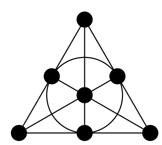
1. Find an *infinite* set of points in the plane (not all collinear) such that every line passing through two of them also passes through a third.

   This shows that the Sylvester–Gallai theorem is false without the requirement that we only use finitely many points.

2. An *incidence structure* consists of a set P (which we call points), and a set L (which we call lines), such that each line is a subset of P. An incidence structure is called an *abstract geometry* if for every pair of distinct points $p_1, p_2 \in P$, there exists a *unique* line $\ell \in L$ such that $p_1 \in \ell$ and $p_2 \in \ell$.

   (a) Persuade yourself that our usual Euclidean notions of points and lines fit into this abstract framework of geometry.

   (b) However, abstract geometries can be pretty wacky! Verify that the following picture defines an abstract geometry, called the *Fano plane*.

   Here, each dot represents a point, and each segment (as well as the circle) represents a line.

   (c) Is the Sylvester–Gallai theorem true in the Fano plane? What about the de Bruijn–Erdős theorem?

   (d) Play around with abstract geometries! Try to construct some weird ones, and see which facts from Euclidean geometry remain true and which ones do not.

3. In this problem, we'll explore things that aren't lines.

   (a) Find a set of points and unit circles in the plane such that every unit circle passes through at least three of the points, but the points do not all lie on a single unit circle. This shows that in some sense, the "circular Sylvester–Gallai" theorem is not true.

   (b) Let's define a *standard parabola* to be the graph of $y = (x - a)^2 + b$, for some $a, b \in \mathbb{R}$; in other words, a standard parabola is a translated copy of the parabola $y = x^2$, translated so its vertex is at the point (a, b).

   Prove that every pair of points in the plane lie on a unique standard parabola, unless they have the same x coordinate.

   (c) Prove that any set of n points in the plane with distinct x coordinates, not all lying on a single standard parabola, define at least n standard parabolas.

   Hint: Recall the notion of abstract geometry from Exercise 2!

   ? (d) Does the "parabolic Sylvester–Gallai" theorem hold? That is, are there n points in the plane with distinct x coordinates, not all lying on a single standard parabola, such that no standard parabola passes through exactly two of them?

$\star$4. In this problem, you'll reproduce the original proof of de Bruijn and Erdős of the de Bruijn–Erdős theorem.

(a) Let the points be $p_1, \dots, p_n$ and the lines $\ell_1, \dots, \ell_m$. Let a_i denote the number of lines containing p_i, and let b_j denote the number of points on the line ℓ_j. Prove that

$$\sum_{i=1}^{n} a_i = \sum_{j=1}^{m} b_j.$$

Hint: Both of these expressions are counting the same thing; what is it?

(b) Prove that if p_i does not lie on the line ℓ_j, then $a_i \geq b_j$.

(c) Assume (without loss of generality) that a_n is the minimum of $a_1, \dots, a_n$, and let $x = a_n$. Prove that $x \geq 2$.

(d) By relabeling the lines, assume that the lines through p_n are $\ell_1, \dots, \ell_x$. For every such line ℓ_j, there is some other point on it, say p_j. Prove that $p_1, \dots, p_x$ are all distinct points.

(e) Conclude from (b) and (d) that

$$a_1 \geq b_2, \qquad a_2 \geq b_3, \qquad \dots \qquad a_{x-1} \geq b_x, \qquad a_x \geq b_1.$$

(f) Prove that $a_i \geq a_n$ for every $i > x$, and that $a_n \geq b_j$ for every $j > x$.

(g) Add up parts (e) and (f) to conclude that if $m < n$, we have

$$\sum_{i=1}^{n} a_i > \sum_{j=1}^{m} b_j.$$

Conclude that $m \geq n$, proving the de Bruijn–Erdős theorem.

(h) Check that you never used any properties of the Euclidean plane, and thus that this proof works for any abstract geometry, in the sense of Exercise 2.

1. Recall that the de Bruijn–Erdős theorem says that every set of $n \geq 3$ non-collinear points in the plane defines at least n lines.

   (a) For every $n \geq 3$, find a set of n non-collinear points in the plane that defines exactly n lines. This shows that the bound in the de Bruijn–Erdős theorem is best possible.

   $\star$ (b) Prove that the construction you found in part (a) is unique, i.e. that any other non-isomorphic set of n points in the plane defines strictly more than n lines.

2. Prove the Sylvester–Gallai theorem in higher dimensions, namely that any n non-collinear points in $\mathbb{R}^d$ define a line passing through exactly two of them, for any $d \geq 2$. Try to prove this in two ways—first by following the proof of the Sylvester–Gallai theorem we saw in class, and second by using the random projection argument we used to prove the higher-dimensional de Bruijn–Erdős theorem.

3. Recall that for a set of points P in $\mathbb{R}^d$, we define $F_i(P)$ to be the number of i-dimensional hyperplanes defined by P.

   (a) Pick your favorite polyhedron in $\mathbb{R}^3$, and let P be its set of vertices. Compute $F_0(P), F_1(P)$, and $F_2(P)$.

   Note that this amounts to more than just counting the edges and faces of the polyhedron! Some of the lines and planes defined by P will not be edges or faces, since they'll go "through" the polyhedron.

   (b) The set of vertices of the *d-dimensional hypercube* consists of the 2^d points in $\mathbb{R}^d$ whose coordinates are 0 or 1. Persuade yourself that the 2-dimensional hypercube is just a square, and that the 3-dimensional hypercube is a cube.

   $\star$ (c) Compute $F_0(P), F_1(P), F_2(P), F_3(P)$, where P is the set of vertices of the 4-dimensional hypercube.

   ? (d) Can you find a formula for $F_i(P)$, where P is the set of vertices d-dimensional hypercube and $0 \leq i \leq d-1$?

   (e) Check that for all the examples above, Rota's conjecture and the Dowling–Wilson conjecture hold.

4. *If you don't know what fields are, you may wish to skip this problem.*

   Given any field $\mathbb{F}$, we may define the *plane* over $\mathbb{F}$ to consist of all ordered pairs of elements of $\mathbb{F}$. Then a *line* over $\mathbb{F}$ is the set of points (x, y) in the plane such that $ax + by = c$, for some fixed $a, b, c \in \mathbb{F}$.

   (a) Prove that for any field $\mathbb{F}$, this notion of points and lines satisfies the property that every two points define a unique line. Thus, this "plane" is an abstract geometry in the sense of exercise 2 from Homework #1.

   (b) Prove that if $\mathbb{F}$ is a finite field, then the Sylvester–Gallai theorem is false in the plane over $\mathbb{F}$.

(c) Prove that the Sylvester–Gallai theorem is true over the rational numbers, i.e. when $\mathbb{F} = \mathbb{Q}$.

$\star\star$ (d) Prove that the Sylvester–Gallai theorem is false over the complex numbers, i.e. when $\mathbb{F} = \mathbb{C}$.

Hint: There is a set of nine points in the plane over $\mathbb{C}$ spanning twelve lines, each containing exactly three of the points. However, the only good way I know to come up with these points involves the theory of elliptic curves, a fascinating area of mathematics at the intersection of algebra, geometry, and number theory. Please try to find such a set—I'd be very interested if you succeed! But if you don't succeed, one such set of points is given below in white text which you can highlight to make visible; feel free to check that it works!

This is often called the Hesse configuration.

1. Consider the coordinate axes in $\mathbb{R}^n$, namely the n lies passing through the origin and the points $(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$, respectively. Additionally, consider the "diagonal" line passing through the origin and the point $(1, 1, \ldots, 1)$.

   Prove that if n is sufficiently large, then these $n + 1$ lines are almost orthogonal.

   In some sense, this explains "why" there can be many almost orthogonal lines in high dimensions—even our standard set of n orthogonal lines can be extended with at least one more almost orthogonal line!

⋆2. Find three random events A, B, C such that A and B are independent, B and C are independent, A and C are independent, but the three events A, B, C are *not* mutually independent.

   This sort of thing is the reason why we have to be careful with the definition of mutual independence: sometimes, the dependencies between events can be pretty tricky to find!

   Hint: The randomness can be two fair coin flips, and the events A, B, C can all have probability $\frac{1}{2}$.

3. Prove that for every $\varepsilon > 0$, there exists some $\delta > 0$ such that the following holds for all n. There are $\lfloor (1 + \delta)^n \rfloor$ lines in $\mathbb{R}^n$, such that each pair forms an angle between $(90 - \varepsilon)^\circ$ and $(90 + \varepsilon)^\circ$. This shows that there was nothing special in our choice of 89° and 91° in the definition of almost orthogonal lines.

⋆4. In this problem, you'll prove the Chernoff bound.

   (a) For a random variable X, let $\mathbb{E}[X]$ denote the *expectation* (or *average*) of X, which is defined by

   $$\mathbb{E}[X] = \sum_x x \Pr(X = x).$$

   Prove that if X and Y are independent random variables, then

   $$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y].$$

   (b) Prove *Markov's inequality*, which states that if Z is a non-negative random variable (i.e. all the values that Z can take are non-negative), then

   $$\Pr(Z > a) < \frac{\mathbb{E}[Z]}{a}$$

   for every $a > 0$.

   ⋆(c) Now let X take on values ± 1 with probability $\frac{1}{2}$. Prove that for any $\lambda \geq 0$,

   $$\mathbb{E}[e^{\lambda X}] \leq e^{\lambda^2/2}.$$

   Hint: First find an expression for $\mathbb{E}[e^{\lambda X}]$ as a function of λ. Then compare the Taylor series for this function and for the function $e^{x^2/2}$.

(d) Now let $X_1, \dots, X_n$ be independent, identically distributed random variables all taking the values ± 1 with probability $\frac{1}{2}$. Prove that

$$\mathbb{E}[e^{\lambda(X_1+\cdots+X_n)}] \leq e^{\lambda^2 n/2}.$$

for any $\lambda \geq 0$.

(e) Combine parts (b) and (d) to show that

$$\Pr(X_1 + \cdots + X_n > a) < e^{\lambda^2 n/2 - \lambda a}$$

for any $a > 0$ and any $\lambda \geq 0$.

(f) Plug in $\lambda = a/n$ to conclude the first part of the Chernoff bound.

(g) Prove the second part of the Chernoff bound.