Ramsey numbers of books and quasirandomness

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Joint work with David Conlon and Jacob Fox

“Outside of a dog, a book is a man’s best friend. Inside of a dog, it is too dark to read.”
–Groucho Marx

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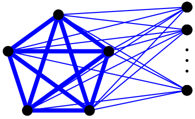

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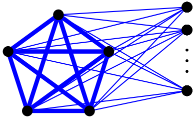

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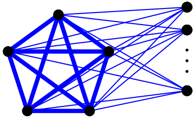

Previous observation: if $n = r(K_t, K_{t-k})$, then $r(K_t) \le r(B_n^{(k)})$.

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A coloring is quasirandom iff at most $o_k(N^k)$ monochromatic K_ks have more than $2^{-k}N + o_k(N)$ extensions to a monochromatic K_{k+1}.

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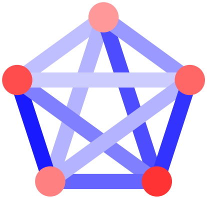
A (k, ε, δ)-**good configuration** is a tuple of disjoint vertex sets $C_1, \ldots, C_k$ such that each pair (C_i, C_j) is ε-regular, $d_R(C_i) \geq \delta$, and $d_B(C_i, C_j) \geq \delta$ for $i \neq j$.



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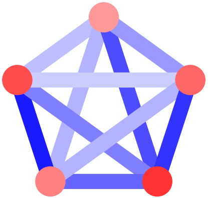

Lemma (Conlon '18, Conlon–Fox–W '19)

Let $N = (2^k + \beta)n$ and $\varepsilon \ll \delta \ll \beta$. If a coloring of K_N has a (k,ε,δ)-good configuration, then it contains a monochromatic $B_n^{(k)}$.

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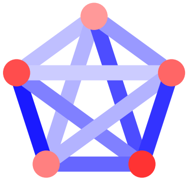

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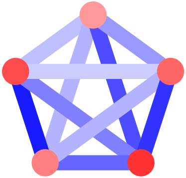

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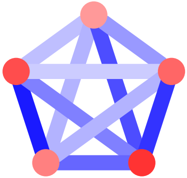

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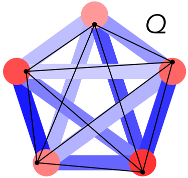

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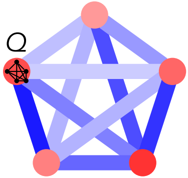

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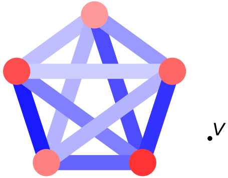

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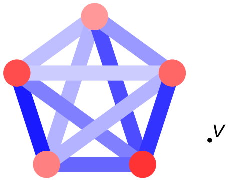

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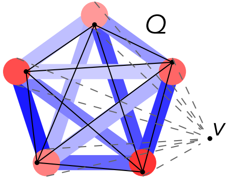

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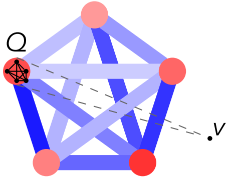

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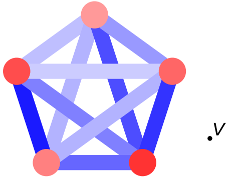

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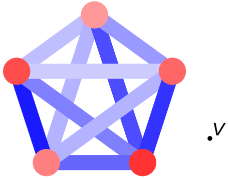

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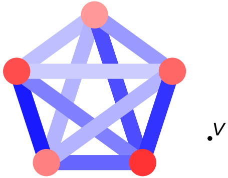

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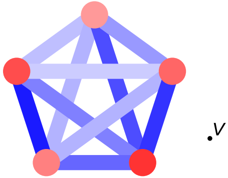

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Adding up over all v shows that Q has $\gtrsim 2^{-k}N$ monochromatic extensions on average $\implies$ monochromatic $B_n^{(k)}$.

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Instead, we use much weaker partitioning results, and thus have to work much harder to find a good configuration.

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So if some x_i is far from $\frac{1}{2}$, then $f(x_1,\dots,x_k) \geq 2^{-k} + c$.
Strengthen the key lemma: if some C_i is not ε-regular to the rest of the graph, we can find a monochromatic $B^{(k)}_{(2^{-k}+c)N}$.

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Recall the inequality

$$f(x_1,\ldots,x_k) := \frac{1}{2}\left(\prod_{i=1}^{k} x_i + \frac{1}{k}\sum_{i=1}^{k}(1-x_i)^k\right) \geq 2^{-k}.$$

For $k \geq 3$, $(\frac{1}{2},\ldots,\frac{1}{2})$ is the unique minimizer of f on $[0,1]^k$.
So if some x_i is far from $\frac{1}{2}$, then $f(x_1,\ldots,x_k) \geq 2^{-k} + c$.
Strengthen the key lemma: if some C_i is not ε-regular to the rest of the graph, we can find a monochromatic $B^{(k)}_{(2^{-k}+c)N}$.

Inductively apply the previous argument to "nibble" out these ε-regular pieces, and conclude quasirandomness. □

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  - Sós conjectured that every extremal coloring with no monochromatic K_t is quasirandom; we prove the book analogue.
- Off-diagonal $r(B_n^{(k)}, B_{cn}^{(k)})$ for fixed $c \in (0, 1)$. We can prove analoguous results when k is sufficiently large.

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- Thomason conjectured that $r(B_n^{(k)}) \leq 2^k(n+k-2)+2$.