Finding large blowups

Yuval Wigderson

ETH Zürich

IBS ECOPRO Workshop on developments in combinatorics

November 27, 2024

Joint with António Girão and Zach Hunter

The Kővári–Sós–Turán theorem

The Kővári–Sós–Turán theorem

Theorem (Kővári–Sós–Turán '54)

Let G be an N-vertex graph with at least εN^2 edges. Then G contains a copy of $K_{k,k}$, where

$$k \geqslant c \frac{\log N}{\log \frac{1}{\varepsilon}}$$

and $c > 0$ is an absolute constant.

The Kővári–Sós–Turán theorem

Theorem (Kővári–Sós–Turán '54)

Let G be an N-vertex graph with at least εN^2 edges. Then G contains a copy of $K_{k,k}$, where

$$k \geqslant c \frac{\log N}{\log \frac{1}{\varepsilon}}$$

and $c > 0$ is an absolute constant.

This is tight up to the value of c

The Kővári–Sós–Turán theorem

Theorem (Kővári–Sós–Turán '54)

Let G be an N-vertex graph with at least εN^2 edges. Then G contains a copy of $K_{k,k}$, where

$$k \geqslant c\frac{\log N}{\log \frac{1}{\varepsilon}}$$

and $c > 0$ is an absolute constant.

This is tight up to the value of c: a random N-vertex graph with εN^2 edges contains no copy of $K_{k,k}$ with $k = C\frac{\log N}{\log \frac{1}{\varepsilon}}$, for some $C > 0$.

The Kővári–Sós–Turán theorem

Theorem (Kővári–Sós–Turán '54)

Let G be an N-vertex graph with at least εN^2 edges. Then G contains a copy of $K_{k,k}$, where

$$k \geqslant c\frac{\log N}{\log \frac{1}{\varepsilon}}$$

and $c > 0$ is an absolute constant.

This is tight up to the value of c: a random N-vertex graph with εN^2 edges contains no copy of $K_{k,k}$ with $k = C\frac{\log N}{\log \frac{1}{\varepsilon}}$, for some $C > 0$.

This holds for all ε and all N.

The Kővári–Sós–Turán theorem

Theorem (Kővári–Sós–Turán '54)

Let G be an N-vertex graph with at least εN^2 edges. Then G contains a copy of $K_{k,k}$, where

$$k \geqslant c \frac{\log N}{\log \frac{1}{\varepsilon}}$$

and $c > 0$ is an absolute constant.

This is tight up to the value of c: a random N-vertex graph with εN^2 edges contains no copy of $K_{k,k}$ with $k = C\frac{\log N}{\log \frac{1}{\varepsilon}}$, for some $C > 0$.

This holds for all ε and all N. For example, taking $\varepsilon = N^{-c/k}$ shows that every graph with $N^{2-c/k}$ edges contains $K_{k,k}$.

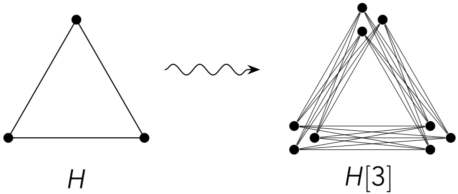
Blowups of graphs

Blowups of graphs

The k-blowup $H[k]$ of a graph H is obtained by replacing each vertex of H by k vertices, and each edge of H by $K_{k,k}$.

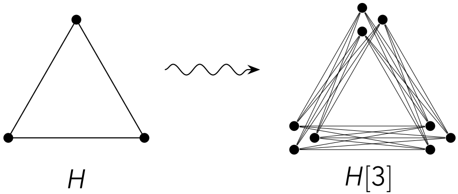
Blowups of graphs

The *k-blowup* $H[k]$ of a graph H is obtained by replacing each vertex of H by k vertices, and each edge of H by $K_{k,k}$.



Blowups of graphs

The k-blowup $H[k]$ of a graph H is obtained by replacing each vertex of H by k vertices, and each edge of H by $K_{k,k}$.



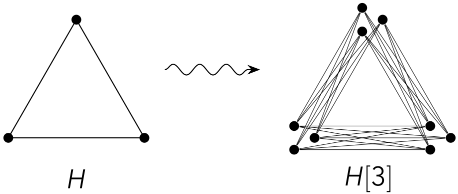
Theorem (Kővári–Sós–Turán '54)

Let G be an N-vertex graph with at least εN^2 edges. Then G contains a copy of $K_{k,k}$, where

$$k \geqslant c\frac{\log N}{\log \frac{1}{\varepsilon}}.$$

Blowups of graphs

The k-blowup $H[k]$ of a graph H is obtained by replacing each vertex of H by k vertices, and each edge of H by $K_{k,k}$.



Theorem (Kővári–Sós–Turán '54)

Let G be an N-vertex graph with at least εN^2 copies of K_2. Then G contains a copy of $K_2[k]$, where

$$k \geqslant c \frac{\log N}{\log \frac{1}{\varepsilon}}.$$

What about other graphs?

Theorem (Kővári–Sós–Turán '54)

Let G be an N-vertex graph with at least εN^2 copies of K_2. Then G contains a copy of $K_2[k]$, where

$$k \geqslant c \frac{\log N}{\log \frac{1}{\varepsilon}}.$$

What about other graphs?

Theorem (Kővári–Sós–Turán '54)

Let G be an N-vertex graph with at least εN^2 copies of K_2. Then G contains a copy of $K_2[k]$, where

$$k \geqslant c\frac{\log N}{\log \frac{1}{\varepsilon}}.$$

What if G contains many copies of some other graph H?

What about other graphs?

Theorem (Kővári–Sós–Turán '54)

Let G be an N-vertex graph with at least εN^2 copies of K_2. Then G contains a copy of $K_2[k]$, where

$$k \geqslant c \frac{\log N}{\log \frac{1}{\varepsilon}}.$$

What if G contains many copies of some other graph H?

Theorem (Erdős '64)

Let H be an h-vertex graph. Let G be an N-vertex graph with at least εN^h copies of H.

What about other graphs?

Theorem (Kővári–Sós–Turán '54)

Let G be an N-vertex graph with at least εN^2 copies of K_2. Then G contains a copy of $K_2[k]$, where

$$k \geqslant c \frac{\log N}{\log \frac{1}{\varepsilon}}.$$

What if G contains many copies of some other graph H?

Theorem (Erdős '64)

Let H be an h-vertex graph. Let G be an N-vertex graph with at least εN^h copies of H. Then G contains a copy of $H[k]$, where

$$k \geqslant c_H \frac{(\log N)^{\frac{1}{h-1}}}{\log \frac{1}{\varepsilon}}.$$

Graphs vs. hypergraphs

Theorem (Erdős '64)

Let H be an h-vertex graph. Let G be an N-vertex graph with at least εN^h copies of H. Then G contains a copy of $H[k]$, where

$$k \geqslant c_H \frac{(\log N)^{\frac{1}{h-1}}}{\log \frac{1}{\varepsilon}}.$$

Graphs vs. hypergraphs

Theorem (Erdős '64)

Let H be an h-vertex graph. Let G be an N-vertex graph with at least εN^h copies of H. Then G contains a copy of $H[k]$, where

$$k \geqslant c_H \frac{(\log N)^{\frac{1}{h-1}}}{\log \frac{1}{\varepsilon}}.$$

For $H = K_3$, this says that if G has $\Omega(N^3)$ triangles, it contains a copy of $K_3[k]$ with $k = \Omega(\sqrt{\log N})$.

Graphs vs. hypergraphs

Theorem (Erdős '64)

Let H be an h-vertex graph. Let G be an N-vertex graph with at least εN^h copies of H. Then G contains a copy of $H[k]$, where

$$k \geqslant c_H \frac{(\log N)^{\frac{1}{h-1}}}{\log \frac{1}{\varepsilon}}.$$

For $H = K_3$, this says that if G has $\Omega(N^3)$ triangles, it contains a copy of $K_3[k]$ with $k = \Omega(\sqrt{\log N})$.

This is not tight for random graphs: a random graph with $\Theta(N^3)$ triangles contains $K_3[k]$ with $k = \Theta(\log N)$.

Graphs vs. hypergraphs

Theorem (Erdős '64)

Let H be an h-vertex graph. Let G be an N-vertex graph with at least εN^h copies of H. Then G contains a copy of $H[k]$, where

$$k \geqslant c_H \frac{(\log N)^{\frac{1}{h-1}}}{\log \frac{1}{\varepsilon}}.$$

For $H = K_3$, this says that if G has $\Omega(N^3)$ triangles, it contains a copy of $K_3[k]$ with $k = \Omega(\sqrt{\log N})$.

This is not tight for random graphs: a random graph with $\Theta(N^3)$ triangles contains $K_3[k]$ with $k = \Theta(\log N)$.

Erdős's theorem holds for arbitrary hypergraphs, and is tight for random hypergraphs.

Graphs vs. hypergraphs

Theorem (Erdős '64)

Let H be an h-vertex graph. Let G be an N-vertex graph with at least εN^h copies of H. Then G contains a copy of $H[k]$, where

$$k \geqslant c_H \frac{(\log N)^{\frac{1}{h-1}}}{\log \frac{1}{\varepsilon}}.$$

For $H = K_3$, this says that if G has $\Omega(N^3)$ triangles, it contains a copy of $K_3[k]$ with $k = \Omega(\sqrt{\log N})$.

This is not tight for random graphs: a random graph with $\Theta(N^3)$ triangles contains $K_3[k]$ with $k = \Theta(\log N)$.

Erdős's theorem holds for arbitrary hypergraphs, and is tight for random hypergraphs.

The triangles in a graph are not a "generic" 3-uniform hypergraph!

Nikiforov's blowup theorem

Theorem (Erdős '64)

Let H be an h-vertex graph. Let G be an N-vertex graph with at least εN^h copies of H. Then G contains a copy of $H[k]$, where

$$k \geqslant c_H \frac{(\log N)^{\frac{1}{h-1}}}{\log \frac{1}{\varepsilon}}.$$

Nikiforov's blowup theorem

Theorem (Erdős '64)

Let H be an h-vertex graph. Let G be an N-vertex graph with at least εN^h copies of H. Then G contains a copy of $H[k]$, where

$$k \geqslant c_H \frac{(\log N)^{\frac{1}{h-1}}}{\log \frac{1}{\varepsilon}}.$$

Theorem (Nikiforov '08)

Fix $\varepsilon > 0$ and an h-vertex graph H. If N is sufficiently large, and G is an N-vertex graph with at least εN^h copies of H, then G contains a copy of $H[k]$, where

$$k \geqslant c_H \, \varepsilon^h \log N.$$

Nikiforov's blowup theorem

Theorem (Erdős '64)

Let H be an h-vertex graph. Let G be an N-vertex graph with at least εN^h copies of H. Then G contains a copy of $H[k]$, where

$$k \geqslant c_H \frac{(\log N)^{\frac{1}{h-1}}}{\log \frac{1}{\varepsilon}}.$$

Theorem (Nikiforov '08)

Fix $\varepsilon > 0$ and an h-vertex graph H. If N is sufficiently large, and G is an N-vertex graph with at least εN^h copies of H, then G contains a copy of $H[k]$, where

$$k \geqslant c_H \, \varepsilon^h \log N.$$

Pros: Right dependence on N! It's about graphs, not hypergraphs.

Nikiforov's blowup theorem

Theorem (Erdős '64)

Let H be an h-vertex graph. Let G be an N-vertex graph with at least εN^h copies of H. Then G contains a copy of $H[k]$, where

$$k \geqslant c_H \frac{(\log N)^{\frac{1}{h-1}}}{\log \frac{1}{\varepsilon}}.$$

Theorem (Nikiforov '08)

Fix $\varepsilon > 0$ and an h-vertex graph H. If N is sufficiently large, and G is an N-vertex graph with at least εN^h copies of H, then G contains a copy of $H[k]$, where

$$k \geqslant c_H \, \varepsilon^h \log N.$$

Pros: Right dependence on N! It's about graphs, not hypergraphs.
Cons: Requires N sufficiently large; wrong dependence on ε.
A random graph has $k = C_H \frac{\log N}{\log \frac{1}{\varepsilon}}$.

Nikiforov's blowup theorem

Theorem (Erdős '64)

Let H be an h-vertex graph. Let G be an N-vertex graph with at least εN^h copies of H. Then G contains a copy of $H[k]$, where

$$k \geqslant c_H \frac{(\log N)^{\frac{1}{h-1}}}{\log \frac{1}{\varepsilon}}.$$

Theorem (Nikiforov '08; Rödl–Schacht '12)

Fix $\varepsilon > 0$ and an h-vertex graph H. If N is sufficiently large, and G is an N-vertex graph with at least εN^h copies of H, then G contains a copy of $H[k]$, where

$$k \geqslant c_H \, \varepsilon^{1+o(1)} \log N.$$

Pros: Right dependence on N! It's about graphs, not hypergraphs.
Cons: Requires N sufficiently large; wrong dependence on ε.
A random graph has $k = C_H \frac{\log N}{\log \frac{1}{\varepsilon}}$.

Nikiforov's blowup theorem

Theorem (Erdős '64)

Let H be an h-vertex graph. Let G be an N-vertex graph with at least εN^h copies of H. Then G contains a copy of $H[k]$, where

$$k \geqslant c_H \frac{(\log N)^{\frac{1}{h-1}}}{\log \frac{1}{\varepsilon}}.$$

Theorem (Nikiforov '08; Rödl–Schacht '12; Fox–Luo–W '21)

Fix $\varepsilon > 0$ and an h-vertex graph H. If N is sufficiently large, and G is an N-vertex graph with at least εN^h copies of H, then G contains a copy of $H[k]$, where

$$k \geqslant c_H \, \varepsilon^{1-1/e(H)+o(1)} \log N.$$

Pros: Right dependence on N! It's about graphs, not hypergraphs.
Cons: Requires N sufficiently large; wrong dependence on ε.
A random graph has $k = C_H \frac{\log N}{\log \frac{1}{\varepsilon}}$.

A bold conjecture

A bold conjecture

Conjecture (Folklore?)

Let H be an h-vertex graph. If G is an N-vertex graph with at least εN^h copies of H, then G contains a copy of $H[k]$, where

$$k \geqslant c_H \frac{\log N}{\log \frac{1}{\varepsilon}}.$$

A bold conjecture

Conjecture (Folklore?)

Let H be an h-vertex graph. If G is an N-vertex graph with at least εN^h copies of H, then G contains a copy of $H[k]$, where

$$k \geqslant c_H \frac{\log N}{\log \frac{1}{\varepsilon}}.$$

This would be tight for random graphs, and is for all ε and all N.

A bold conjecture

Conjecture (Folklore?)

Let H be an h-vertex graph. If G is an N-vertex graph with at least εN^h copies of H, then G contains a copy of $H[k]$, where

$$k \geqslant c_H \frac{\log N}{\log \frac{1}{\varepsilon}}.$$

This would be tight for random graphs, and is for all ε and all N.

Partial progress:

A bold conjecture

Conjecture (Folklore?)

Let H be an h-vertex graph. If G is an N-vertex graph with at least εN^h copies of H, then G contains a copy of $H[k]$, where

$$k \geqslant c_H \frac{\log N}{\log \frac{1}{\varepsilon}}.$$

This would be tight for random graphs, and is for all ε and all N.

Partial progress:

- Kővári–Sós–Turán: the conjecture is true for $H = K_2$.

A bold conjecture

Conjecture (Folklore?)

Let H be an h-vertex graph. If G is an N-vertex graph with at least εN^h copies of H, then G contains a copy of $H[k]$, where

$$k \geqslant c_H \frac{\log N}{\log \frac{1}{\varepsilon}}.$$

This would be tight for random graphs, and is for all ε and all N.

Partial progress:

- Kővári–Sós–Turán: the conjecture is true for bipartite H.

A bold conjecture

Conjecture (Folklore?)

Let H be an h-vertex graph. If G is an N-vertex graph with at least εN^h copies of H, then G contains a copy of $H[k]$, where

$$k \geqslant c_H \frac{\log N}{\log \frac{1}{\varepsilon}}.$$

This would be tight for random graphs, and is for all ε and all N.

Partial progress:

- Kővári–Sós–Turán: the conjecture is true for bipartite H.
- That's it.

A bold conjecture

Conjecture (Folklore?)

Let H be an h-vertex graph. If G is an N-vertex graph with at least εN^h copies of H, then G contains a copy of $H[k]$, where

$$k \geqslant c_H \frac{\log N}{\log \frac{1}{\varepsilon}}.$$

This would be tight for random graphs, and is for all ε and all N.

Partial progress:

- Kővári–Sós–Turán: the conjecture is true for bipartite H.
- That's it.

Theorem (Girão–Hunter–W '24+)

The conjecture is true if H is triangle-free.

Proof sketch I

Let H be an h-vertex triangle-free graph.

Theorem (Girão–Hunter–W '24+)

If G is an N-vertex graph with at least εN^h copies of H, then G contains a copy of $H[k]$, where

$$k \geqslant c \frac{\log N}{\log \frac{1}{\varepsilon}}.$$

Proof sketch I

Let H be an h-vertex triangle-free graph. We'll focus on $H = C_5$.

Theorem (Girão–Hunter–W '24+)

If G is an N-vertex graph with at least εN^5 copies of C_5, then G contains a copy of $C_5[k]$, where

$$k \geqslant c \frac{\log N}{\log \frac{1}{\varepsilon}}.$$

Proof sketch I

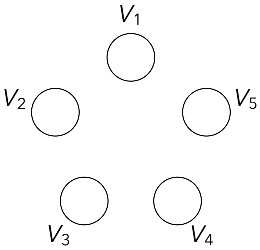
Let H be an h-vertex triangle-free graph. We'll focus on $H = C_5$.

Theorem (Girão–Hunter–W '24+)

If G is an N-vertex graph with at least εN^5 copies of C_5, then G contains a copy of $C_5[k]$, where

$$k \geqslant c \frac{\log N}{\log \frac{1}{\varepsilon}}.$$

We may assume that G is 5-partite.



Proof sketch I

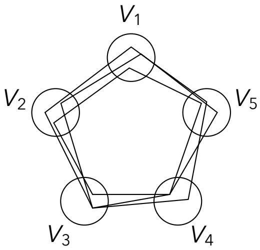
Let H be an h-vertex triangle-free graph. We'll focus on $H = C_5$.

Theorem (Girão–Hunter–W '24+)

If G is an N-vertex graph with at least εN^5 copies of C_5, then G contains a copy of $C_5[k]$, where

$$k \geqslant c \frac{\log N}{\log \frac{1}{\varepsilon}}.$$

We may assume that G is 5-partite.
$|V_i| = N$, and εN^5 canonical copies of C_5.



Proof sketch I

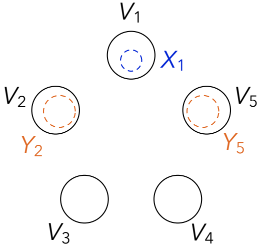
Let H be an h-vertex triangle-free graph. We'll focus on $H = C_5$.

Theorem (Girão–Hunter–W '24+)

If G is an N-vertex graph with at least εN^5 copies of C_5, then G contains a copy of $C_5[k]$, where

$$k \geqslant c \frac{\log N}{\log \frac{1}{\varepsilon}}.$$

We may assume that G is 5-partite.
$|V_i| = N$, and εN^5 canonical copies of C_5.
Key lemma: There exist
$X_1 \subseteq V_1$, $Y_2 \subseteq V_2$, $Y_5 \subseteq V_5$ such that:



Proof sketch I

Let H be an h-vertex triangle-free graph. We'll focus on $H = C_5$.

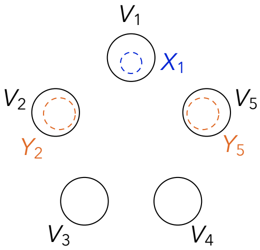
Theorem (Girão–Hunter–W '24+)

If G is an N-vertex graph with at least εN^5 copies of C_5, then G contains a copy of $C_5[k]$, where

$$k \geqslant c \frac{\log N}{\log \frac{1}{\varepsilon}}.$$

We may assume that G is 5-partite.
$|V_i| = N$, and εN^5 canonical copies of C_5.
Key lemma: There exist
$X_1 \subseteq V_1$, $Y_2 \subseteq V_2$, $Y_5 \subseteq V_5$ such that:

- $|X_1| \geqslant c \frac{\log N}{\log \frac{1}{\varepsilon}}$ and $|Y_i| \geqslant \sqrt{N}$



Proof sketch I

Let H be an h-vertex triangle-free graph. We'll focus on $H = C_5$.

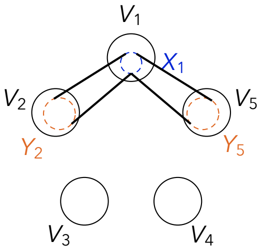
Theorem (Girão–Hunter–W '24+)

If G is an N-vertex graph with at least εN^5 copies of C_5, then G contains a copy of $C_5[k]$, where

$$k \geqslant c \frac{\log N}{\log \frac{1}{\varepsilon}}.$$

We may assume that G is 5-partite.
$|V_i| = N$, and εN^5 canonical copies of C_5.
Key lemma: There exist
$X_1 \subseteq V_1$, $Y_2 \subseteq V_2$, $Y_5 \subseteq V_5$ such that:

- $|X_1| \geqslant c \frac{\log N}{\log \frac{1}{\varepsilon}}$ and $|Y_i| \geqslant \sqrt{N}$
- X_1 is complete to Y_2, Y_5



Proof sketch I

Let H be an h-vertex triangle-free graph. We'll focus on $H = C_5$.

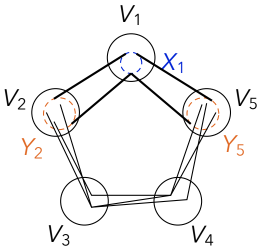
Theorem (Girão–Hunter–W '24+)

If G is an N-vertex graph with at least εN^5 copies of C_5, then G contains a copy of $C_5[k]$, where

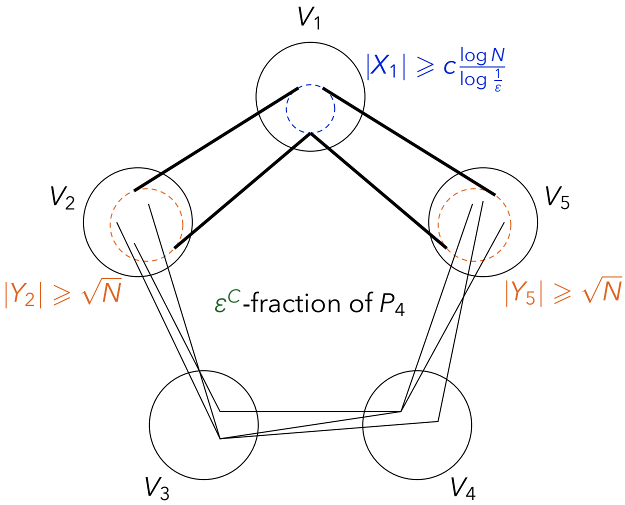
$$k \geqslant c\frac{\log N}{\log \frac{1}{\varepsilon}}.$$

We may assume that G is 5-partite.
$|V_i| = N$, and εN^5 canonical copies of C_5.
Key lemma: There exist $X_1 \subseteq V_1$, $Y_2 \subseteq V_2$, $Y_5 \subseteq V_5$ such that:

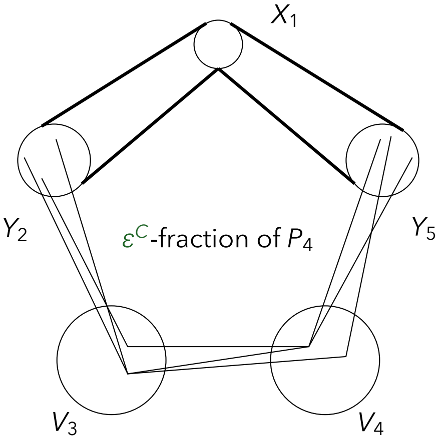
- $|X_1| \geqslant c\frac{\log N}{\log \frac{1}{\varepsilon}}$ and $|Y_i| \geqslant \sqrt{N}$
- X_1 is complete to Y_2, Y_5
- There are $\varepsilon^C|Y_2||V_3||V_4||Y_5|$ canonical P_4



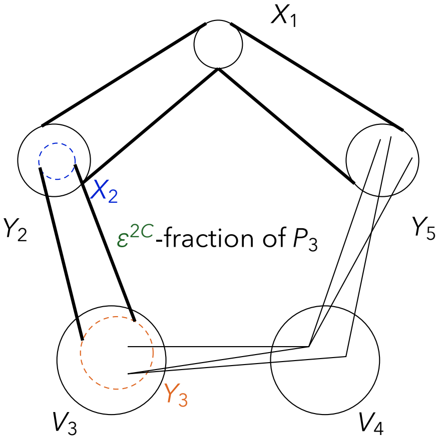
Proof sketch II: Using the key lemma



Proof sketch II: Using the key lemma

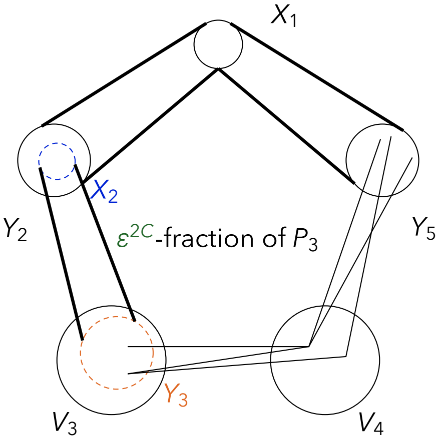


Proof sketch II: Using the key lemma

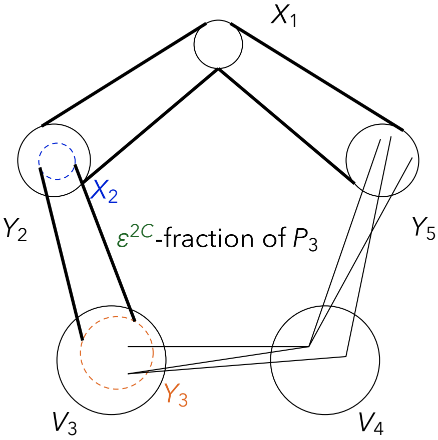


Proof sketch II: Using the key lemma

$$|X_2| \geqslant c\frac{\log|Y_2|}{\log\frac{1}{\varepsilon^C}}$$



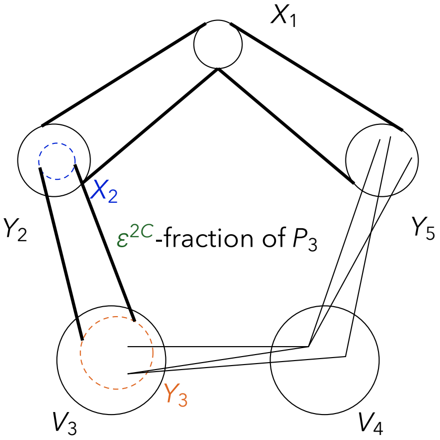
Proof sketch II: Using the key lemma



$$|X_2| \geqslant c\frac{\log|Y_2|}{\log \frac{1}{\varepsilon^C}}$$

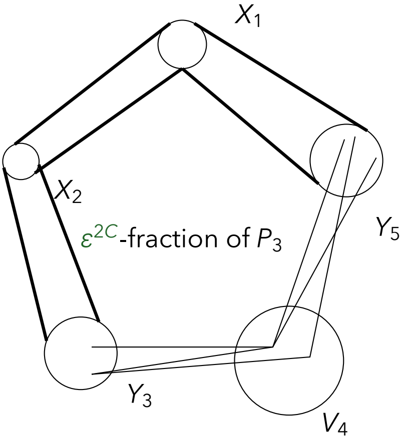
$$\geqslant c\frac{\log\sqrt{N}}{\log \frac{1}{\varepsilon^C}}$$

Proof sketch II: Using the key lemma



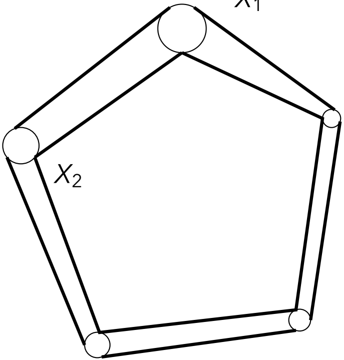
$$
\begin{aligned}
|X_2| &\geqslant c\frac{\log|Y_2|}{\log \frac{1}{\varepsilon^C}} \\
&\geqslant c\frac{\log \sqrt{N}}{\log \frac{1}{\varepsilon^C}} \\
&= c'\frac{\log N}{\log \frac{1}{\varepsilon}}
\end{aligned}
$$

Proof sketch II: Using the key lemma



$$|X_2| \geqslant c\frac{\log|Y_2|}{\log\frac{1}{\varepsilon^C}} \geqslant c\frac{\log\sqrt{N}}{\log\frac{1}{\varepsilon^C}} = c'\frac{\log N}{\log\frac{1}{\varepsilon}}$$

Proof sketch II: Using the key lemma



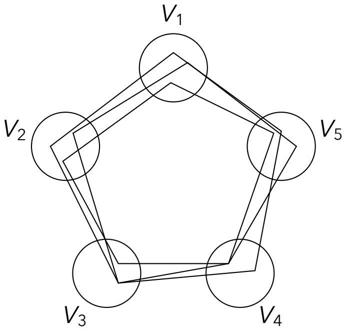
$$
\begin{aligned}
|X_2| &\geqslant c\frac{\log|Y_2|}{\log\frac{1}{\varepsilon^C}} \\
&\geqslant c\frac{\log\sqrt{N}}{\log\frac{1}{\varepsilon^C}} \\
&= c'\frac{\log N}{\log\frac{1}{\varepsilon}}
\end{aligned}
$$

$$|X_i| \geqslant c''\frac{\log N}{\log\frac{1}{\varepsilon}}$$

Proof sketch III: Proof of the key lemma

Key lemma: There exist $X_1 \subseteq V_1$, $Y_2 \subseteq V_2$, $Y_5 \subseteq V_5$ such that:

- $|X_1| \geqslant c\frac{\log N}{\log \frac{1}{\varepsilon}}$ and $|Y_i| \geqslant \sqrt{N}$
- X_1 is complete to Y_2, Y_5
- There are $\varepsilon^C|Y_2||V_3||V_4||Y_5|$ canonical P_4

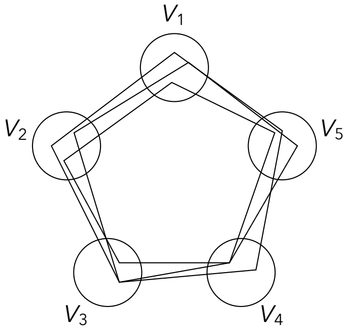


Proof sketch III: Proof of the key lemma

Key lemma: There exist $X_1 \subseteq V_1$, $Y_2 \subseteq V_2$, $Y_5 \subseteq V_5$ such that:

- $|X_1| \geqslant c\frac{\log N}{\log \frac{1}{\varepsilon}}$ and $|Y_i| \geqslant \sqrt{N}$
- X_1 is complete to Y_2, Y_5
- There are $\varepsilon^C |Y_2||V_3||V_4||Y_5|$ canonical P_4

Let $s = c\frac{\log N}{\log \frac{1}{\varepsilon}}$.

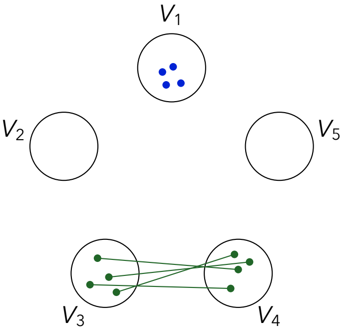


Proof sketch III: Proof of the key lemma

Key lemma: There exist $X_1 \subseteq V_1$, $Y_2 \subseteq V_2$, $Y_5 \subseteq V_5$ such that:

- $|X_1| \geqslant c\frac{\log N}{\log \frac{1}{\varepsilon}}$ and $|Y_i| \geqslant \sqrt{N}$
- X_1 is complete to Y_2, Y_5
- There are $\varepsilon^C |Y_2||V_3||V_4||Y_5|$ canonical P_4

Let $s = c\frac{\log N}{\log \frac{1}{\varepsilon}}$.
Pick random vertices $a_1, \ldots, a_s \in V_1$ and random edges $b_1c_1, \ldots, b_sc_s \in V_3 \times V_4$.

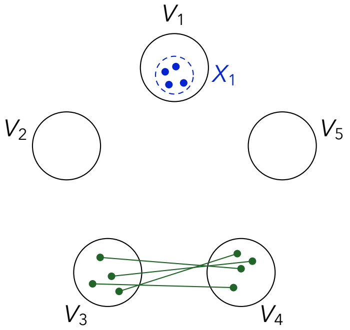


Proof sketch III: Proof of the key lemma

Key lemma: There exist $X_1 \subseteq V_1$, $Y_2 \subseteq V_2$, $Y_5 \subseteq V_5$ such that:

- $|X_1| \geqslant c\frac{\log N}{\log \frac{1}{\varepsilon}}$ and $|Y_i| \geqslant \sqrt{N}$
- X_1 is complete to Y_2, Y_5
- There are $\varepsilon^C |Y_2||V_3||V_4||Y_5|$ canonical P_4

Let $s = c\frac{\log N}{\log \frac{1}{\varepsilon}}$.
Pick random vertices $a_1, \ldots, a_s \in V_1$ and random edges $b_1c_1, \ldots, b_sc_s \in V_3 \times V_4$.
Let $X_1 = \{a_1, \ldots, a_s\}$.

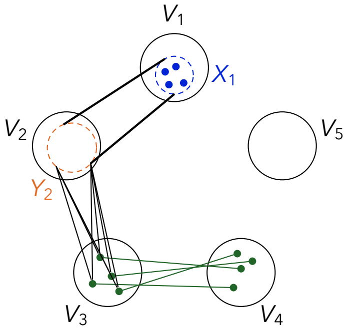


Proof sketch III: Proof of the key lemma

Key lemma: There exist $X_1 \subseteq V_1$, $Y_2 \subseteq V_2$, $Y_5 \subseteq V_5$ such that:

- $|X_1| \geqslant c\frac{\log N}{\log \frac{1}{\varepsilon}}$ and $|Y_i| \geqslant \sqrt{N}$
- X_1 is complete to Y_2, Y_5
- There are $\varepsilon^C |Y_2||V_3||V_4||Y_5|$ canonical P_4

Let $s = c\frac{\log N}{\log \frac{1}{\varepsilon}}$.
Pick random vertices $a_1, \dots, a_s \in V_1$ and random edges $b_1c_1, \dots, b_sc_s \in V_3 \times V_4$.
Let $X_1 = \{a_1, \dots, a_s\}$.
Let Y_2 be the common neighborhood of $a_1, \dots, a_s, b_1, \dots, b_s$

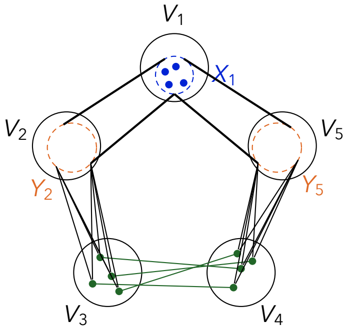


Proof sketch III: Proof of the key lemma

Key lemma: There exist $X_1 \subseteq V_1$, $Y_2 \subseteq V_2$, $Y_5 \subseteq V_5$ such that:

- $|X_1| \geqslant c\frac{\log N}{\log \frac{1}{\varepsilon}}$ and $|Y_i| \geqslant \sqrt{N}$
- X_1 is complete to Y_2, Y_5
- There are $\varepsilon^C |Y_2||V_3||V_4||Y_5|$ canonical P_4

Let $s = c\frac{\log N}{\log \frac{1}{\varepsilon}}$.
Pick random vertices $a_1, \ldots, a_s \in V_1$ and random edges $b_1c_1, \ldots, b_sc_s \in V_3 \times V_4$.
Let $X_1 = \{a_1, \ldots, a_s\}$.
Let Y_2 be the common neighborhood of $a_1, \ldots, a_s, b_1, \ldots, b_s$; similarly for Y_5.

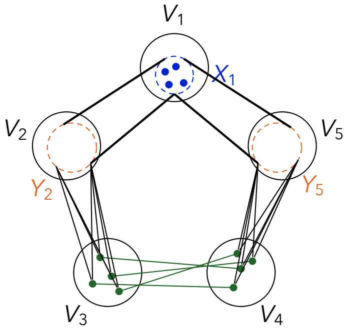


Proof sketch III: Proof of the key lemma

Key lemma: There exist $X_1 \subseteq V_1$, $Y_2 \subseteq V_2$, $Y_5 \subseteq V_5$ such that:

- $|X_1| \geqslant c\frac{\log N}{\log \frac{1}{\varepsilon}}$ and $|Y_i| \geqslant \sqrt{N}$
- ✓ X_1 is complete to Y_2, Y_5
- There are $\varepsilon^C |Y_2||V_3||V_4||Y_5|$ canonical P_4

Let $s = c\frac{\log N}{\log \frac{1}{\varepsilon}}$.
Pick random vertices $a_1, \ldots, a_s \in V_1$ and random edges $b_1c_1, \ldots, b_sc_s \in V_3 \times V_4$.
Let $X_1 = \{a_1, \ldots, a_s\}$.
Let Y_2 be the common neighborhood of $a_1, \ldots, a_s, b_1, \ldots, b_s$; similarly for Y_5.

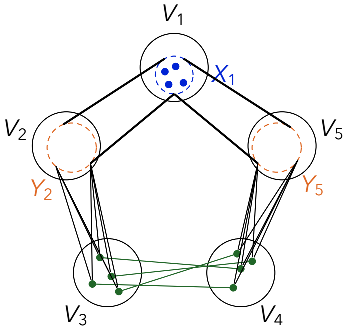


Proof sketch III: Proof of the key lemma

Key lemma: There exist $X_1 \subseteq V_1$, $Y_2 \subseteq V_2$, $Y_5 \subseteq V_5$ such that:

- ✓ $|X_1| \geqslant c\frac{\log N}{\log \frac{1}{\varepsilon}}$ and $|Y_i| \geqslant \sqrt{N}$
- ✓ X_1 is complete to Y_2, Y_5
- There are $\varepsilon^C |Y_2||V_3||V_4||Y_5|$ canonical P_4

Let $s = c\frac{\log N}{\log \frac{1}{\varepsilon}}$.
Pick random vertices $a_1, \ldots, a_s \in V_1$ and random edges $b_1c_1, \ldots, b_sc_s \in V_3 \times V_4$.
Let $X_1 = \{a_1, \ldots, a_s\}$.
Let Y_2 be the common neighborhood of $a_1, \ldots, a_s, b_1, \ldots, b_s$; similarly for Y_5.

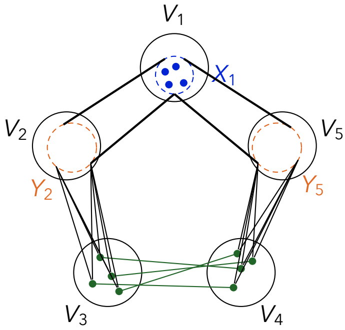


Proof sketch III: Proof of the key lemma

Key lemma: There exist $X_1 \subseteq V_1$, $Y_2 \subseteq V_2$, $Y_5 \subseteq V_5$ such that:

- ✓ $|X_1| \geqslant c\frac{\log N}{\log \frac{1}{\varepsilon}}$ and $|Y_i| \geqslant \sqrt{N}$
- ✓ X_1 is complete to Y_2, Y_5
- There are $\varepsilon^C|Y_2||V_3||V_4||Y_5|$ canonical P_4

Let $s = c\frac{\log N}{\log \frac{1}{\varepsilon}}$.
Pick random vertices $a_1, \ldots, a_s \in V_1$ and random edges $b_1c_1, \ldots, b_sc_s \in V_3 \times V_4$.
Let $X_1 = \{a_1, \ldots, a_s\}$.
Let Y_2 be the common neighborhood of $a_1, \ldots, a_s, b_1, \ldots, b_s$; similarly for Y_5.
Crucially: $Y_2 \times Y_5$ is the set of pairs completing all a_i, b_i, c_i to a C_5.

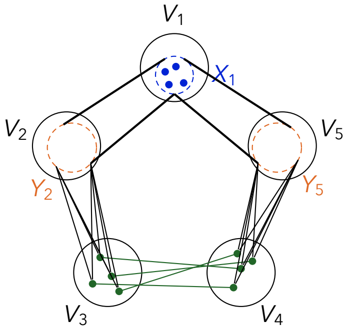


Proof sketch III: Proof of the key lemma

Key lemma: There exist $X_1 \subseteq V_1$, $Y_2 \subseteq V_2$, $Y_5 \subseteq V_5$ such that:

- ✓ $|X_1| \geqslant c\frac{\log N}{\log \frac{1}{\varepsilon}}$ and $|Y_i| \geqslant \sqrt{N}$
- ✓ X_1 is complete to Y_2, Y_5
- ✓ There are $\varepsilon^C|Y_2||V_3||V_4||Y_5|$ canonical P_4

Let $s = c\frac{\log N}{\log \frac{1}{\varepsilon}}$.
Pick random vertices $a_1, \ldots, a_s \in V_1$ and random edges $b_1c_1, \ldots, b_sc_s \in V_3 \times V_4$.
Let $X_1 = \{a_1, \ldots, a_s\}$.
Let Y_2 be the common neighborhood of $a_1, \ldots, a_s, b_1, \ldots, b_s$; similarly for Y_5.
Crucially: $Y_2 \times Y_5$ is the set of pairs completing all a_i, b_i, c_i to a C_5.



Thank you!