Ramsey numbers of books and quasirandomness

Yuval Wigderson

Joint work with David Conlon and Jacob Fox

"Outside of a dog, a book is a man's best friend. Inside of a dog, it is too dark to read."
–Groucho Marx

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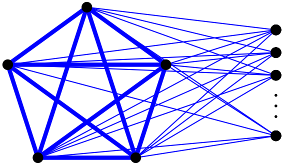

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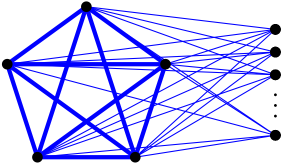

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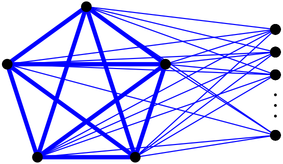

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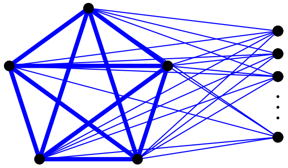

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Conlon: $r(K_t) \leq t^{-c \log t / \log \log t} 4^t$.

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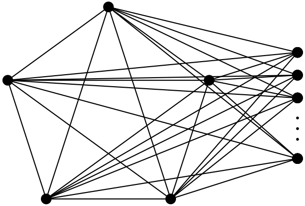

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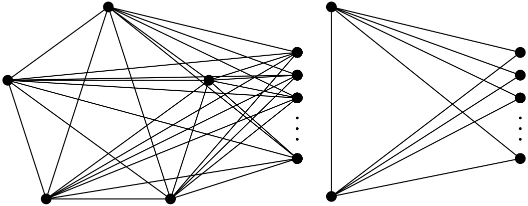

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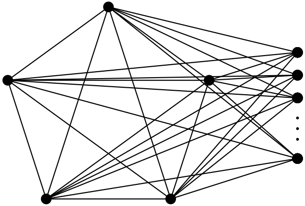

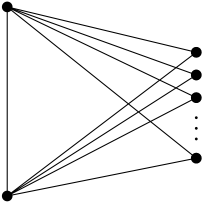

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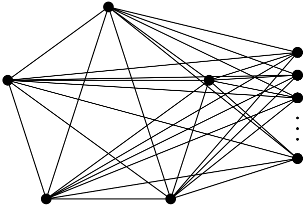

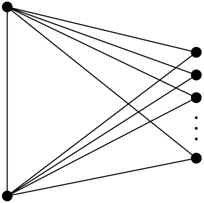

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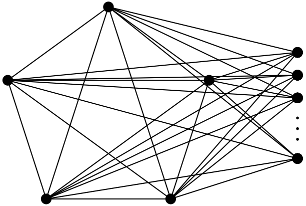

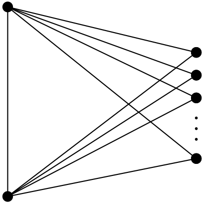

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A coloring of K_N is quasirandom iff at most $o(N^k)$ monochromatic K_k have more than $2^{-k}N + o(N)$ extensions to a monochromatic K_{k+1}.

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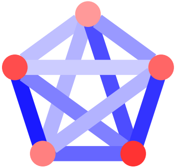

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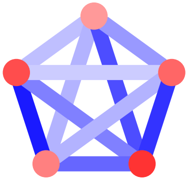

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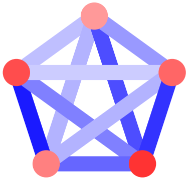

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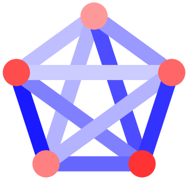

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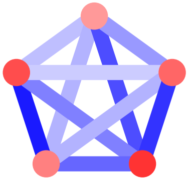

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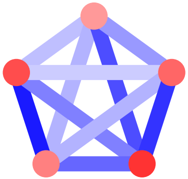

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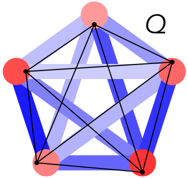

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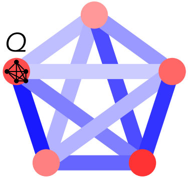

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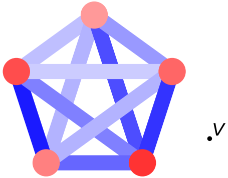

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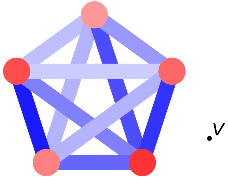

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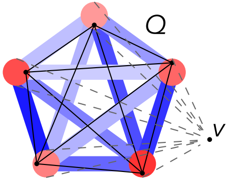

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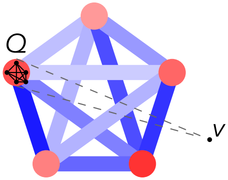

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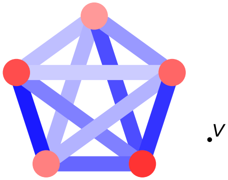

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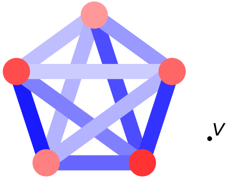

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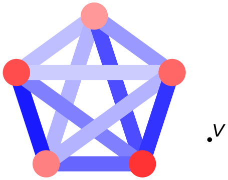

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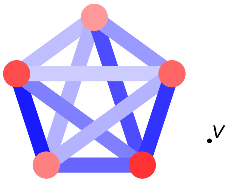

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Adding up over all v shows that Q has $\gtrsim 2^{-k}N$ monochromatic extensions on average $\Longrightarrow$ monochromatic $B_n^{(k)}$.

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Instead, we use much weaker partitioning results, and thus have to work much harder to find a good configuration.

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Inductively apply the previous argument to "nibble" out these ε-regular pieces, and conclude quasirandomness. □

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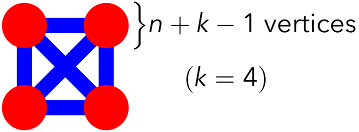

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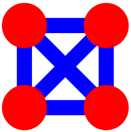

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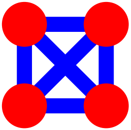

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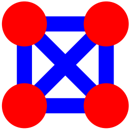

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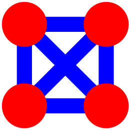

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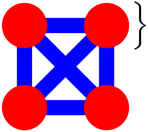

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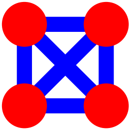

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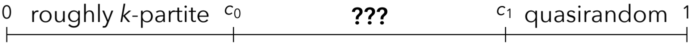

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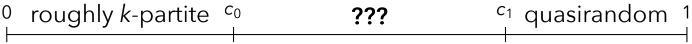

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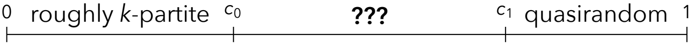

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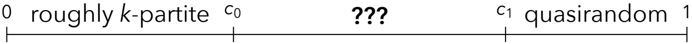

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