All the homework is optional, which is one reason why there's a lot of it: if you're interested in doing some problems related to this class, find the ones that seem most interesting and work on those! In particular, the first two problems are only recommended if you've never seen a proof of Euler's formula or of linearity of expectation.

Results I mentioned in class but didn't prove

Problem 1 (Euler's formula)**.** In class, I mentioned Euler's formula. Here is the statement again:

Theorem. *Suppose G is a connected graph drawn in the plane with no crossing edges, and let this drawing have v vertices, e edges, and f faces. Then $\mathsf{v} - \mathsf{e} + \mathsf{f} = 2$.*

In this problem, we will prove Euler's formula. Feel free to skip any steps of the proof that you already know; I'm just including all of them for completeness.

(a) Prove that the connectivity assumption on G is necessary. In other words, show that if a *disconnected* graph is drawn in the plane, then $\mathsf{v} - \mathsf{e} + \mathsf{f} \neq 2$. If you know some topology, can you come up with a topological explanation for what's going on?

(b) Recall that a *tree* is a connected graph with no cycles, and a *leaf* is a vertex with only one neighbor. Prove that every tree has at least one leaf.

(c) Using (b), prove by induction that if a tree has n vertices, then it has $n-1$ edges.

(d) Prove that every tree is planar, and that no matter how we draw a tree in the plane, it will have one face. Conclude from this and (c) that Euler's formula holds for trees.

(e) Prove that every connected graph G has a *spanning tree*, namely a subgraph which is a tree and which contains all the vertices of G.

(f) Prove Euler's formula using parts (d) and (e), and by induction on e.

Problem 2 (Linearity of expectation)**.** Recall that the expectation of a random quantity Z is defined by

$$\mathbb{E}[Z] = \sum_z z \cdot \Pr(Z = z),$$

where the sum is over all values z which Z might take. Suppose X, Y are random quantities, and let $Z = X + Y$. Prove that $\mathbb{E}[Z] = \mathbb{E}[X] + \mathbb{E}[Y]$.

Hint: By definition, we know that $\mathbb{E}[Z] = \sum_z z \cdot \Pr(Z = z)$. Prove that we can also write $\mathbb{E}[Z] = \sum_x \sum_y (x+y) \Pr(X = x, Y = y)$, where the sums are over all values x, y that X, Y can take, respectively.

Variants of the crossing number

Problem 3. The *rectilinear crossing number* of a graph G, denoted $\mathrm{rcr}(G)$, is defined as the minimum number of crossings in any *straight-line* drawing of G in the plane.

(a) Prove that $\mathrm{cr}(G) \le \mathrm{rcr}(G)$ for any graph G.

(b) Does equality always hold in (a)?

$\star$ (c) Prove that $\mathrm{cr}(G) = 0$ if and only if $\mathrm{rcr}(G) = 0$.

Problem 4. Let us define a *stupid drawing* of a graph in the plane to be a drawing where we allow more than two edges to all cross at a single point. Let us also define the *stupid crossing number* $\mathrm{scr}(G)$ to be the minimum number of crossing points in a stupid drawing of G. Note that we only count the number of crossing points, not the number of pairs of edges which cross there.

Prove that

$$\mathrm{scr}(G) = \begin{cases} 0 & \text{if } G \text{ is planar} \\ 1 & \text{otherwise.} \end{cases}$$

Conclude that this is not a particularly interesting concept.

Nice drawings of graphs

Problem 5. Let's find some explicit drawings of graphs with few crossings!

(a) Pick your favorite non-planar graph and try to draw it in the plane with as few crossings as possible. Can you prove that your drawing uses the minimum number of crossings?

(b) Find a drawing of $K_{m,n}$ in the plane with $\left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$ crossings.

(c) Find a drawing of K_n in the plane with $\frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$ crossings.

$\star\star$ (d) Prove matching lower bounds for (b) or (c).

Problem 6$\star$ (In memory of John Conway). A *thrackle* is a drawing of a graph in the plane such that every pair of edges meets exactly once, either at a common endpoint or at an internal crossing. For instance, the 5-pointed star is a thrackle drawing of the cycle C_5.

(a) Prove that C_4 cannot be drawn in the plane as a thrackle.

(b) Prove that every cycle graph C_k other than C_4 *can* be drawn as a thrackle.

(c) A *linear thrackle* is a thrackle in which all the edges are straight lines. Prove that any linear thrackle has $\mathsf{e} \le \mathsf{v}$.

$\star\star$ (d) Prove that in any thrackle, $\mathsf{e} \le \mathsf{v}$.

Problem 1. In class, I mentioned the Szemerédi–Trotter theorem, which was used by Elekes in his work on the sum-product problem. Here is the precise statement.

Theorem. *There exists an absolute constant $C > 0$ such that the following holds. Let $P \subset \mathbb{R}^2$ be a set of n points in the plane, and let L be a set of m lines in the plane. Let $I(P, L)$ denote the number of incidences of P and L, namely the number of pairs $(p, \ell) \in P \times L$ such that p lies on ℓ. Then*

$$I(P, L) \leq C(m^{2/3}n^{2/3} + m + n).$$

Prove the Szmerédi–Trotter theorem, using the crossing number lemma.
Hint: Base your proof on our upper bound for the unit distance problem. When choosing which graph to apply the crossing number lemma, think about what graph you'd see if you just drew a bunch of points and a bunch of lines in the plane!

Problem 2⋆ (Unit distances in higher dimensions)**.** Given a finite set $S \subset \mathbb{R}^d$, define

$$u(S) = |\{\{x, y\} \subset S : \|x - y\| = 1\}|,$$

where $\|\cdot\|$ denotes the usual Euclidean distance in $\mathbb{R}^d$. Then, as we did with the plane, define

$$u_d(n) = \max_{\substack{S \subset \mathbb{R}^d \\ |S|=n}} u(S)$$

to be the maximum number of unit distances among n points in $\mathbb{R}^d$.

(a) Prove that $u_1(n) = n - 1$.

⋆(b) The first new interesting case is $d = 3$, which appears to be roughly as hard as the case $d = 2$ that we discussed. The best known bounds are

$$cn^{4/3} \log \log n \leq u_3(n) \leq Cn^{3/2}$$

for some absolute constants $c, C > 0$. Prove either of these bounds.

Hint: The lower bound is due to Erdős, and uses similar ideas to the lower bound I sketched in class; to obtain this bound, you'll probably have to look up some results in number theory. The upper bound is much harder, and I've never actually seen a full proof of it.

⋆⋆(c) Improve either of the bounds from (b).

(d) Weirdly, things get a lot easier for $d \geq 4$. Prove that

$$u_d(n) \geq \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil \approx \frac{n^2}{4}$$

for all $d \geq 4$ and all n. Since we also have an upper bound $u_d(n) \leq \binom{n}{2} \approx \frac{n^2}{2}$, this determines $u_d(n)$ up to the constant factor for all $d \geq 4$.

Hint: Consider two orthogonal 2-dimensional planes in $\mathbb{R}^d$, and a circle in each of them.

(e) Modify your construction from (d) to prove that

$$u_d(n) \gtrsim \frac{1}{2}\left(1 - \frac{1}{\lfloor d/2 \rfloor}\right) n^2,$$

where the $\gtrsim$ denotes that this holds up to a lower-order term.

Hint: Why restrict yourself to only two orthogonal circles?

$\star$ (f) Prove a matching upper bound to (e), namely

$$u_d(n) \lesssim \frac{1}{2}\left(1 - \frac{1}{\lfloor d/2 \rfloor}\right) n^2.$$

Thus, up to the lower-order terms, the problem is fully solved for $d \geq 4$! (excitement, not factorial)

Hint: Apply the Erdős–Stone–Simonovits theorem from extremal graph theory; if you don't know what that is, consider skipping this part.

(g) Actually, there is no reason to restrict our attention to $\mathbb{R}^d$. For any metric space X, one can define $u_X(n)$ to denote the maximum number of unit distances among n points in X. Can you say interesting things about $u_X(n)$ for your favorite metric spaces?

Problem 3. I mentioned in class that the upper bound $u(n) \leq Cn^{4/3}$ hasn't budged in nearly 40 years. There is actually a good reason why, which is the following construction due to Valtr.

$\star$ (a) *Feel free to skip this part because it's not very interesting, but make sure you understand the metric constructed here before moving on to the next parts.*

Let

$$C = \{(x, y) \in \mathbb{R}^2 : |y| = 1 - x^2\} \subset \mathbb{R}^2.$$

We can define a metric ρ on $\mathbb{R}^2$ such that the unit circle around every point is a translate of C, i.e. a metric so that the points at distance 1 from (x_0, y_0) are the points

$$C + (x_0, y_0) = \{(x, y) : |y - y_0| = 1 - (x - x_0)^2\}.$$

Concretely, we define the distance $\rho((x_0, y_0), (x_1, y_1))$ between distinct points (x_0, y_0) and (x_1, y_1) to be the unique real number $\alpha > 0$ such that

$$\frac{|y_1 - y_0|}{\alpha} = 1 - \frac{(x_1 - x_0)^2}{\alpha^2}.$$

Prove that ρ is a metric on $\mathbb{R}^2$.

Remark: This actually has nothing to do with the specifics of C: given any convex, centrally symmetric shape around the origin, it defines a metric on $\mathbb{R}^2$ in this way.

(b) Let $u_\rho(n)$ be the maximum number of ρ-unit distances among n points in the plane, where by a ρ-unit distance I mean two points $(x_0, y_0), (x_1, y_1)$ with $\rho((x_0, y_0), (x_1, y_1)) = 1$. Verify that our proof in class that $u(n) \le 8n^{4/3}$ also implies $u_\rho(n) \le 8n^{4/3}$.

Remark: Again, there is nothing special about C here: as you'll find when doing this proof, the only property we need is that C can intersect any translate of it in at most two points. This holds for any "strictly convex metric" on $\mathbb{R}^2$.

(c) Suppose for simplicity that $n = (2k+1)(2k^2+1)$ for some integer k. Consider the n-point set

$$S = \left\{ \left(\frac{i}{k}, \frac{j}{k^2} \right) : |i| \le k, |j| \le k^2 \right\}.$$

Prove that every point in S is at ρ-unit distance away from at least k other points of S.

(d) Conclude from (c) that

$$u_\rho(n) \ge u_\rho(S) \ge \frac{kn}{2} > \frac{n^{4/3}}{4}.$$

Thus, up to the constant factor, we see that our upper bound for $u_\rho(n)$ is basically correct.

(e) Conclude that no proof like the one we saw in class could ever prove a bound better than $u(n) \le Cn^{4/3}$. In fact, the metric ρ is very similar to the Euclidean metric on $\mathbb{R}^2$, and no one really knows what properties of the Euclidean metric might be needed to prove a better upper bound. Can you think of any "nice" differences between ρ and the Euclidean metric?