1 You probably know that the exponential function $f(z) = e^z$ has no roots anywhere in $\mathbb{C}$. But have you seen a proof of this? Using the Argument Principle, show that this is indeed the case.

[Also, if you know of a less stupid proof, come tell me about it!]

2 In this problem, we will see how to use Rouché's Theorem to conclude powerful results about roots of polynomials.

(a) The polynomial $p(z) = z^5 - z - 1$ is a famous example of a quintic polynomial that has no solutions by radicals. However, by writing

$$f(z) = z^5 \qquad\qquad g(z) = -z - 1$$

prove that every root of p has absolute value at most 2. What's the best bound you can get on the absolute value of the roots of p?

(b) Consider the polynomial $p(z) = z^5 + 15z + 1$. Prove that four of its five roots lie in the annulus

$$\left\{ z : \frac{3}{2} < |z| < 2 \right\}$$

by applying Rouché's Theorem twice in two different ways.

(c) Write down other polynomials and see what you can prove about their roots in this way.

3 Prove the Fundamental Theorem of Algebra, which states that every polynomial $P(z)$ of degree n has exactly n roots (counted with multiplicity) in $\mathbb{C}$.

Hint: Write your polynomial as

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

Then apply Rouché's Theorem by writing

$$f(z) = a_n z^n \qquad\qquad g(z) = a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

and picking a very big circle.

4* In this problem, we will explore the complex logarithm and its relation to the Argument Principle.

(a) Recall that every non-zero complex number can be written as $z = re^{i\theta}$, where $r > 0$ and $\theta \in \mathbb{R}$; θ is called the *argument* of z. If this were to have a well-defined logarithm, we'd want it to be

$$\log z = \log r + i\theta$$

where $\log r$ is the usual logarithm of the positive number r. However, this is not well-defined, since θ is only defined up to adding $2\pi n$, where $n \in \mathbb{Z}$. Because of this, we say that the complex logarithm has "multiple branches" (and if you're in Riemann Surfaces, you'll learn more about what this means). However, if we declare a value for θ at some point, e.g. by setting it to be $\pi/2$ at the number i, then we can try to extend this definition to other points in $\mathbb{C}$. Prove that we can do this so long as we don't "go around" the origin.

(b) On the other hand, prove that if we do go around the origin k times, then our value of the logarithm will be different by $2\pi ik$ once we come back to where we started.

(c) Prove that, assuming log is well-defined, we have that

$$(\log f(z))' = \frac{f'(z)}{f(z)}$$

(d) Conclude that

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} \, \mathrm{d}z$$

should be counting how many times we go around roots of f inside C, and thus that the Argument Principle is at least plausible.

1 Prove the Fundamental Theorem of Algebra again, in a different way.

Hint: Assume that a polynomial P has no roots, and apply the Maximum Modulus Principle to the analytic function $1/P(z)$ on some big circle.

2 In class, we pointed out that if C is a simple closed curve containing a region U, and if f is analytic on U and C, then f attains its maximum on C, rather than in U.

One must be very careful to apply this statement correctly. For instance, consider the function $f(z) = e^{iz^2}$, and let U be the first quadrant of the plane (namely all points with positive real and imaginary parts). Prove that on the boundary of U (namely the positive real and imaginary axes), $|f|$ is exactly 1, despite the fact that $|f|$ is unbounded on U. Why does this not contradict the Maximum Modulus Principle?

3 Prove the following strengthening of the Maximum Modulus Principle, which was mentioned in class: if f is analytic in an open set U, then $|f|$ never attains a *local* maximum inside U (note that what was proved in class was for a *global* maximum).

Hint: Apply the Maximum Modulus Principle to a small subset of U.

4* In this problem, we'll try to understand the angle-preserving property of analytic functions that was mentioned in class. This problem will require a bit of linear algebra to do.

(a) A matrix of the form

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

is called a *rotation matrix*. Prove that if $v = \begin{pmatrix} x \\ y \end{pmatrix}$ is a vector in $\mathbb{R}^2$, then Rv is just the result of rotating v by the angle θ.

(b) Prove that any matrix of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

where $a, b \in \mathbb{R}$ can be written as the product of a rotation matrix and some multiple of the identity matrix. Conclude that such a matrix acts by rotating and scaling.

(c) In case you've forgotten or haven't learned it, here's an important version of the multivariate chain rule: if $\gamma : \mathbb{R} \to \mathbb{R}^2$ is a differentiable function, and $F : \mathbb{R}^2 \to \mathbb{R}^2$ is another differentiable function, and they're given by

$$\gamma(t) = (\gamma_1(t), \gamma_2(t)) \qquad\qquad F(x, y) = (u(x, y), v(x, y))$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t} F(\gamma(t)) = \begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix} \begin{pmatrix} \gamma_1'(t) \\ \gamma_2'(t) \end{pmatrix}$$

This first matrix, consisting of the partial derivatives of u and v, is called the *Jacobian matrix* of F, or the *total derivative* of F.

Using the Cauchy-Riemann equations, prove that if F is actually an analytic function of $z = x+iy$, then its Jacobian matrix is of the form in part (b).

(d) Now suppose that $\gamma, \eta : \mathbb{R} \to \mathbb{R}^2$ are two curves (namely differentiable functions), and suppose that $\gamma(t_0) = \eta(t_0) = (x_0, y_0)$, meaning that the two curves intersect at some point. Then the angle they form is defined to be the angle between the vectors $\gamma'(t_0)$ and $\eta'(t_0)$. Prove that if $F : \mathbb{R}^2 \to \mathbb{R}^2$ is an analytic function of $z = x + iy$ and if the Jacobian of F at (x_0, y_0) is not the zero matrix, then the angle between γ and η is equal to the angle between $F(\gamma)$ and $F(\eta)$. Convince yourself that this is precisely the angle-preserving property we discussed in class.

(e)* Try to understand what part (d) actually means.

1 The Brouwer Fixed-Point Theorem is a famous result in topology. A special case of it states that if $\overline{\mathbb{D}}$ is the closed unit disk, then any continuous function $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ has a fixed point, namely some $z \in \overline{\mathbb{D}}$ with $f(z) = z$.

Prove that the use of the closed unit disk is necessary. Namely, exhibit a continuous (in fact, analytic) map $f : \mathbb{D} \to \mathbb{D}$ that has no fixed point.

Hint: First, construct an analytic map $\mathbb{H} \to \mathbb{H}$ that has no fixed point, then use the conformal equivalence between $\mathbb{H}$ and $\mathbb{D}$.

2 Given some $\alpha \in \mathbb{D}$, we define a map ψ_α, sometimes called a *Blaschke factor*, as follows:

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha}z}$$

(a) Prove that ψ_α is analytic on $\mathbb{D}$.

(b) Prove that ψ_α is a map $\mathbb{D} \to \mathbb{D}$, namely check that $|\psi_\alpha(z)| < 1$ for any $z \in \mathbb{D}$.
Hint: Use the Maximum Modulus Principle.

(c) Prove that $\psi_\alpha(0) = \alpha$ and $\psi_\alpha(\alpha) = 0$.

(d) Prove that $\psi_\alpha \circ \psi_\alpha = \mathrm{id}_{\mathbb{D}}$, and thus conclude that ψ_α is a conformal equivalence between $\mathbb{D}$ and itself.

3 Suppose $f : \mathbb{D} \to \mathbb{D}$ is analytic and has two distinct fixed points. Prove that f must be the identity map.

Hint: If one of the fixed points is $\alpha \in \mathbb{D}$, consider $\psi_\alpha \circ f \circ \psi_\alpha$. Then apply the Schwarz Lemma.

4 Do you expect there to be a conformal equivalence between a square and a rectangle that is not a square? For intuition, think about whether or not such a map would be able to send small circles to small circles, namely stretch both the x and y axes by the same amount. Try to prove your conjecture.

5* In class, we proved that if f, g are both analytic and are inverses of one another, then the derivatives of both never vanish. This is a very strong assumption, and we can get away with less. It turns out that if $f : U \to \mathbb{C}$ is an *injective* analytic map, then its derivative never vanishes on U, which implies that its inverse is also analytic. Prove this.

Hint: Suppose the derivative vanishes at some point, and then apply Rouché's Theorem in a neighborhood of that point to contradict injectivity.