Blowups of triangle-free graphs

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Joint with António Girão and Zach Hunter

In memory of Gábor Simonyi

Gábor Simonyi (1963–2025)

The Kővári–Sós–Turán theorem

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Theorem (Kővári–Sós–Turán '54)

Let G be an N-vertex graph with at least εN^2 edges. Then G contains a copy of $K_{k,k}$, where

$$k \geqslant c \frac{\log N}{\log \frac{1}{\varepsilon}}$$

and $c > 0$ is an absolute constant.

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This holds for all ε and all N. For example, taking $\varepsilon = N^{-c/k}$ shows that every graph with $N^{2-c/k}$ edges contains $K_{k,k}$.

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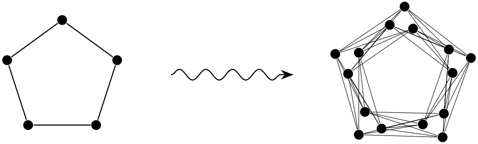

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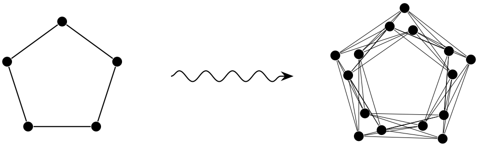

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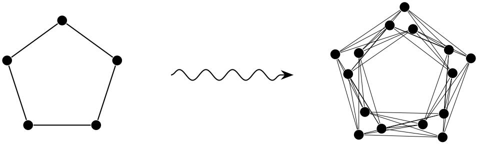

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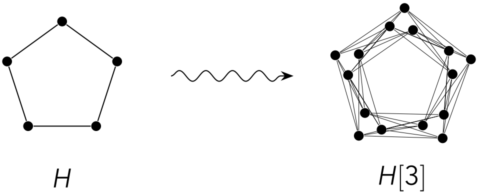

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The triangles in a graph are not a "generic" 3-uniform hypergraph!

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Fix $\varepsilon > 0$ and an h-vertex graph H. If N is sufficiently large, and G is an N-vertex graph with at least εN^h copies of H, then G contains a copy of $H[k]$, where

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Theorem (Nikiforov '08; Rödl–Schacht '12; Fox–Luo–W '21)

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Theorem (Girão–Hunter–W '24+)

The conjecture is true if H is triangle-free.

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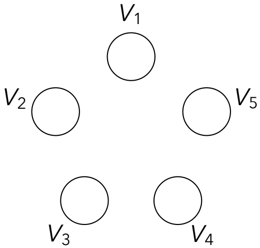

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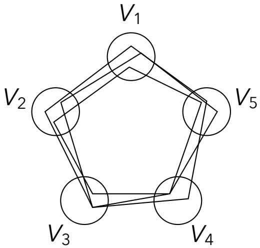

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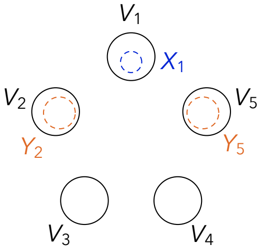

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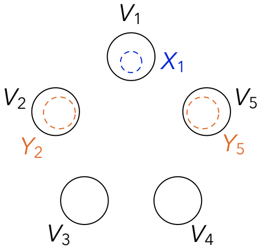

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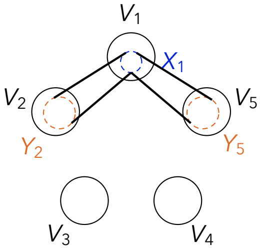

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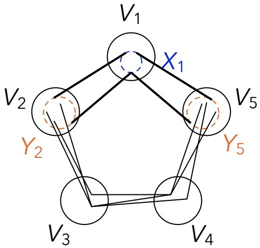
- There are $\varepsilon^C|Y_2||V_3||V_4||Y_5|$ canonical P_4



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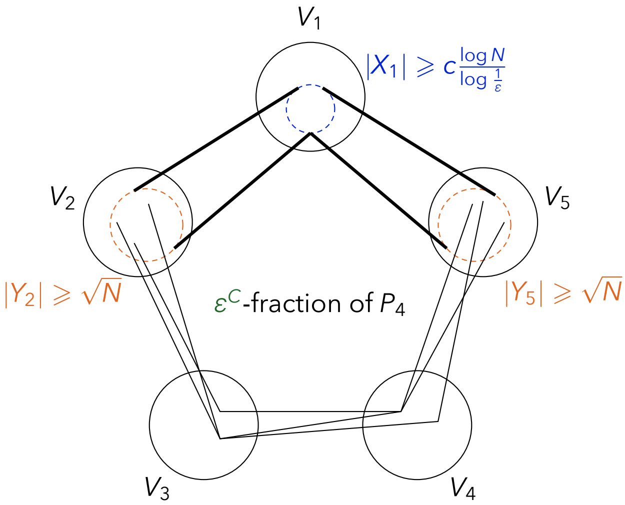

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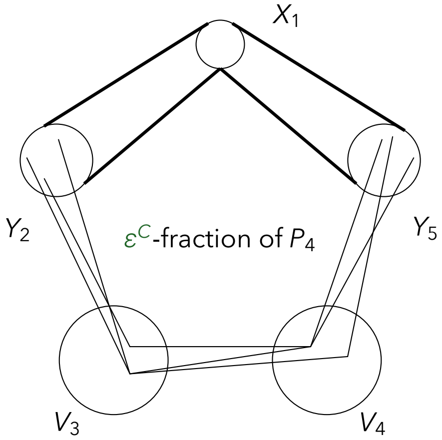

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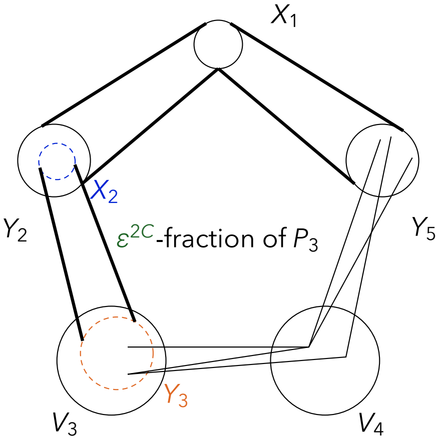

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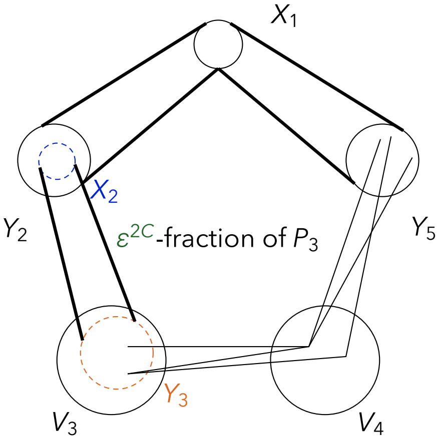

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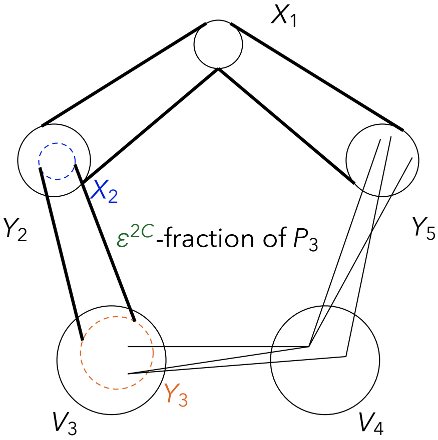

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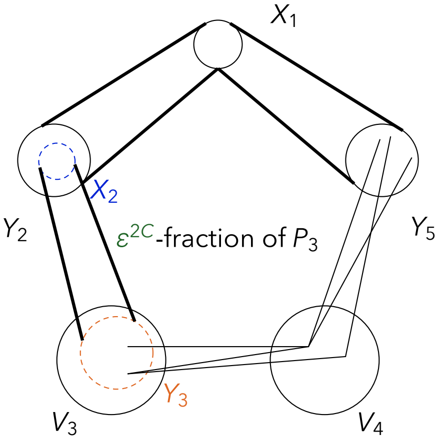

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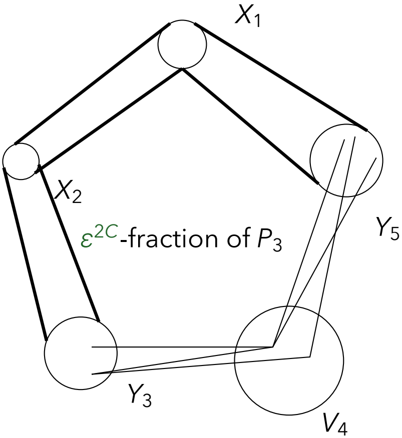

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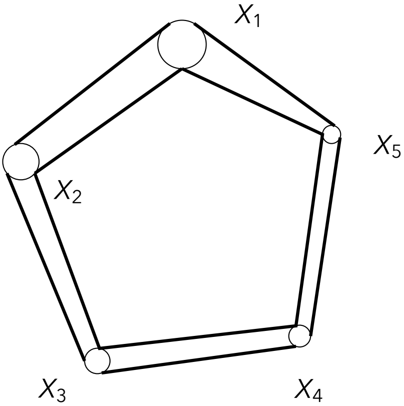

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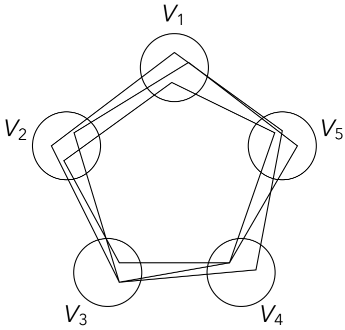
$$|X_i| \geqslant c''\frac{\log N}{\log\frac{1}{\varepsilon}}$$

Proof sketch III: Proof of the key lemma

Key lemma: There exist $X_1 \subseteq V_1$, $Y_2 \subseteq V_2$, $Y_5 \subseteq V_5$ such that:

- $|X_1| \geqslant c\frac{\log N}{\log \frac{1}{\varepsilon}}$ and $|Y_i| \geqslant \sqrt{N}$
- X_1 is complete to Y_2, Y_5
- There are $\varepsilon^C |Y_2||V_3||V_4||Y_5|$ canonical P_4



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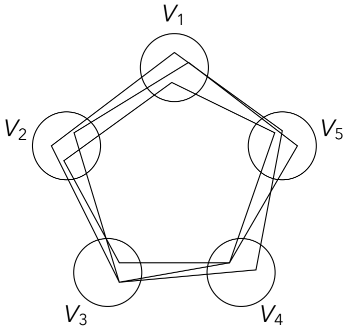
Let $s = c\frac{\log N}{\log \frac{1}{\varepsilon}}$.



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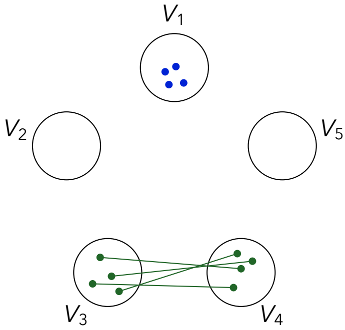

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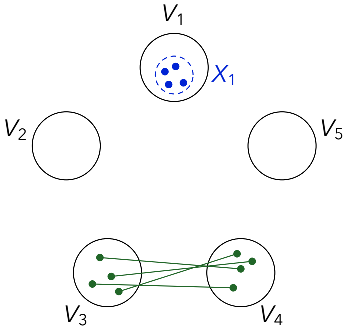

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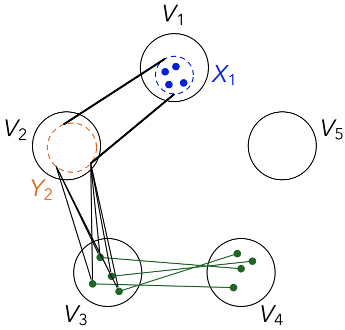

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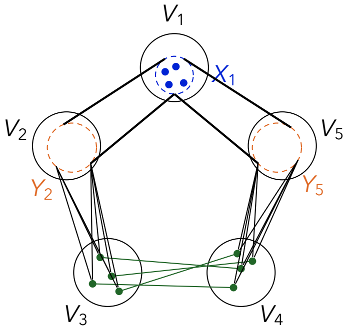

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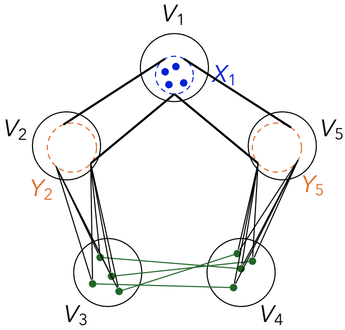

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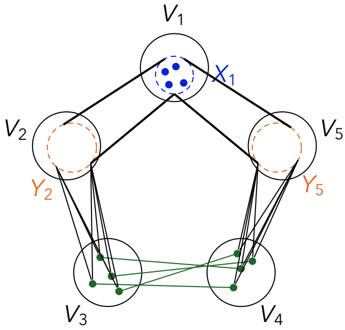

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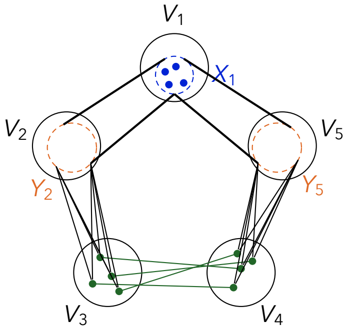
Crucially: $Y_2 \times Y_5$ is the set of pairs completing all a_i, b_i, c_i to a C_5.



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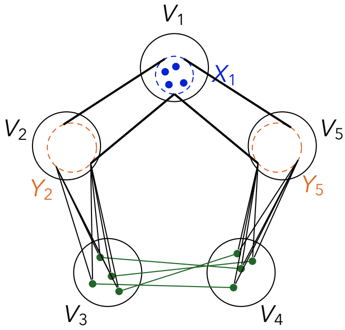

Thank you!