Ramsey goodness of books revisited

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Abstract

The Ramsey number $r(G, H)$ is the minimum N such that every graph on N vertices contains G as a subgraph or its complement contains H as a subgraph. For integers $n \geq k \geq 1$, the k-book $B_{k,n}$ is the graph on n vertices consisting of a copy of K_k, called the *spine*, as well as $n - k$ additional vertices each adjacent to every vertex of the spine and non-adjacent to each other. A connected graph H on n vertices is called *p-good* if $r(K_p, H) = (p-1)(n-1)+1$. Nikiforov and Rousseau proved that if n is sufficiently large in terms of p and k, then $B_{k,n}$ is p-good. Their proof uses Szemerédi's regularity lemma and gives a tower-type bound on n. We give a short new proof that avoids using the regularity method and shows that every $B_{k,n}$ with $n \geq 2^{k^{10p}}$ is p-good.

Using Szemerédi's regularity lemma, Nikiforov and Rousseau also proved much more general goodness-type results, proving a tight bound on $r(G, H)$ for several families of sparse graphs G and H as long as $|V(G)| < \delta|V(H)|$ for a small constant $\delta > 0$. Using our techniques, we prove a new result of this type, showing that $r(G, H) = (p-1)(n-1)+1$ when $H = B_{k,n}$ and G is a complete p-partite graph whose first $p - 1$ parts have constant size and whose last part has size δn, for some small absolute constant $\delta > 0$.

1 Introduction

For two graphs G, H, their *Ramsey number* $r(G, H)$ is the smallest N such that every graph Γ on N vertices contains G as a subgraph, or its complement contains H as a subgraph. The existence of $r(G, H)$ is guaranteed by Ramsey's theorem [24]. The most well-studied Ramsey number is the *diagonal* Ramsey number $r(K_k, K_k)$. One of the oldest (and easiest) results in Ramsey theory is the fact that $r(K_k, K_k) \geq (k-1)^2 + 1$, which is proved by taking Γ to be the complete balanced $(k-1)$-partite graph on $(k-1)^2$ vertices.

This quadratic lower bound is far from best possible. Indeed, it is known [13, 18] that $r(K_k, K_k)$ must grow exponentially in k, though the exact exponential rate remains unknown despite decades of intense research. Nonetheless, it is an instance of a much more general inequality which can be tight. Write $\chi(G)$ for the chromatic number of G. The inequality in question is then

$$r(G, H) \geq (p-1)(n-1) + a, \tag{1}$$

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which holds under the conditions that $\chi(G) = p$ and a is the minimum size of a color class among all proper p-colorings of G, and H is a connected graph with n vertices. Inequality (1) was first proved by Burr [3], by taking Γ to be a complete p-partite graph with $p-1$ parts of size $n-1$ and one part of size $a-1$.

Burr and Erdős [4] initiated the study of when (1) is tight; following their terminology, one says that a connected n-vertex graph H is *G-good* if (1) is tight when $\chi(G) = p$ and a is the minimum size of a color class among all proper p-colorings of G. In case $G = K_p$, one says that H is *p-good*, rather than K_p-good.

While the Ramsey goodness bound (1) is far from tight in the case of cliques, it turns out that many interesting graphs are p-good, and that the theory of Ramsey goodness generalizes many important results in graph theory. For example, Turán's theorem, which states that the balanced complete $(p-1)$-partite graph has the most edges among all K_p-free graphs on N vertices, is equivalent to the fact that stars are p-good. Extending this fact, Chvátal [7] proved that all trees are p-good for all $p \geq 3$, and this theorem inspired Burr and Erdős to define Ramsey goodness. At this point, there is a rich theory of Ramsey goodness, about which we refer the interested reader to the survey [10, Section 2.5].

For $n \geq k \geq 1$, the k-book $B_{k,n}$ on n vertices consists of a copy of K_k, called the *spine*, as well as $n-k$ additional vertices each joined to every vertex of the spine; equivalently, $B_{k,n}$ consists[1] of $n-k$ cliques of order $k+1$ sharing a common K_k. Book graphs arise naturally in the study of Ramsey numbers. Indeed, Ramsey [24] originally proved the finiteness of $r(K_k, K_k)$ by proving the finiteness of $r(B_{k,n}, B_{k,n})$ for every n, and it was observed by Erdős, Faudree, Rousseau, and Schelp [16] that the classical Erdős–Szekeres [18] upper bound on Ramsey numbers can also be framed as an upper bound on certain book Ramsey numbers. This connection yields an important approach to improving upper bounds on $r(K_k, K_k)$; for more details, see e.g. [8, 12].

In [22], Nikiforov and Rousseau used Szemerédi's regularity lemma to prove that for every $k, p \geq 1$ and every sufficiently large n, the book $B_{k,n}$ is p-good. One consequence of applying the regularity method is that their proof yields tower-type bounds on how large n must be in terms of k and p, and they raised the question of what the best possible n is. Our first main result is a new proof of p-goodness for books which avoids the use of the regularity lemma, and thus gets a much better dependence for n on k and p.

Theorem 1.1. *If $n \geq 2^{k^{10p}}$, then $B_{k,n}$ is p-good.*

Our main technique is a novel variant of the greedy embedding strategy, which allows us to build a large induced copy of a complete multipartite graph inside a K_p-free graph whose complement does not contain a very large book.

Extending the techniques from [22], Nikiforov and Rousseau [23] were later able to prove a remarkable theorem, which remains the most general result in the field of Ramsey goodness. As the result in its full generality requires some notation, we state only the following special case.

Theorem 1.2 (Nikiforov and Rousseau [23, Theorem 2.12])**.** *For every $k, p \geq 2$, there exists some $\delta > 0$ such that for all sufficiently large n,*

$$r(B_{p-1,\delta n}, B_{k,n}) = (p-1)(n-1) + 1.$$

[1] We remark that other notation exists for book graphs; notably, some other papers (e.g. [8, 12, 22]) use $B_{n-k}^{(k)}$ to denote what is $B_{k,n}$ in our notation.

In other words, the Ramsey goodness result $r(K_p, B_{k,n}) = (p-1)(n-1)+1$ remains true even if we replace K_p by the much larger graph $B_{p-1,\delta n}$ containing it. This result goes beyond the basic Ramsey goodness framework introduced by Burr and Erdős, to show that $r(G,H) = (p-1)(n-1)+1$ in cases even when G is not a fixed graph.

Just as before, the proof of [23] uses Szemerédi's regularity lemma, and hence the bound on $1/\delta$ in Theorem 1.2 is of tower type. In order to demonstrate the flexibility of our proof technique, we prove the following generalization of Theorem 1.2, which again goes beyond the basic Ramsey goodness framework of Burr and Erdős.

Theorem 1.3. *For every $k, p, t \geq 2$, there exists $\delta > 0$ such that the following holds for all $n \geq 1$. Let $1 \leq a_1 \leq \cdots \leq a_{p-1} \leq t$ and $a_p \leq \delta n$ be positive integers. Let G be the complete p-partite graph with parts of sizes $a_1, \ldots, a_p$, and let $H = B_{k,n}$. Then $r(G,H) = (p-1)(n-1) + a_1$ if and only if $a_1 = a_2 = 1$.*

Note that Theorem 1.3 is vacuously true unless n is sufficiently large, as otherwise there does not exist a positive integer $a_p \leq \delta n$. Our proof shows that one may take $1/\delta$ to be double-exponential in k, p, and t. Additionally, once n is double-exponential in k, p, and t, the statement holds with $1/\delta$ merely single-exponential in k, p, and t.

Although Theorem 1.3 has not appeared in the literature, the "if" direction (which is the harder one) can be deduced from the general theorem of Nikiforov and Rousseau [23, Theorem 2.1]. Nonetheless, the main novelty is not the statement of Theorem 1.3, but rather the fact that our proof again avoids the use of the regularity lemma, so that the bounds on $1/\delta$ are not of tower-type. It would be very interesting to see how far one can push these ideas; for example, is it possible to completely eliminate the use of the regularity lemma from the proof of [23, Theorem 2.1]?

Organization. In Section 2, we warm up by proving Theorem 1.2; in fact, we prove a generalization that sets the groundwork for Theorem 1.3. In Section 3, we prove a stability-supersaturation version of Turán's theorem, and use that to prove a variant of the Andrásfai–Erdős–Sós theorem, Theorem 4.1, in Section 4. Theorem 4.1 is an important ingredient in the proof of Theorem 1.3, as it essentially allows us to reduce to the case that Γ is $(p-1)$-partite. While both such results are relatively standard, the specific statements we need are apparently new. Finally, the proof of Theorem 1.3 is completed in Section 5, and we collect some interesting open problems in Section 6.

For the sake of clarity of presentation, we omit floor and ceiling signs when they are not crucial.

2 Ramsey goodness of books

Let $K_r(t)$ denote the complete r-partite graph with parts of size t. The following result is the greedy embedding lemma that we use. Given a graph Γ, it allows us to find a large book in $\overline{\Gamma}$ or find a large induced complete multipartite subgraph of Γ.

Lemma 2.1. *Let k, r, s, t be positive integers with $s \leq t$ and $2k \leq t$, and let G be any graph. Let Γ be a G-free graph with $N \geq \binom{t}{s}^r \frac{t}{2ks} r(G, K_s)$ vertices which contains $K_r(t)$ as an induced subgraph, with parts $V_1, \ldots, V_r$. If $\overline{\Gamma}$ does not contain a book $B_{k,n}$ with $n \geq (1 - 4ks/t)N/r$ vertices, then Γ contains an induced copy of $K_{r+1}(s)$ with parts $W_0, \ldots, W_r$, where $W_i \subseteq V_i$ for every $1 \leq i \leq r$.*

Proof. Let $\varepsilon = s/t$. Partition the vertex set of Γ into $r+1$ parts $U_0, U_1, \ldots, U_r$, where, for each $i \in [r]$, every vertex in U_i has degree at most εt to V_i, and every vertex in U_0 has degree at least εt to each V_j. Note that by construction, $V_i \subseteq U_i$ for $i \in [r]$.

Suppose there is $i \in [r]$ such that $|U_i| \geq (1-2k\varepsilon)N/r$. Let X denote the set of all vertices $v \in V_i$ with at most $2\varepsilon|U_i \setminus V_i|$ neighbors in $U_i \setminus V_i$. Since each vertex in U_i has density at most ε to V_i, we have $|X| \geq |V_i|/2 = t/2 \geq k$. Let Q be any k vertices in X. Then all but at most a $2k\varepsilon$ fraction of the vertices in $U_i \setminus V_i$ are empty to Q. So Q together with the vertices of U_i that have have no neighbors in Q form a k-book in $\overline{\Gamma}$ with at least $(1-2k\varepsilon)|U_i \setminus V_i| + |V_i| \geq (1-4k\varepsilon)N/r$ vertices.

So we may assume that there is no $i \in [r]$ with $|U_i| \geq (1-2k\varepsilon)N/r$. In this case, we have $|U_0| \geq N - r(1-2k\varepsilon)N/r = 2k\varepsilon N$. By the pigeonhole principle, there is a subset $T \subset U_0$ of size at least $\binom{t}{s}^{-r}|U_0| \geq r(G, K_s)$ such that there are subsets $W_i \subseteq V_i$ with $|W_i| = s$ for $i \geq 1$ such that every vertex in T is complete to each W_i. As Γ and hence the induced subgraph $\Gamma[T]$ is G-free and $|T| \geq r(G, K_s)$, we know that T contains an independent set W_0 of order s. Then $W_0, W_1, \ldots, W_r$ form a complete induced $(r+1)$-partite subgraph of Γ with parts of size s. $\square$

Our next lemma shows that, once we find a large induced complete multipartite subgraph of Γ, we can find a large book in $\overline{\Gamma}$.

Lemma 2.2. *If a K_p-free graph Γ on n vertices contains $K_{p-1}(k)$ as an induced subgraph, then its vertex set can be partitioned into $p-1$ subsets that each span a k-book in $\overline{\Gamma}$.*

Proof. Let $V_1, \ldots, V_{p-1}$ be the $p-1$ parts of the induced $K_{p-1}(k)$. As Γ is K_p-free, each vertex in Γ has no neighbors in some V_i. Partition the vertex set of Γ into $p-1$ parts $U_1, \ldots, U_{p-1}$, where, for each $i \in [p-1]$, each vertex in U_i has no neighbors in V_i. Then each U_i spans a k-book in $\overline{\Gamma}$ with spine V_i. $\square$

Our next result is the main form in which we use Lemma 2.1, and follows from it by a simple inductive argument.

Lemma 2.3. *Let k, p, x be positive integers, and let $z = x \cdot (20k)^p$. Let Γ be a K_p-free graph on at least $N = (p-1)(n-1)+1$ vertices, and suppose $S \subseteq V(\Gamma)$ satisfies $|S| \geq z^z \cdot r(K_p, K_z)$. Then either $\overline{\Gamma}$ contains a copy of $B_{k,n}$, or else Γ contains $K_{p-1}(x)$ as an induced subgraph, one part of which is a subset of S.*

Proof. For $r = 1, \ldots, p-2$, let $\varepsilon_r = (1 - r/(p-1))/(4k)$ so that $(1-4k\varepsilon_r)/r = 1/(p-1)$. Let $t_{p-1} = x$ and $t_r = t_{r+1}/\varepsilon_r$ for $r = p-2, \ldots, 1$. Observe that

$$t_1 = t_{p-1} / \prod_{r=1}^{p-2} \varepsilon_r = x(4k)^{p-2}(p-1)^{p-2}/(p-2)! < (20k)^p x = z.$$

Since $t_1 \geq t_2 \geq \cdots \geq t_{p-1}$, this implies that $t_r < z$ for all r. We now prove by induction on r for $r \in [p-1]$ that Γ contains $K_r(t_r)$ as an induced subgraph, with the first part of $K_r(t_r)$ being a subset of S.

For the base case $r = 1$, we have $|S| \geq r(K_p, K_z) > r(K_p, K_{t_1})$, so Γ contains an independent set of order t_1, that is, $\Gamma[S]$ contains $K_r(t_r)$ with $r = 1$ as an induced subgraph.

Now suppose Γ contains $K_r(t_r)$ as an induced subgraph, with the first part a subset of S. We apply Lemma 2.1 with $s = t_{r+1}$, $t = t_r$, and $G = K_p$. Observe that

$$\begin{aligned} \binom{t_{r+1}}{t_r}^r (2kt_{r+1}/t_r)^{-1} r(K_p, K_{t_{r+1}}) &\leq (e/\varepsilon_r)^{rt_r} (2kt_{r+1}/t_r)^{-1} r(K_p, K_{t_{r+1}}) \\ &< z^z \cdot r(K_p, K_z) \leq |S|. \end{aligned}$$

So either $\overline{\Gamma}$ contains a k-book with at least $(1-4k\varepsilon_r)N/r = N/(p-1) \geq n$ vertices, in which case we are done, or Γ contains an induced $K_{r+1}(t_{r+1})$ whose first r parts are subsets of the r parts of the $K_r(t_r)$. In particular, the first part of this induced $K_{r+1}(t_r)$ is a subset of S. This proves the claimed inductive statement. The desired statement is just then the case $r = p-1$. $\square$

We are now ready to prove Theorem 1.1, whose statement we now recall.

Theorem 1.1. *If $n \geq 2^{k^{10p}}$, then $B_{k,n}$ is p-good, that is, $r(K_p, B_{k,n}) = (p-1)(n-1)+1$.*

Proof. Let $N = (p-1)(n-1)+1$. Our choice of n guarantees that if $z = k(20k)^p$, then $N \geq z^z \cdot r(K_p, K_z)$. Suppose for the sake of contradiction that there is a K_p-free graph on N vertices such that $\overline{\Gamma}$ does not contain a k-book with n vertices. By Lemma 2.3, applied with $S = V(\Gamma)$ and $x = k$, we see that Γ must contain $K_{p-1}(k)$ as an induced subgraph. But then Lemma 2.2 implies that $\overline{\Gamma}$ contains a k-book with n vertices as a subgraph, completing the proof. $\square$

3 A stability-supersaturation theorem

One of our main tools is a version of the Erdős–Simonovits stability version of Turán's theorem. While many variants of the stability theorem are known, we were not able to find the following result in the literature, though its proof is similar to the proofs of several known results. Roughly speaking, this result combines two types of well-known variants of Turán's theorem. The first, namely the Erdős–Simonovits stability theorem [15, 25], says that if Γ is a K_p-free graph with slightly fewer edges than the Turán graph, then Γ can be turned into the Turán graph by changing a small number of edges. The second, often known as a supersaturation result [17], says that if Γ is an m-vertex graph with slightly *more* edges than the K_p-free Turán graph, then it actually contains many (that is, $\Omega(m^p)$) copies of K_p. Contrapositively, this latter result says that if Γ has few copies of K_p, then it cannot have substantially more edges than the Turán graph.

The result that we need, a combination of the two mentioned above, is the following. It asserts that if Γ has slightly fewer edges than the Turán graph (the stability regime) and has few copies of K_p (the supersaturation regime), then it is close to the Turán graph.

Theorem 3.1. *For every $\varepsilon > 0$ and every integer $p \geq 3$, there exist $\eta, \gamma > 0$ such that the following holds for all $m \geq 5$. Suppose Γ is a graph on m vertices with minimum degree at least $(1-\frac{1}{p-1}-\gamma)m$ and at most ηm^p copies of K_p. Then $V(\Gamma)$ can be partitioned into $V_1 \sqcup \cdots \sqcup V_{p-1}$, such that the total number of internal edges in $V_1, \ldots, V_{p-1}$ is at most $\varepsilon\binom{m}{2}$.*

Moreover, we may take $\gamma = \min\{1/(2p^2), \varepsilon/2\}$ and $\eta = p^{-10p}\varepsilon$.

A natural approach to prove Theorem 3.1 is to first apply the celebrated graph removal lemma (see the survey [9]). This allows us to pass to a K_p-free subgraph Γ' of Γ which still has very many edges. At this point, we can apply the standard stability theorem to deduce that Γ' is nearly $(p-1)$-partite; since we deleted few edges to go from Γ to Γ', we must also have that Γ is nearly $(p-1)$-partite. This proof technique was used to prove [11, Corollary 3.4], which is a very similar result to Theorem 3.1. This proof technique actually proves a stronger theorem than Theorem 3.1, weakening the minimum degree condition to an average degree condition.

However, since the known bounds in the graph removal lemma are very weak, this proof technique would yield a tower-type dependence in the parameters ε and η in the statement of Theorem 3.1. Moreover, a super-polynomial dependence on the parameters is unavoidable if one only assumes an average degree condition. Indeed, let Γ be the disjoint union of a Turán graph on

$(1-\gamma)m$ vertices and a graph Γ_0 on γm vertices which is extremal for the K_p removal lemma, so that Γ has at least $(1-\frac{1}{p-1}-\gamma)\binom{m}{2}$ edges. Then the distance of Γ from being $(p-1)$-partite is roughly the same as the distance of Γ_0 from being K_p-free, and it is known that the clique removal lemma requires super-polynomial bounds in general. Such a construction shows that the clique removal lemma and stability-supersaturation theorems like Theorem 3.1 are very closely related.

The Γ constructed has high average degree but low minimum degree, and this distinction turns out to be crucial. Indeed, in [19], Fox and Wigderson proved that the K_p removal lemma has *linear* bounds if the minimum degree of Γ is above a certain threshold, namely $(1-\frac{2}{2p-3})m$. This allows us to prove Theorem 3.1 using the technique outlined above, while obtaining much stronger quantitative control.

The first tool we need to prove Theorem 3.1 is the high-degree removal lemma with linear bounds mentioned above, from [19, Theorem 2.1]. We remark that the explicit p-dependence of the constant is not given in [19, Theorem 2.1], but it is easy to verify that the proof yields the following result.

Theorem 3.2. *Let Γ be an m-vertex graph with with minimum degree at least $(1-\frac{2}{2p-3}+\beta)m$ and with at most $(10p)^{-2p}\beta\lambda m^p$ copies of K_p. Then Γ can be made K_p-free by deleting at most λm^2 edges.*

We also use the following quantitative form of the stability theorem, due to Füredi [20].

Theorem 3.3. *Let Γ be an m-vertex K_p-free graph with at least $(1-\frac{1}{p-1})\frac{m^2}{2}-\ell$ edges. Then Γ can be made $(p-1)$-partite by deleting at most ℓ edges.*

With these preliminaries, we can now prove Theorem 3.1.

Proof of Theorem 3.1. Since $\gamma \le 1/(2p^2)$, we see that

$$1-\frac{1}{p-1}-\gamma \ge 1-\frac{1}{p-1}-\frac{1}{2p^2} = 1-\frac{2}{2p-3}+\frac{5p-3}{4p^4-10p^3+6p^2} \ge 1-\frac{2}{2p-3}+\frac{1}{p^3}.$$

Therefore, we may apply Theorem 3.2 with $\beta = 1/p^3$. We also set $\lambda = \varepsilon/10$, and note that the number of K_p in Γ is at most

$$\eta m^p = p^{-10p}\varepsilon m^p \le (10p)^{-2p}\cdot\frac{1}{p^3}\cdot\frac{\varepsilon}{10}\cdot m^p = (10p)^{-2p}\beta\lambda m^p.$$

This implies that we may delete at most $\frac{\varepsilon}{10}m^2$ edges from Γ to obtain a K_p-free graph Γ'. Since Γ has minimum degree at least $(1-\frac{1}{p-1}-\gamma)n$, we see that Γ' has at least $(1-\frac{1}{p-1}-\gamma)\frac{m^2}{2}-\frac{\varepsilon}{10}m^2$ edges. Therefore, by Theorem 3.3, we see that Γ' can be made $(p-1)$-partite by deleting at most $(\frac{\gamma}{2}+\frac{\varepsilon}{10})m^2$ edges. Let Γ'' be this $(p-1)$-partite subgraph, and let $V_1\sqcup\cdots\sqcup V_{p-1}$ be its $(p-1)$-partition. Since each V_i is an independent set in Γ'', we see that the total number of edges of Γ contained in $V_1,\dots,V_{p-1}$ is at most

$$\frac{\varepsilon}{10}m^2+\left(\frac{\gamma}{2}+\frac{\varepsilon}{10}\right)m^2 \le \left(\frac{\varepsilon}{5}+\frac{\varepsilon}{4}\right)m^2 \le \varepsilon\binom{m}{2}$$

by our choice of $\gamma \le \varepsilon/2$ and $m \ge 5$. □

4 A blowup variant of the Andrásfai–Erdős–Sós theorem

The Andrásfai–Erdős–Sós theorem [1] is a minimum-degree stability version of Turán's theorem. It says that if an m-vertex K_p-free graph has minimum degree greater than $\frac{3p-7}{3p-4}m$, then it is $(p-1)$-partite; moreover, the constant $\frac{3p-7}{3p-4}$ is best possible. We need a related result, which says that if a graph has high minimum degree and does not contain some blowup of K_p, then it is $(p-1)$-partite. We remark that unlike Andrásfai, Erdős, and Sós, we do not obtain the exact minimum degree threshold for being $(p-1)$-partite; for more on such refined questions, see e.g. [21].

Theorem 4.1. *For every $p, t \geq 2$, every $1 = a_1 = a_2 \leq a_3 \leq \cdots \leq a_{p-1} \leq t$, there exist some $\gamma, \delta > 0$ such that if m is large enough in terms of a_{p-1} and p, $a_p \leq \delta m$, and Γ is a $K_p(a_1, a_2, \ldots, a_p)$-free graph on m vertices with minimum degree at least $(1 - 1/(p-1) - \gamma)m$, then Γ is $(p-1)$-partite.*

We need the following lemma, which is essentially due to Erdős [14].

Lemma 4.2. *For every $\eta > 0$ and $p, t \geq 2$, and $1 \leq a_1 \leq \cdots \leq a_{p-1} \leq t$, there exists some $\delta > 0$ such that the following holds for large enough m. If $a_p \leq \delta m$ and Γ is a $K_p(a_1, a_2, \ldots, a_p)$-free graph on m vertices, then Γ has at most ηm^p copies of K_p.*

Proof. We proceed by induction on p. The base case $p = 2$ just says that a $K_{a_1,\delta m}$-free graph has at most ηm^2 edges. We double-count the number of copies of K_{1,a_1} in Γ. On the one hand, every a_1-set has at most δm common neighbors, so there are at most $\delta m \binom{m}{a_1} < \delta m^{a_1+1}$ copies of K_{1,a_1}. On the other hand, a vertex of degree d contributes $\binom{d}{a_1}$ many copies. Therefore,

$$\delta m^{a_1+1} > \sum_{v \in V(G)} \binom{\deg(v)}{a_1} \geq m \binom{2e(\Gamma)/m}{a_1} \geq m \left(\frac{2e(\Gamma)}{e a_1 m} \right)^{a_1}$$

where the second inequality uses Jensen's inequality. Rearranging, we find that $e(\Gamma) < 2a_1 \delta^{1/a_1} m^2$. If we let $\delta = (\eta/(2a_1))^{a_1}$, this gives the desired result.

We now proceed with the inductive step. For every $(p-1)$-set of vertices S, let $\operatorname{ext}(S)$ denote the set of vertices v such that $S \cup \{v\}$ is a K_p. Note that the sum of $|\operatorname{ext}(S)|$ over all $(p-1)$-sets S is exactly p times the number of K_p in Γ. By assumption, this sum is therefore more than $p\eta m^p$. Thus, the average value of $|\operatorname{ext}(S)|$ is greater than $p\eta m^p / \binom{m}{p-1} > \eta m$. Again by convexity

$$\sum_{S \in \binom{V(\Gamma)}{p-1}} \binom{\operatorname{ext}(S)}{a_1} > \binom{m}{p-1} \binom{\eta m}{a_1}$$

Therefore, there is some a_1-set A such that the common neighborhood of A has at least

$$\binom{m}{p-1} \binom{\eta m}{a_1} \Big/ \binom{m}{a_1} \geq \eta' m^{p-1}$$

copies of K_{p-1}, for some η' depending only on η, t, and p. By induction, the common neighborhood of A must have a copy of $K_{p-1}(a_2, \ldots, a_p)$, which is a contradiction. $\square$

We can now prove Theorem 4.1.

Proof of Theorem 4.1. Fix some small $\varepsilon > 0$ depending on p and t. Let γ, η be the parameters given in Theorem 3.1, depending only on ε and p, and recall that $\gamma \le \varepsilon$. Finally, let $\delta > 0$ be the parameter in Lemma 4.2. By Lemma 4.2, we see that since Γ is a $K_p(a_1, a_2, \ldots, a_p)$-free graph on m vertices, it must have at most ηm^p copies of K_p, and it has minimum degree at least $(1 - 1/(p-1) - \gamma)m$ by assumption. Therefore, Theorem 3.1 implies that Γ has a partition into parts $V_1, \ldots, V_{p-1}$ such that the total number of internal edges is at most $\varepsilon\binom{m}{2}$. We fix such a partition with the minimum number of total internal edges. In particular, every vertex must have at least as many neighbors in every other part as it does in its own part.

Since Γ has minimum degree at least $(1-1/(p-1)-\gamma)m$, it must have at least $(1-1/(p-1)-\gamma)\frac{m^2}{2}$ edges. Therefore, since there are at most $\varepsilon\frac{m^2}{2}$ internal edges in $V_1, \ldots, V_{p-1}$, we must have that

$$\sum_{1 \le i < j \le p-1} e(V_i, V_j) \ge \left(1 - \frac{1}{p-1} - \gamma - \varepsilon\right)\frac{m^2}{2} \ge \left(1 - \frac{1}{p-1} - 2\varepsilon\right)\frac{m^2}{2} \tag{2}$$

since $\gamma \le \varepsilon$. We note that

$$\sum_{i=1}^{p-1}\left(|V_i| - \frac{m}{p-1}\right)^2 = \sum_{i=1}^{p-1}|V_i|^2 - \frac{2m}{p-1}\sum_{i=1}^{p-1}|V_i| + \frac{m^2}{p-1} = \sum_{i=1}^{p-1}|V_i|^2 - \frac{m^2}{p-1}. \tag{3}$$

Since the left-hand side of (3) is non-negative, we see that

$$\sum_{1 \le i < j \le p-1} |V_i||V_j| = \frac{1}{2}\left(m^2 - \sum_{i=1}^{p-1}|V_i|^2\right) \le \frac{1}{2}\left(m^2 - \frac{m^2}{p-1}\right) = \left(1 - \frac{1}{p-1}\right)\frac{m^2}{2}.$$

We can conclude from this that each V_i has cardinality $\frac{m}{p-1} \pm \sqrt{2\varepsilon}m$. For if not, then the left-hand side of (3) would be larger than $2\varepsilon m^2$, and the above computation would contradict (2).

Now, suppose that for some $1 \le a < b \le p-1$, we have that $e(V_a, V_b) < (1 - p^2\varepsilon)|V_a||V_b|$. Then we would find that

$$\sum_{1 \le i < j \le p-1} e(V_i, V_j) < \sum_{1 \le i < j \le p-1} |V_i||V_j| - p^2\varepsilon|V_a||V_b| \le \left(1 - \frac{1}{p-1} - 2\varepsilon\right)\frac{m^2}{2},$$

contradicting (2), using the bound $|V_a| \ge \frac{m}{p-1} - \sqrt{2\varepsilon}m \ge m/p$ for sufficiently small ε. Therefore, we find that for all $i \ne j$,

$$e(V_i, V_j) \ge (1 - p^2\varepsilon)|V_i||V_j|. \tag{4}$$

Now suppose that some vertex $v \in V_i$ has more than $2p^2\sqrt{\varepsilon}|V_i|$ neighbors in its own part V_i. By our assumption above, this means that v also has more than $2p^2\sqrt{\varepsilon}|V_i| \ge p^2\sqrt{\varepsilon}|V_j|$ neighbors in each part V_j for $j \ne i$, where we used the fact that $|V_j| = \frac{m}{p-1} \pm \sqrt{2\varepsilon}m$ and the fact that ε is sufficiently small to conclude that $|V_i| \ge \frac{1}{2}|V_j|$. Let $U_j = N(v) \cap V_j$ denote the neighbors of v in V_j. For every $1 \le a \ne b \le p-1$, we have by (4) that

$$e(U_a, U_b) \ge |U_a||U_b| - p^2\varepsilon|V_a||V_b| \ge \left(1 - \frac{p^2\varepsilon}{p^4\varepsilon}\right)|U_a||U_b| = \left(1 - \frac{1}{p^2}\right)|U_a||U_b|,$$

where the second inequality uses our assumption that $|U_a| \ge p^2\sqrt{\varepsilon}|V_a|$, and similarly for U_b. By the union bound, if we pick a random vertex from U_a for each $1 \le a \le p-1$, then they span a copy of K_{p-1} with probability at least $1 - \binom{p}{2}/p^2 \geq \frac{1}{2}$. Therefore, the neighborhood of v contains at least

$$\frac{1}{2}\prod_{a=1}^{p-1}|U_a| \geq \frac{(p^2\sqrt{\varepsilon})^{p-1}}{2}\prod_{a=1}^{p-1}|V_a| \geq \frac{(p^2\sqrt{\varepsilon})^{p-1}}{2p^{p-1}}m^{p-1} = \eta' m^{p-1}$$

copies of K_{p-1}, for η' depending only on p and ε. By Lemma 4.2, this implies that if δ is sufficiently small in terms of p, t, and ε, then the neighborhood of v contains a copy of $K_{p-1}(a_2, \dots, a_p)$. Since $a_1 = 1$, this implies that Γ contains a copy of $K_p(a_1, a_2, \dots, a_p)$, which is a contradiction. Thus, we conclude that every vertex $v \in V_i$ has at most $2p^2\sqrt{\varepsilon}|V_i|$ neighbors in its own part V_i, for every $1 \leq i \leq p-1$.

We now claim that for every $1 \leq i \neq j \leq p-1$, every vertex $v \in V_i$ has at least $(1-1/(2pt))|V_j|$ neighbors in V_j, as long as ε is sufficiently small in terms of p and t. Indeed, if not, then v has at least $|V_j|/(2pt)$ non-neighbors in V_j, and at least $(1-2p^2\sqrt{\varepsilon})|V_i| - 1$ non-neighbors in V_i. In total, the number of non-neighbors of v is at least

$$\frac{1}{2pt}|V_j| + (1-2p^2\sqrt{\varepsilon})|V_i| - 1 > \left(1 + \frac{1}{2pt} - 2p^2\sqrt{\varepsilon} - 4\sqrt{\varepsilon}\right)\frac{m}{p-1} > \left(\frac{1}{p-1} + \varepsilon\right)m,$$

as long as ε is sufficiently small in terms of p and t. This contradicts the assumption that the minimum degree of Γ is at least $(1 - 1/(p-1) - \varepsilon)m$.

Now suppose that there is some edge vw inside some part V_i, and assume without loss of generality that $i = 1$. The vertices v and w have at least $(1-1/(pt))|V_2| > a_3$ common neighbors in V_2, so we may pick some set of a_3 common neighbors in V_2. Then v, w, and these a_3 common neighbors have at least $(1-(a_3+2)/(2pt))|V_3| > (1-2t/(2pt))|V_3| > a_4$ common neighbors in V_3, so we may pick a_4 such common neighbors in V_3. Continuing in this way, we can greedily pick a_{j+1} vertices from V_j which are common neighbors of the previously chosen vertices, for each $j \leq p-2$. Having done this, we have picked at most pt vertices, so they still have at least $(1-pt/(2pt))|V_{p-1}| = \frac{1}{2}|V_{p-1}| > \delta m$ common neighbors in V_{p-1}. Thus, we have built a copy of $K_p(a_1, \dots, a_p)$ in Γ, a contradiction. This shows that there can be no edge inside any V_i, and thus that Γ is $(p-1)$-partite. $\square$

5 Proof of Theorem 1.3

In this section, we prove Theorem 1.3. Recall the statement: for any $k, p, t \geq 2$, there exists $\delta > 0$ such that the following holds. If n is large enough in terms of k, p and t, $1 \leq a_1 \leq a_2 \leq \cdots \leq a_{p-1} \leq t \leq a_p = \delta n$, $G = K_p(a_1, a_2, \dots, a_p)$, and $H = B_{k,n}$, then $r(G, H) = (p-1)(n-1) + a_1$ if and only if $a_1 = a_2 = 1$. Here $K_p(a_1, a_2, \dots, a_p)$ is the complete p-partite graph with part sizes $a_1, \dots, a_p$.

We start with the construction for the "only if" direction.

Proof of "only if" direction of Theorem 1.3. It suffices to show that if $a_2 \geq 2$, then $r(G, H) > (p-1)(n-1) + a_1$.

Let Γ be a graph on $N = (p-1)(n-1) + a_1$ vertices which are divided into $p-1$ parts $U_1, \dots, U_{p-1}$ with $|U_1| = n + a_1 - 1$ and $|U_2| = \cdots = |U_{p-1}| = n-1$. The edges of Γ are defined as follows. First, all pairs of vertices in two different parts are adjacent. Second, U_1 induces a C_4-free subgraph $A = \Gamma[U_1]$ which is almost a_1-regular. This means that either A is a_1-regular (if $|U_1|$ or

a_1 is even), or else all but one vertices of A have degree a_1 and one vertex has degree $a_1 - 1$ (if $|U_1|$ and a_1 are both odd). Such a graph A always exists if n is large enough in terms of a_1.

It remains to show that Γ is G-free and $\overline{\Gamma}$ is H-free. Suppose Γ contained a copy of G. Since G is complete p-partite and $U_2, \dots, U_{p-1}$ are independent sets of Γ, each of these sets can contain only vertices from one part of this copy of G. Thus, at least two parts of G must be entirely contained inside U_1, which means that $\Gamma[U_1]$ must contain a copy of the complete bipartite graph K_{a_1,a_2}. By construction, $A = \Gamma[U_1]$ is C_4-free, so this is impossible unless $a_1 = 1$. When $a_1 = 1$, A has maximum degree 1 and thus cannot contain a copy of K_{a_1,a_2} either, when $a_2 \geq 2$. In all cases, Γ is G-free.

The complement $\overline{\Gamma}$ is a disjoint union of $\overline{A}$ and $p-2$ copies of K_{n-1}. The book H is connected and has n vertices, so K_{n-1} cannot contain a copy of H. Also, $k \geq 2$, so H contains at least two vertices of degree $n-1$, whereas $\overline{A}$ has either one or zero vertices of degree at least $n-1$. It follows that $\overline{A}$ contains no copies of H either, completing the proof. $\square$

The proof of the "if" direction of Theorem 1.3 divides into three parts. First, we show Lemma 5.1 below (which uses Lemma 2.3) that under the assumptions of the theorem, we can assume that most vertices of Γ have degree at least $(1 - 1/(p-1) - o(1))N$. We then apply Theorem 4.1, which proves that except for the small number of low-degree vertices, Γ is $(p-1)$-partite. Finally, we use a careful averaging argument to show that under these assumptions, $\overline{\Gamma}$ must contain a copy of $H = B_{k,n}$, completing the proof.

We begin by proving that most vertices of Γ have high degree.

Lemma 5.1. *Under the assumptions of Theorem 1.3, if $a_1 = a_2 = 1$, then the following holds for any $\alpha > 0$, assuming that n is sufficiently large in terms of α. If Γ is a graph on $N = (p-1)(n-1)+1$ vertices such that Γ is G-free and $\overline{\Gamma}$ is H-free, then at most αN vertices of Γ have degree at most $d = (1 - 1/(p-1) - \alpha)N$.*

Proof. Let $\varepsilon = \varepsilon(k, \alpha) > 0$ be small in terms of k and α, and let $x = x(k, \varepsilon) \geq 1$ be large in terms of k and ε. We also assume that $\delta = \delta(k, p, t) > 0$ is chosen to be small in terms of ε and α.

Let $S \subset V(\Gamma)$ be the set of vertices of degree less than d. We proceed by contradiction and assume that $|S| \geq \alpha N$. By Lemma 2.3, there is an induced copy of $K_{p-1}(x)$ in Γ whose parts are $V_1, \dots, V_{p-1}$ with $V_1 \subseteq S$. Thus, all the vertices of V_1 have degree less than d. We remark that Lemma 2.3 requires $|S|$ to be double-exponentially large in p, so we require n to be at least double-exponentially large in p for this step. This is the only place where a double-exponential dependence is needed.

Partition the vertices of Γ into p parts $U_0, \dots, U_{p-1}$, where for $i \geq 1$, each vertex in U_i has at most εx neighbors in V_i, and U_0 consists of all vertices more than εx neighbors in each V_i.

First, if $|U_0| \geq (e/\varepsilon)^{tp} \cdot \delta n$, then we can find a copy of G in Γ as follows. If we pick a set W by taking a_i vertices uniformly at random from V_i for $i = 1, \dots, p-1$, then the expected number of vertices of U_0 complete to W is at least

$$\prod_{i=1}^{p-1} \frac{\binom{\varepsilon x}{a_i}}{\binom{x}{a_i}} \cdot |U_0| \geq (\varepsilon/e)^{tp} \cdot |U_0| \geq \delta n.$$

Thus, there exists a W for which we can find δn vertices of U_0 which together with W form a copy of $G = K_p(a_1, \dots, a_p)$, which is impossible.

Next, suppose $|U_i| \geq (1 - 2k\varepsilon)^{-1}(n - k)$ for some $i \geq 1$. Every vertex in U_i has at most εx neighbors in V_i, so we may remove half the vertices of V_i (the ones with highest degree to U_i)

to find a subset V_i' such that every $v \in V_i'$ has at most $2\varepsilon|U_i|$ neighbors in U_i. Take W to be any k-subset of V_i', and let $U_i' \subseteq U_i$ be the set of vertices with no neighbors in W. We have $|U_i'| \geq (1-2k\varepsilon)|U_i| \geq n-k$, and so W and U_i' form a copy of $B_{k,n}$ in $\overline{\Gamma}$, which is again impossible. We henceforth assume that $|U_i| < (1-2k\varepsilon)^{-1}(n-k)$ for all $i \geq 1$.

Finally, suppose $|U_1| \geq (1-2k\varepsilon)^{-1}(n-k) - (\alpha/10)^k N$. We seek to find a copy of $B_{k,n}$ in $\overline{\Gamma}$ again, this time using the degree condition on V_1. As before, we may pass to a subset V_1' of half the vertices of V_1 such that each has at least $(1-2\varepsilon)|U_1|$ non-neighbors in U_1. Each vertex of V_1' has degree at most $d = (1-1/(p-1)-\alpha)N$, and so has at least $N-1-d = N/(p-1)+\alpha N - 1 \geq n + \alpha N - 2$ non-neighbors in total. In particular, since $|U_1| < (1-2k\varepsilon)^{-1}(n-k) < (1+3k\varepsilon)n-2$ and $\alpha \geq 6k\varepsilon$, each vertex of V_1' has at least $(n+\alpha N-2) - |U_1| \geq \alpha N/2$ non-neighbors in $\overline{U_1}$.

Pick a random k-subset W of V_1' to form the spine of the book. The number of common non-neighbors the vertices of W have inside U_1 is at least $(1-2k\varepsilon)|U_1|$. We now count the expected number of common non-neighbors the vertices of W have in $\overline{U_1}$. For the convenience of the following calculation, we insert phantom vertices to $\overline{U_1}$, each complete to W, until $|\overline{U_1}| = N$; this has no effect on the common non-neighborhood we care about. If $u \in \overline{U_1}$ has y non-neighbors in V_1', then the probability that W is chosen entirely among these y vertices is $\binom{y}{k}/\binom{x/2}{k}$. Since vertices in V_1' have at most $(1-\alpha/2)N$ neighbors in $\overline{U_1}$, the average value of y over a random $u \in \overline{U_1}$ is at least $\alpha/2 \cdot |V_1'| = \alpha x/4$. By linearity of expectation and convexity we find that the expected number of common non-neighbors of W in $\overline{U_1}$ is at least

$$\binom{\alpha x/4}{k}\binom{x/2}{k}^{-1}|\overline{U_1}| \geq \left(\frac{\alpha x}{4k}\right)^k \left(\frac{2k}{ex}\right)^k |\overline{U_1}| \geq (\alpha/(2e))^k \cdot (N/2) \geq (\alpha/10)^k N.$$

Thus, there exists some particular W with at least $(1-2\varepsilon)|U_1| + (\alpha/10)^k N \geq n-k$ non-neighbors, forming the desired $B_{k,n}$ in $\overline{\Gamma}$. This contradicts our assumptions on Γ.

We conclude that the partition $V(\Gamma) = U_0 \sqcup \cdots \sqcup U_{p-1}$ satisfies

$$\begin{aligned}
|U_0| &< (e/\varepsilon)^{tp} \cdot \delta n \\
|U_1| &< \frac{n-k}{1-2k\varepsilon} - (\alpha/10)^k N \\
|U_i| &< \frac{n-k}{1-2k\varepsilon} \text{ if } i \geq 2.
\end{aligned}$$

Adding these together, we obtain that the number N of vertices in Γ is

$$\begin{aligned}
\sum_{i=0}^{p-1} |U_i| &< (e/\varepsilon)^{tp} \cdot \delta n + (p-1)\frac{n-k}{1-2k\varepsilon} - (\alpha/10)^k N \\
&< (1+3k\varepsilon)N + (e/\varepsilon)^{tp} \cdot \delta n - (\alpha/10)^k N \\
&< N,
\end{aligned}$$

if ε is small enough compared to k and α, and δ is small enough compared to ε, t, and p. This is a contradiction and we are done. □

We now have all the tools to complete the proof.

Proof of "if" direction of Theorem 1.3. Recall that $H = B_{k,n}$, and $G = K_p(a_1, \ldots, a_p)$, where $1 \leq a_1 \leq \cdots \leq a_{p-1} \leq t$, $a_p = \delta n$, and n is sufficiently large in terms of t, k, and p. We are given a G-free graph Γ on $N = (p-1)(n-1)+1$ vertices, and we wish to show that $\overline{\Gamma}$ contains a copy of H. We have already proved in Lemma 5.1 that at most αN vertices of Γ have degree at most $d = (1 - 1/(p-1) - \alpha)N$, for any fixed $\alpha > 0$ and sufficiently large N. If we let T be the set of vertices of degree greater than d, then the induced subgraph $\Gamma[T]$ has at least $(1-\alpha)N$ vertices and thus minimum degree at least $(1 - 1/(p-1) - 2\alpha)|T|$. Applying Theorem 4.1 to the graph $\Gamma[T]$, we find that as long as α is sufficiently small in terms of p and t, we have that $\Gamma[T]$ is $(p-1)$-partite. Let the parts of $\Gamma[T]$ be $T_1, \ldots, T_{p-1}$. We now argue roughly as in the proof of Theorem 4.1.

Recall that for a vertex v and a vertex set W, we denote by $d(v, W)$ the *density* of v to W, namely the number of neighbors of v in W divided by $|W|$.

Claim 5.2. *Let $T_1, \ldots, T_{p-1}$ be as defined above. Let $\xi = 4p^2\alpha$. Then for every $1 \le i \neq j \le p-1$, we have that*

$$\left(\frac{1}{p-1} - \xi\right) N \le |T_i| \le \left(\frac{1}{p-1} + \xi\right) N \tag{5}$$

and

$$d(w, T_j) \ge 1 - \xi \text{ for every } w \in T_i. \tag{6}$$

Proof. Since T_i is an independent set, every vertex in T_i has degree at most $N - |T_i|$. Since every vertex in T_i has degree at least d, this implies that $|T_i| \le N - d = (1/(p-1) + \alpha)N$. Since $T_1, \ldots, T_{p-1}$ partition T, which has size at least $(1-\alpha)N$, this implies that $|T_i| = |T| - \sum_{j \neq i} |T_j| \ge (1/(p-1) - p\alpha)N$, which proves (5) since $p\alpha < \xi$.

For (6), we recall that the induced subgraph $\Gamma[T]$ has minimum degree at least $(1 - 1/(p-1) - 2\alpha)|T|$. So any $w \in T_i$ has at most $(1/(p-1) + 2\alpha)|T|$ non-neighbors in T. Additionally, since T_i is an independent set, every $w \in T_i$ has $|T_i| - 1$ non-neighbors in T_i. If $d(w, T_j) < 1 - \xi$, then the total number of non-neighbors of w is at least

$$\xi|T_j| + |T_i| - 1 \ge (1+\xi)\left(\frac{1}{p-1} - p\alpha\right) N - 1 > \left(\frac{1+\xi}{p-1} - 2p\alpha\right) N > \left(\frac{1}{p-1} + 2\alpha\right)|T|,$$

using the computations above and our choice of $\xi = 4p^2\alpha$. This is a contradiction. $\square$

Let S be the complement of T, i.e. the set of vertices in Γ with degree less than d, and recall that $|S| \le \alpha N$.

Claim 5.3. *Let $\zeta = pt\xi = 4p^3t\alpha$. For every $v \in S$, at least one of the following is true. Either v has no edges to some T_i, or else $d(v, T_i) < \zeta$ for at least two different choices of $i \in [p-1]$.*

Proof. Suppose for contradiction that this is false for some $v \in S$. Thus, $d(v, T_i) \ge \zeta$ for all but at most one choice of $i \in [p-1]$, and additionally v has a neighbor in each T_i. By relabeling the parts, we may assume that $d(v, T_i) \ge \zeta$ for all $i \in [p-2]$. Let w be a neighbor of v in T_{p-1}. By (6), we see that v and w have at least $(\zeta - \xi)|T_1| > \xi|T_1| > a_3$ common neighbors in T_1, for N sufficiently large. Pick any a_3 common neighbors in T_1. Then v, w, and these a_3 common neighbors have at least $(\zeta - (a_3+1)\xi)|T_2| > \xi|T_2| > a_4$ common neighbors in T_2. Continuing in this way, we can pick out a_i vertices in T_{i+2} which are common neighbors of all previously-chosen vertices. At the end of this process, we can still pick at least $(\zeta - (p-1)t\xi)|T_{p-2}| \ge \delta n$ common neighbors in T_{p-2}, and thus we can build a copy of G, contradicting our assumption that Γ is G-free. $\square$

We partition S into $S_1 \cup S_2$, where S_1 consists of all vertices in S that are empty to some part T_i, and S_2 consists of the remaining vertices v, namely those satisfying $d(v, T_i) < \zeta$ for at least two choices of $1 \leq i \leq p-1$.

Now, we pick an index $i \in [p-1]$ uniformly at random, and then pick a k-set $Q \subset V_i$ uniformly at random. By doing so, we obtain a (non-uniform) distribution on the set of k-cliques in $\overline{\Gamma}$. For a vertex $v \in V(\Gamma)$, let us say that v *extends* Q if $Q \cup \{v\}$ is also a clique in $\overline{\Gamma}$, or equivalently if v is not adjacent in Γ to any vertex of Q. Note that if $v \in Q$, then we still say that v extends Q, even though this is not really an extension per se. We observe that if $v \in T$, then the probability that v extends Q is at least $1/(p-1)$. Indeed, the probability that v extends Q is at least the probability that $v \in T_i$ for the randomly chosen index i, which is exactly $1/(p-1)$ since we pick the index i uniformly at random.

Next, if $v \in S_1$, then we again have that the probability that v extends Q is at least $1/(p-1)$. Indeed, if $v \in S_1$, then v has no edges to T_j for at least one index j. The probability that v extends Q is then at least the probability that j is the randomly chosen index, which equals $1/(p-1)$.

Finally, if $v \in S_2$, then without loss of generality, $d(v, T_1) < \zeta$ and $d(v, T_2) < \zeta$. If the randomly chosen index i is 1 or 2, then the probability that v has an edge to Q is at most $k\zeta$, by the union bound. Therefore, if $v \in S_2$, then

$$\Pr(v \text{ extends } Q) \geq \frac{2}{p-1} \cdot (1 - k\zeta) \geq \frac{1}{p-1},$$

since we may pick α sufficiently small so that $k\zeta \leq 1/2$. By putting all of this together, we find that $\Pr(v \text{ extends } Q) \geq 1/(p-1)$ for every vertex $v \in V(\Gamma)$. By linearity of expectation, this implies that

$$\mathbb{E}[|\{v : v \text{ extends } Q\}|] = \sum_{v \in V(\Gamma)} \Pr(v \text{ extends } Q) \geq \frac{N}{p-1}$$

Therefore, there exists some clique Q in $\overline{\Gamma}$ which has at least $\lceil N/(p-1) \rceil = n$ extensions. Since exactly k of these extensions are the degenerate ones coming from vertices in Q itself, we find that $\overline{\Gamma}$ contains a copy of $H = B_{k,n}$. This completes the proof. $\square$

6 Concluding remarks

In this section we collect a few of the tantalizing open questions remaining in this area.

Removing regularity. Note that the full Ramsey goodness results of Nikiforov and Rousseau [23] hold in greater generality than our results Theorem 1.2 and Theorem 1.3. However, due to the dependence of their arguments on Szemerédi's regularity lemma, the quantitative dependence between the graph sizes involved are tower-type. It would be interesting to find a direct proof of their goodness results without regularity, as this would likely lead to superior quantitative bounds.

Near Ramsey goodness. In Theorem 1.3, we study the Ramsey number $r(K_p(a_1, \ldots, a_p), B_{k,n})$ for sufficiently large n, where $a_1, \ldots, a_{p-1}$ are fixed and $a_p \leq \delta n$ for some absolute constant $\delta > 0$. We are able to determine this Ramsey number in the case $a_1 = a_2 = 1$ (in which case the answer is given by the Ramsey goodness bound), but it is natural to ask what happens for larger values of a_1 and a_2. In this case, there is a natural lower bound, generalizing the proof of the "only if" direction of Theorem 1.3, and which shows a surprising connection to an analogue of the classical extremal problem for complete bipartite graphs. To explain this connection, we first define the following Dirac-type extremal function.

Definition 6.1. Given a graph H and integers k, n, let $d_k(n, H)$ be the maximum d for which there is an $(n+d-1)$-vertex H-free graph, at most $k-1$ vertices of which have degree less than d.

Now let $d = d_k(n, K_{a_1,a_2})$, and let Γ_0 be a K_{a_1,a_2}-free graph on $n+d-1$ vertices, at most $k-1$ of which have degree less than d. Let Γ be a graph with $N = (p-1)(n-1)+d$ vertices, whose vertex set is divided into $p-1$ parts $U_1, \dots, U_{p-1}$ with $|U_1| = n+d-1$ and $|U_2| = \cdots = |U_{p-1}| = n-1$, such that $\Gamma[U_1]$ is isomorphic to Γ_0, and such that all pairs of vertices in different parts are adjacent. Then Γ is $K_p(a_1, \dots, a_p)$-free, since $U_2, \dots, U_{p-1}$ are independent sets, and $\Gamma[U_1]$ is K_{a_1,a_2}-free. Additionally, $\overline{\Gamma}$ is a disjoint union of $\overline{\Gamma_0}$ and $p-2$ cliques of order $n-1$. The cliques are too small to contain a copy of $B_{k,n}$, and all but at most $k-1$ vertices of $\overline{\Gamma_0}$ have degree at most $(n+d-1)-1-d = n-2$. Since $B_{k,n}$ has k vertices of degree $n-1$, this shows that $\overline{\Gamma}$ is $B_{k,n}$-free. Thus, we conclude that

$$r(K_p(a_1, \dots, a_p), B_{k,n}) > (p-1)(n-1) + d_k(n, K_{a_1,a_2}). \tag{7}$$

Our proof of the "only if" direction of Theorem 1.3 used the same argument, and we simply noted that if $a_2 > 1$, then for sufficiently large n, we have $d_k(n, K_{a_1,a_2}) \geq a_1$ for all $k \geq 2$. We conjecture that the lower bound (7) is tight for sufficiently large n, if $a_1, \dots, a_{p-1}$ are fixed, and $a_p \leq \delta n$.

Conjecture 6.2. *For all integers $k, p, t \geq 2$, there exists some $\delta > 0$ such that the following holds for all $n \geq 1$. For positive integers $a_1 \leq \cdots \leq a_{p-1} \leq t$ and $a_p \leq \delta n$, we have*

$$r(K_p(a_1, \dots, a_p), B_{k,n}) = (p-1)(n-1) + d_k(n, K_{a_1,a_2}) + 1.$$

Thus, Theorem 1.3 verifies Conjecture 6.2 in the case $a_1 = a_2 = 1$.

Disconnected graphs. Ramsey goodness results are some of the rare examples in graph Ramsey theory where exact values of Ramsey numbers are known. Another such example is an old result of Burr, Erdős, and Spencer [5], recently improved by Bucić and Sudakov [2], which shows

$$r(nG, nG) = 2(|G| - \alpha(G))n + c$$

for n sufficiently large and some constant $c = c(G)$. Here, G is a fixed graph, nG is a vertex disjoint union of n copies of G, and $\alpha(G)$ is the independence number of G. Does there exist a theory of Ramsey goodness for disconnected graphs, giving a common generalization of the Burr–Erdős–Spencer result and our theorems?

Empty pairs in triangle-free graphs. Motivated by a well-studied approach to the famous Erdős–Hajnal conjecture, the following conjecture was proposed by Conlon, Fox, and Sudakov.

Conjecture 6.3 ([10, Conjecture 3.14])**.** *There exists some $\varepsilon > 0$ such that every N-vertex triangle complete graph contains two vertex subsets A, B with $|A| \geq \varepsilon N$, $|B| \geq N^{\varepsilon}$, and with no edges between A and B.*

For more on this conjecture and its variants, see also [6]. Conjecture 6.3 remains open. The strongest result in this direction, due independently to Fox and Shapira (unpublished) says that one may take $|A| \geq \varepsilon N$ and $|B| \geq \varepsilon \log N / \log \log N$. One consequence of Theorem 1.1 is that we may take $|A| \geq \varepsilon N$ and $|B| \geq (\log N)^{\varepsilon}$, for $\varepsilon = 1/31$. Indeed, Theorem 1.1 with $p = 3$ says that if $n \geq 2^{k^{10p}} = 2^{k^{30}}$ and if $N = (p-1)(n-1)+1 = 2n-1$, then for every N-vertex triangle-free graph Γ, its complement $\overline{\Gamma}$ contains a copy of $B_{k,n}$. Let A be the set of leaves of this book and B be its spine, so that $|A| = n \geq N/31$ and $|B| = k \geq (\log N)^{1/31}$. Since $A \cup B$ span a book in $\overline{\Gamma}$, there are no edges between A and B in Γ.

By the same argument, we see that improving the bounds in Theorem 1.1 could yield progress on Conjecture 6.3. For example, improving the bound $n \geq 2^{k^{10p}}$ in Theorem 1.1 to a bound that is single-exponential in both k and p would allow one to take $|A| \geq \varepsilon N$ and $|B| \geq \varepsilon \log N$ in Conjecture 6.3.

Ramsey goodness threshold. More generally, it is natural to ask what the "Ramsey goodness threshold" is in Theorem 1.1. That is, what is the smallest n (in terms of k and p) such that $r(K_p, B_{k,n}) = (p-1)(n-1)+1$? A simple random construction shows that this threshold is at least $(k/\log p)^{cp}$, for an absolute constant $c > 0$. Indeed, let $n = (k/\log p)^{cp}$ and $N = (p-1)(n-1)+1$, and let Γ be an Erdős–Rényi random graph on N vertices with edge probability[2] $C(\log p)/k$, for an absolute constant $C > 0$. Then a first moment estimate shows that with positive probability, Γ does not contain a copy of K_p and its complement does not contain a copy of $B_{k,n}$.

However, there remains a rather large gap between the lower bound of $(k/\log p)^{cp}$ and the upper bound of $2^{k^{10p}}$ for this threshold. In particular, it would be interesting to determine if, for p fixed, the correct behavior is polynomial or exponential in k.

References

[1] B. Andrásfai, P. Erdős, and V. T. Sós, On the connection between chromatic number, maximal clique minimal degree of a graph, *Discrete Math.* **8** (1974), 205–218.

[2] M. Bucić and B. Sudakov, Tight Ramsey bounds for multiple copies of a graph, 2021. Preprint available at arXiv:2108.11946.

[3] S. A. Burr, Ramsey numbers involving graphs with long suspended paths, *J. London Math. Soc.* **24** (1981), 405–413.

[4] S. A. Burr and P. Erdős, Generalizations of a Ramsey-theoretic result of Chvátal, *J. Graph Theory* **7** (1983), 39–51.

[5] S. A. Burr, P. Erdős, and J. H. Spencer, Ramsey theorems for multiple copies of graphs, *Trans. Amer. Math. Soc.* **209** (1975), 87–99.

[6] M. Chudnovsky, J. Fox, A. Scott, P. Seymour, and S. Spirkl, Pure pairs. III. Sparse graphs with no polynomial-sized anticomplete pairs, *J. Graph Theory* **95** (2020), 315–340.

[7] V. Chvátal, Tree-complete graph Ramsey numbers, *J. Graph Theory* **1** (1977), 93.

[2] If the quantity $C(\log p)/k$ is greater than 1, then the result we are trying to prove is vacuously true, since $(k/\log p)^{cp}$ is then less than 1. Thus we may assume that this is a valid edge probability.

[8] D. Conlon, The Ramsey number of books, *Adv. Comb.* (2019), Paper No. 3, 12pp.

[9] D. Conlon and J. Fox, Graph removal lemmas, in *Surveys in combinatorics 2013*, *London Math. Soc. Lecture Note Ser.*, vol. 409, Cambridge Univ. Press, Cambridge, 2013, pp. 1–49.

[10] D. Conlon, J. Fox, and B. Sudakov, Recent developments in graph Ramsey theory, in *Surveys in combinatorics 2015, London Math. Soc. Lecture Note Ser.*, vol. 424, Cambridge Univ. Press, Cambridge, 2015, pp. 49–118.

[11] D. Conlon, J. Fox, and B. Sudakov, Short proofs of some extremal results III, *Random Structures Algorithms* **57** (2020), 958–982.

[12] D. Conlon, J. Fox, and Y. Wigderson, Ramsey numbers of books and quasirandomness, *Combinatorica*, to appear.

[13] P. Erdős, Some remarks on the theory of graphs, *Bull. Amer. Math. Soc.* **53** (1947), 292–294.

[14] P. Erdős, On extremal problems of graphs and generalized graphs, *Israel J. Math.* **2** (1964), 183–190.

[15] P. Erdős, On some new inequalities concerning extremal properties of graphs, in *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, Academic Press, New York, 1968, pp. 77–81.

[16] P. Erdős, R. J. Faudree, C. C. Rousseau, and R. H. Schelp, The size Ramsey number, *Period. Math. Hungar.* **9** (1978), 145–161.

[17] P. Erdős and M. Simonovits, Supersaturated graphs and hypergraphs, *Combinatorica* **3** (1983), 181–192.

[18] P. Erdős and G. Szekeres, A combinatorial problem in geometry, *Compos. Math.* **2** (1935), 463–470.

[19] J. Fox and Y. Wigderson, Minimum degree and the graph removal lemma, 2021. Preprint available at arXiv:2105.09194.

[20] Z. Füredi, A proof of the stability of extremal graphs, Simonovits' stability from Szemerédi's regularity, *J. Combin. Theory Ser. B* **115** (2015), 66–71.

[21] F. Illingworth, Minimum degree stability of H-free graphs, 2021. Preprint available at arXiv:2102.11104.

[22] V. Nikiforov and C. C. Rousseau, Large generalized books are p-good, *J. Combin. Theory Ser. B* **92** (2004), 85–97.

[23] V. Nikiforov and C. C. Rousseau, Ramsey goodness and beyond, *Combinatorica* **29** (2009), 227–262.

[24] F. P. Ramsey, On a problem of formal logic, *Proc. London Math. Soc (2)* **30** (1930), 264–286

[25] M. Simonovits, A method for solving extremal problems in graph theory, stability problems, in *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, Academic Press, New York, 1968, pp. 279–319.