end{array} \right] + \left[\begin{array}{c} \bigcirc^x \\ \bigcirc^1 \end{array} \right] \right) \bullet^x \right] \\
&\quad + \left[\sum_{\bigcirc^x a_1 \dots \bigcirc^x a_i} \bigcirc^1 a_1 \dots \bigcirc^1 a_i \begin{array}{c} \bigcirc^1 \\ \bigcirc^1 \end{array} \bullet^x \right]
\end{aligned}$$

$$\begin{aligned}
d\left(K_{i+1}\left(\bigoplus^x\right)\right) &= d\left(\left[\sum_{\substack{a_1^x \dots a_{i+2}^x}} \bigoplus^1 \dots \bigoplus^1 \bigoplus^x \bigoplus^x \right] \right. \\
&\quad \left. + \left[\sum_{\substack{a_1^x \dots a_{i+1}^x}} \bigoplus^1 \dots \bigoplus^1 \bigoplus^1 \bigoplus^x \right]\right) \\
&= \left[\sum_{\substack{a_1^x \dots a_{i+2}^x}} \bigoplus^1 \dots \bigoplus^1 \bigoplus^x \bigoplus^x \right] \\
&\quad + \left[\sum_{\substack{a_1^x \dots a_{i+1}^x}} \bigoplus^1 \dots \bigoplus^1 \left(\left[\begin{smallmatrix} \bigoplus^1 \\ \bigoplus^x \end{smallmatrix}\right] + \left[\begin{smallmatrix} \bigoplus^x \\ \bigoplus^1 \end{smallmatrix}\right]\right) \bigoplus^x \right]
\end{aligned}$$

Therefore, the only term that remains from the splits is

$$[K_{i+1}, d]\left(\bigoplus^x\right) = \sum_{\substack{a_1^x \dots a_i^x}} \bigoplus^1 \dots \bigoplus^1 \bigoplus^1 \bigoplus^x$$

which precisely equals (3).

The computations for the merges proceed analogously, so we omit the explicit calculations. However, they work out as well, and we see that

$$[K_i, h] = [K_{i+1}, d]$$

on any configuration. This means that if we define $K = K_0 + HK_1 + H^2K_2 + \dots$, then we must have $f = [d + Hh, K]$, which means that the Bar-Natan theory splits as the direct sum of its two reduced versions.

4 Isomorphism

So far, we have proved that the Bar-Natan homology splits as a direct sum $H(\mathcal{C}_1) \oplus H(\mathcal{C}_x)$ of the two reduced theories. Moreover, we can say something stronger:

Theorem 4. *As graded chain complexes over $\mathbb{F}_2[H]$, $\mathcal{C}_1 \cong \mathcal{C}_x$. Therefore, the Bar-Natan homology is the direct sum of two copies of the same (that is, isomorphic) reduced theories.*

Proof. We wish to construct a graded $\mathbb{F}_2[H]$ -module isomorphism of chain complexes between \mathcal{C}_1 and \mathcal{C}_x . To do this, we first recall that in the ordinary Khovanov case of [11], this isomorphism is given by the map $I : \mathcal{C}_1 \rightarrow \mathcal{C}_x$ that simply relabels the basepoint circle from $\mathbf{1}$ to \mathbf{x} . This is clearly an isomorphism (as it maps the basis vectors of \mathcal{C}_1 to the basis vectors of \mathcal{C}_x). Moreover, it is straightforward to check that I is a chain map for the Khovanov differential, meaning that $[I, d] = 0$.

For the Bar-Natan case, we define $\iota : \mathcal{C}_1 \rightarrow \mathcal{C}_x$ by

$$\iota := I + HK = I + HK_0 + H^2K_1 + H^3K_2 + \dots$$

First, we show that ι is a chain map, which means that it commutes with the differential. So we want to show that $[\iota, d + Hh] = 0$. However, we have that

$$\begin{aligned}
[\iota, d + Hh] &= [I + HK, d + Hh] \\
&= [I, d + Hh] + [HK, d + Hh] \\
&= [I, d] + [I, Hh] + Hf
\end{aligned}$$

where the last step uses what we proved above, namely that $[K, d + Hh] = f$. Since $[I, d] = 0$, it suffices to prove that

$$f = [I, h]$$

We proceed as in Section 3, again leaving out all cases where the basepoint circle is labelled x , as both f and I will vanish on these. For the split map, we have

$$f \left(\bigcirc^1 \right) = \bigcirc^1_x$$

and

$$\begin{aligned} I \left(h \left(\bigcirc^1 \right) \right) &= I \left(\bigcirc^1_x \right) = \bigcirc^1_x \\ h \left(I \left(\bigcirc^1 \right) \right) &= h \left(\bigcirc^x \right) = 0 \end{aligned}$$

so $f = [I, h]$ for the splits. For the merges, we have

$$\begin{aligned} f \left(\bigcirc^1 \text{---} \bigcirc^1 \right) &= 0 \\ f \left(\bigcirc^1 \text{---} \bigcirc^x \right) &= \bigcirc^x \end{aligned}$$

For both of these, h will evaluate to 0. So it suffices to check that $h \circ I$ agrees with f on them. Indeed,

$$\begin{aligned} h \left(I \left(\bigcirc^1 \text{---} \bigcirc^1 \right) \right) &= h \left(\bigcirc^x \text{---} \bigcirc^1 \right) = 0 \\ h \left(I \left(\bigcirc^1 \text{---} \bigcirc^x \right) \right) &= h \left(\bigcirc^x \text{---} \bigcirc^x \right) = \bigcirc^x \end{aligned}$$

Thus, we see that $f = [I, h]$. Note that unlike in the previous sections, we don't need to check that this equality holds on all surgeries not involving the basepoint circle, since f , I , and h are all defined locally. So we see that the identity $f = [I, h]$ holds always, which means that ι is indeed a chain map.

Now, we must check that ι is an isomorphism of chain complexes. We first prove that ι is injective. For suppose that we had some homogeneous $\alpha \in \ker \iota$. That means that each homogeneous component of $\iota(\alpha)$ is 0, so in particular $I(\alpha) = 0$. But I is injective, so we must have that $\alpha = 0$. Therefore, ι is injective as well.

Proving directly that ι is surjective is significantly more tedious, since ι is defined in terms of various sums. Luckily, we don't need to do this. For we note that since H has grading -2 , we must have that ι is a homogeneous map that always decreases quantum grading by 2. Moreover, at each fixed quantum grading, \mathcal{C}_1 and \mathcal{C}_x are finite-dimensional \mathbb{F}_2 -vector spaces of the same dimension, since there is a bijection between their bases. So at each fixed quantum grading, ι is an injective homomorphism between two vector spaces of the same dimension, so it must be surjective as well. Since this is true for all quantum gradings, ι itself must be surjective. So ι is bijective, and as it was an $\mathbb{F}_2[H]$ -module chain map, its inverse will be an $\mathbb{F}_2[H]$ -module chain map as well. So ι is a graded $\mathbb{F}_2[H]$ -module isomorphism of chain complexes, so $\mathcal{C}_1 \cong \mathcal{C}_x$. \square

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