1 Background

In this class, we'll be investigating drawings of graphs in the plane. For our purposes, a *drawing* of a graph $G = (V, E)$ will be an assignment of a point in $\mathbb{R}^2$ for every vertex in V, as well as a curve in $\mathbb{R}^2$ connecting any two vertices that are joined by an edge in E. We will allow edges to cross each other, though we require that at most two edges meet at any given point. We will also insist that edges don't go through vertices other than their endpoints, and that no edge crosses itself.

Recall that a graph is called *planar* if it can be drawn in the plane with no crossing edges. Our main object of study in this class is the *crossing number* of a graph, which can be thought of as a refined way of asking the question "is a graph planar?" In essence, it asks "how non-planar is a graph?"

Definition 1. Given a graph G, its *crossing number* $\mathrm{cr}(G)$ is defined as the minimum number of edge crossings among all drawings of G in the plane.

Thus, we see that G is planar if and only if $\mathrm{cr}(G) = 0$, since $\mathrm{cr}(G) = 0$ precisely means that there is a drawing of G with no crossing edges. In general, we should think that the larger $\mathrm{cr}(G)$ is, the "more non-planar" G is.

Despite the simple definition, there is a lot we don't know about crossing numbers. We only know the precise value of $\mathrm{cr}(G)$ for special graphs G, and even among the simplest families of graphs, we don't know the value of the crossing number except for small cases. For instance, given that the two simplest non-planar graphs are $K_{3,3}$ and K_5, it is natural to ask about the crossing numbers of the complete bipartite graphs $K_{m,n}$ and of the complete graphs K_n.

Conjecture 2 (Turán, Zarankiewicz)**.**

$$\mathrm{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Conjecture 3 (Hill)**.**

$$\mathrm{cr}(K_n) = \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$$

In both cases, the conjectured values come from specific drawings. Namely, we know how to draw $K_{m,n}$ and K_n in the plane with $\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ and $\frac{1}{4} \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{n-2}{2} \rfloor \lfloor \frac{n-3}{2} \rfloor$ crossings, respectively. Therefore, all that remains to prove these conjectures is to prove a matching lower bound, but this appears difficult in general, and we only know how to do it for small values of m and n, where some casework can be done by hand or by computer. In general, it's not really clear what to do: how do you prove that a graph cannot be drawn in the plane with a small number of crossings? A priori, it seems like the only thing to do is to try all the drawings, which is impossible in general.

In this class, we'll see a way of doing this, which is called the crossing number lemma. It establishes a lower bound on $\mathrm{cr}(G)$ that depends only on the number of vertices and edges of G. While it is not precise enough to establish either of the above conjectures, it turns out to be enormously useful in a number of applications, as we will also see.

Before doing so, though, let me mention one final beautiful conjecture about crossing numbers. Recall that $\chi(G)$ denotes the *chromatic number* of G, namely the minimum number of colors one needs to color the vertices of G so that adjacent vertices receive different colors.

Conjecture 4 (Albertson)**.** *If* $\chi(G) = k$*, then* $\mathrm{cr}(G) \geq \mathrm{cr}(K_k)$*.*

In other words, Albertson's conjecture says that K_k has the smallest crossing number among all graphs of chromatic number k. Despite its simple statement, it contains within it some interesting content. For instance, the $k = 5$ version asserts that if a graph has chromatic number 5, then its crossing number is at least $\mathrm{cr}(K_5)$, which is 1. Thus, the $k = 5$ case of Albertson's conjecture is precisely the four-color theorem! Because this case is already so hard, one might expect the conjecture to just get increasingly harder for larger k, but this is not really the case; certain tools, such as the crossing number lemma, allow one to reduce higher cases of Albertson's conjecture to specific finite checks, and with computers, Albertson's conjecture has now been verified for $k \leq 18$.

2 The basics

Before we can prove the crossing number lemma, we will begin by proving a number of simpler and weaker facts about planar graphs and about crossing numbers. Eventually, we'll be able to boost these weaker results into the stronger result we're looking for. From this point onwards, the letter v will always indicate the number of vertices of a graph G, and the letter e will always indicate the number of edges of G. If G is not clear from context, we'll write $\mathsf{v}(G), \mathsf{e}(G)$, respectively.

Theorem 5. *If G is a planar graph, then* $\mathsf{e} < 3\mathsf{v}$*.*

Proof. If $\mathsf{v} = 1$ then G has no edges, and we're done. Similarly, if $\mathsf{v} = 2$, then G has at most one edge, and we're again done. So we may assume from now on that $\mathsf{v} \geq 3$.

Fix a drawing of G in the plane with no crossing edges. Recall that the *faces* of this drawing are the regions surrounded by the vertices and edges, as well as the one infinite region outside the graph. If any face is not a triangle, we may add some edges to G to divide it into triangles; note that this will increase e and not change v, so if we prove the desired bound for this "triangulated" graph, it will imply the bound for our original graph. Therefore, we will assume from now on that every face of G (including the infinite face) is a triangle. Let f denote the number of faces.

We will now import an important fact from topology, Euler's formula, which tells us that $\mathsf{v} - \mathsf{e} + \mathsf{f} = 2$. As it turns out, this is the only time during this class where we will use anything specific about the structure of the plane. We won't prove Euler's formula, though you can find a proof on the homework.

Note that every edge lies on exactly two faces. Moreover, since we assumed that all faces were triangles, we have that every face contains three edges. Therefore, we can conclude that

$3\mathsf{f} = 2\mathsf{e}$: this is because $3\mathsf{f}$ counts the three edges from each face, and doing so double-counts the number of edges, since every edge appears on exactly two faces. Plugging this equality into Euler's formula $\mathsf{v} - \mathsf{e} + \mathsf{f} = 2$, we get that

$$2 = \mathsf{v} - \mathsf{e} + \mathsf{f} = \mathsf{v} - \mathsf{e} + \frac{2}{3}\mathsf{e} = \mathsf{v} - \frac{\mathsf{e}}{3}.$$

Rearranging this shows that $\mathsf{e} = 3\mathsf{v} - 6$, and in particular that $\mathsf{e} < 3\mathsf{v}$, as claimed. □

Using this fact, we can obtain our first general lower bound on $\mathrm{cr}(G)$.

Corollary 6. *For any graph G,*

$$\mathrm{cr}(G) > \mathsf{e} - 3\mathsf{v}.$$

Proof. Fix a drawing of G in the plane with the minimum number of crossings, namely with exactly $\mathrm{cr}(G)$ crossings. For each crossing, we (arbitrarily) pick one of the edges that participates in it, and delete it. Doing so gives us a new graph G'. Note that $\mathsf{v}(G') = \mathsf{v}(G)$ and that $\mathsf{e}(G') \geq \mathsf{e}(G) - \mathrm{cr}(G)$, since we delete at most one[1] edge for each crossing of G. Moreover, we now have a drawing of G' with no crossings, which implies that G' is planar. By the previous theorem, this tells us that $\mathsf{e}(G') < 3\mathsf{v}(G')$, and so

$$\mathrm{cr}(G) \geq \mathsf{e}(G) - \mathsf{e}(G') > \mathsf{e}(G) - 3\mathsf{v}(G') = \mathsf{e}(G) - 3\mathsf{v}(G),$$

as claimed. □

3 The crossing number lemma

We will now prove the crossing number lemma, which will give us a much stronger lower bound on $\mathrm{cr}(G)$ than the one given above.

Theorem 7 (Crossing number lemma)**.** *If G is a graph with $\mathsf{e} \geq 4\mathsf{v}$, then*

$$\mathrm{cr}(G) \geq \frac{\mathsf{e}^3}{64\mathsf{v}^2}.$$

Before we prove this, there are a few remarks to make. First, don't worry too much about the constant 64. It is the constant that comes out of the proof, and more involved proofs give better constants (the current record is around 29). However, as we will see, the constant just doesn't really matter: if we think of our graph G as enormous, then the only thing that really matters are the exponents on v and e: if these numbers are both in the trillions, then a number like 64 is totally insignificant relative to another factor of a trillion.

[1]We may delete strictly fewer edges if some of the edges participate in more than one crossing, but in either case we get the claimed inequality.

The crossing number lemma was proven independently by Ajtai–Chvátal–Newborn–Szemerédi and by Leighton in the early 1980s. Their original proofs were somewhat complicated; however, we're about to see a much simpler proof that was discovered somewhat later by a number of people[2].

The basic idea in the proof is to again leverage a weaker result by cleverly deleting something. Earlier, we applied this idea by first proving the (weak) result that if G is planar then $\mathsf{e}(G) < 3\mathsf{v}(G)$, and then extending this to the (stronger) result that $\mathrm{cr}(G) > \mathsf{e}(G) - 3\mathsf{v}(G)$ by cleverly deleting edges. To obtain the even stronger bound on $\mathrm{cr}(G)$, we will delete vertices, and then apply the weaker bound. However, we run into a problem, which is that earlier we had a natural choice of which edges to delete, namely the ones that participated in the crossings. Here, it is not clear which vertices to delete—if we delete vertices whose edges participate in lots of crossings, we may accidentally end up also deleting a bunch of inoffensive edges, and it's not clear how to gain anything from this argument.

As it turns out, the right way to deal with this conundrum is to not make these choices ourselves. Instead, we will delete some of the vertices *at random*. This seems totally crazy at first—the whole point is to pick something cleverly, and how on earth will doing it randomly help us? However, as it turns out, it is often the case in mathematics that random choices are better-behaved than "clever" non-random choices, and this is an instance of that.

Before seeing the proof, recall that if X is a random quantity[3], then its *expectation* (or *average*) $\mathbb{E}[X]$ is the sum

$$\mathbb{E}[X] = \sum_x x \cdot \Pr(X = x).$$

Thus, for instance, if X denotes the outcome of a fair die roll, then

$$\mathbb{E}[X] = \sum_{x \in \{1,2,3,4,5,6\}} x \cdot \Pr(X = x) = \sum_{x \in \{1,2,3,4,5,6\}} x \cdot \frac{1}{6} = \frac{7}{2}.$$

The crucial fact that we will need about expectation is that it is *linear*: if X and Y are two random quantities, then $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$. Note that this holds even if X and Y are dependent (e.g. even if the same randomness determines both of them); if you've never seen a proof of this, you should check the homework.

Proof of the crossing number lemma. We fix a parameter $p \in [0,1]$, which we will select later. Given our graph G, let us also fix a drawing of it in the plane with exactly $\mathrm{cr}(G)$ crossings. Now, we form a random induced subgraph of G, which we denote G_p, by keeping each vertex with probability p, independently of all the other vertices. In other words, we go vertex by vertex, and for each vertex we flip a p-biased coin to determine whether or not we keep that vertex: we keep it with probability p, and delete it with probability $1-p$. If

[2] This proof is attributed to Lovász, Matoušek, Pach, and sometimes others. I've heard a story that there was an email chain involving some of these people, and that this proof was discovered through this email chain, but I'm not sure if this story is true.

[3] The technical name for X is a *random variable*, but I'm intentionally avoiding this terminology because I don't want to get caught up in technicalities about what exactly X is.

we delete a vertex, then we also delete all the edges coming off of it; thus, we keep an edge if and only if both of its endpoints are kept.

Note that $\mathsf{v}(G_p), \mathsf{e}(G_p)$, and $\operatorname{cr}(G_p)$ are all random quantities, and they are quite difficult to understand. However, it turns out that the expectations of these random quantities are actually quite simple to determine (or at least to bound). For instance, we have that

$$\mathbb{E}[\mathsf{v}(G_p)] = p\mathsf{v}(G).$$

Indeed, since every vertex is kept with probability p, we can add up over all vertices by linearity of expectation and get this bound. Similarly, since an edge is kept if and only if both its endpoints are kept, we see that each edge survives in G_p with probability p^2. Thus, again by linearity of expectation,

$$\mathbb{E}[\mathsf{e}(G_p)] = p^2\mathsf{e}(G).$$

Finally, let's consider $\operatorname{cr}(G_p)$. In our drawing of G in the plane, every crossing will survive with probability p^4, since a crossing will survive if and only if both edges which participate in it survive, and each of these edges survives with probability p^2. Thus, our drawing of G yields a drawing of G_p which has, on average, $p^4 \operatorname{cr}(G)$ crossings. This drawing may not be the *best* drawing of G_p, but it is *a* drawing, and this implies that

$$\mathbb{E}[\operatorname{cr}(G_p)] \leq p^4 \operatorname{cr}(G).$$

Finally, we apply Corollary 6 to G_p. It tells us that

$$\operatorname{cr}(G_p) \geq \mathsf{e}(G_p) - 3\mathsf{v}(G_p).$$

Taking expectations of both sides, we find that

$$\mathbb{E}[\operatorname{cr}(G_p)] \geq \mathbb{E}[\mathsf{e}(G_p)] - 3\mathbb{E}[\mathsf{v}(G_p)],$$

and plugging in our values above, we find that

$$p^4 \operatorname{cr}(G) \geq p^2\mathsf{e}(G) - 3p\mathsf{v}(G), \qquad \text{or equivalently} \qquad \operatorname{cr}(G) \geq p^{-2}\mathsf{e} - 3p^{-3}\mathsf{v}.$$

Note that this bound holds for all $p \in [0, 1]$, so it makes sense to pick the best choice of p, namely the choice of p that yields the strongest lower bound on $\operatorname{cr}(G)$. Finding this best p can be done with calculus, but let's skip that; it turns out that a pretty good choice is $p = 4\mathsf{v}/\mathsf{e}$. Note that by our assumption that $\mathsf{e} \geq 4\mathsf{v}$, this quantity is indeed in $[0, 1]$, which is good: our probabilistic argument only works if p is a valid probability. Plugging in $p = 4\mathsf{v}/\mathsf{e}$, we find that

$$\operatorname{cr}(G) \geq p^{-2}\mathsf{e} - 3p^{-3}\mathsf{v} = \frac{\mathsf{e}^3}{16\mathsf{v}^2} - \frac{3\mathsf{e}^3}{64\mathsf{v}^2} = \frac{\mathsf{e}^3}{64\mathsf{v}^2}. \qquad \square$$

4 Applications of the crossing number lemma

For a long time after its proof in the early 1980s, many mathematicians didn't know or care about the crossing number lemma. However, this all changed with a paper of Székely from 1997. This is one of my favorite math papers of all time. In it, he shows how a wide array of different theorems in discrete geometry[4] can all be solved using a simple application of the crossing number lemma. Though most of the theorems he proves had been known before, their earlier proofs were usually very involved. Since this paper, the crossing number lemma has become a standard and important tool in discrete geometry.

At its core, the method that Székely came up with is extremely simple. One wishes to prove some statement about, say, a collection of points in the plane. One then usually defines a graph with these points as vertices, and the geometric nature of the problem also means that we get a drawing of this graph in the plane. We then find upper and lower bounds for the crossing number of this graph; the lower bound will come from the crossing number lemma, while the upper bound will come from the specific drawing that we have, together with the geometric assumptions about the problem. If we've done everything well, we can put these bounds together and derive an interesting result.

4.1 Erdős's unit distance problem

One of my favorite applications of the crossing number lemma is to Erdős's unit distance problem, which is one of my favorite open questions in all of math. The question asks for the maximum number of unit distances that can be found among a set of n points in the plane. More precisely, given a finite set $S \subset \mathbb{R}^2$, let

$$u(S) = |\{\{x, y\} \subset S : \|x - y\| = 1\}|$$

denote the number of (unordered) pairs of points in S whose Euclidean distance is exactly 1. Next, we define

$$u(n) = \max_{|S|=n} u(S);$$

thus, $u(n)$ is exactly the maximum number of unit distances we can see among n points in the plane. In 1946, Erdős asked to determine $u(n)$, or at least to find good upper and lower bounds for it.

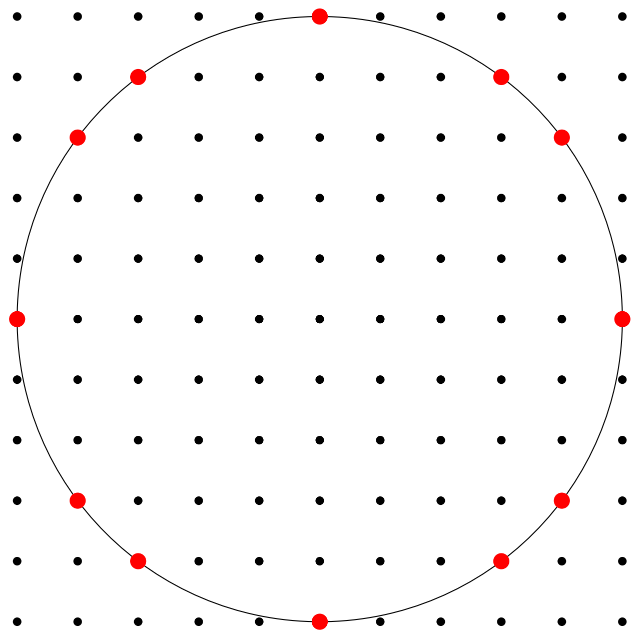
Let us begin with lower bounds. To find a lower bound for $u(n)$, it suffices to find a configuration $S \subset \mathbb{R}^2$ of n points that span many unit distances. Perhaps the simplest thing to try is to let S be a square lattice, namely a $\sqrt{n} \times \sqrt{n}$ grid of points (let's assume for simplicity that n is a perfect square). Then every point in S (except for a few exceptions near the boundary) are at a unit distance from exactly 4 other points. Note that if we add this up over all points, we'll double-count all the unit distances, and we find that $u(S) \approx 2n$, where the $\approx$ again comes from the exceptions near the boundary, which are insignificant if n is large. Thus, we find that $u(n) \gtrsim 2n$. Can we do better?

[4]Discrete geometry is a field of math where one studies questions about the geometry of finite collections of points, lines, etc. in the plane or in $\mathbb{R}^d$.

Actually, we can, using a beautiful trick discovered by Erdős when he came up with this problem. Let us define S_5 to be another $\sqrt{n} \times \sqrt{n}$ square grid of points, except that now we make the distance between adjacent points $1/5$, rather than 1, as it was before. In other words, we get S_5 from S by dilating the plane by a factor of $1/5$. The reason we picked $1/5$ is that 5 is the hypotenuse of a Pythagorean triple: the point $(0,0)$ is at unit distance from the 12 points

$$(\pm 1, 0), (\pm \tfrac{3}{5}, \pm \tfrac{4}{5}), (\pm \tfrac{4}{5}, \pm \tfrac{3}{5}), (0, \pm 1).$$

We can see this visually: the unit circle around $(0,0)$ passes through the 12 red points in the grid S_5 below.



This shows that $u(S_5) \approx 6n$, since we again double-count the unit distances when we add up 12 over all n points. Thus, $u(n) \gtrsim 6n$.

Of course, there's no reason to stop at S_5. Suppose we pick a number k such that k^2 can be represented as a sum of two perfect squares in ℓ different ways. Then if we define S_k to be a square grid dilated down by a factor of $1/k$, then almost every point in S_k will be at unit distance from ℓ others, and we will thus get that $u(S_k) \approx \ell n$. However, we have to make sure that k is not too large relative to n, for otherwise the edge effects will start getting problematic: for instance, if we take $k > \sqrt{2n}$, then the entire grid S_k will have diagonal length less than 1, and will have zero unit distances!

But essentially, this all boils down to the question of which number k (which is not too large relative to n) can be expressed as the sum of two squares in the maximum number of ways. This is a number theory question, and it was already essentially answered by Fermat and Lagrange by the 17th century. By using their results, Erdős was able to show that we can pick such a k for which $\ell = n^{c/\log \log n}$, where $c > 0$ is some absolute constant. By the using this rescaling trick, he concluded that

$$u(n) \geq n^{1+c/\log \log n}.$$

Note that this function $n^{1+c/\log\log n}$ is somewhat strange: it grows faster than any linear function in n, but slower than any other power of n. More precisely,

$$\lim_{n\to\infty} \frac{n^{1+c/\log\log n}}{n} = \infty \qquad \text{but} \qquad \lim_{n\to\infty} \frac{n^{1+c/\log\log n}}{n^{\alpha}} = 0 \qquad \text{for any } \alpha > 1$$

Erdős published this result in 1946, and to this day, $n^{1+c/\log\log n}$ is the best known lower bound on $u(n)$; in fact, Erdős conjectured that this lower bound is essentially correct, and most of the subsequent work has focused on improving the upper bound.

The best upper bound we have on $u(n)$ is $u(n) \le Cn^{4/3}$ for some constant $C > 0$. This result was first proved by Spencer, Szemerédi, and Trotter in 1984 using somewhat complicated techniques; I will present the simple proof of Székely, using the crossing number lemma. At this point, there are at least three different proofs of this result, all using different ideas, but no one has been able to improve on it, and getting beyond the 4/3 barrier is a major open problem.

Theorem 8. $u(n) \le 8n^{4/3}$.

Proof (Székely). Let $S \subset \mathbb{R}^2$ be an arbitrary set of n points in the plane; it suffices to prove the desired upper bound on $u(S)$. Following the general strategy, we will construct a graph whose vertex set is S, whose edges "know something about" the unit distances in S, and whose crossing number we can understand.

To do this, let us first draw a unit circle around every point in S. Then the number of unit distances is exactly half of the number of incidences between the points of S and these unit circles, where an *incidence* is a pair consisting of a circle and a point lying on it. Note that we need to take half the incidences because every unit distance defines two incidences. We define a graph G_0 whose vertex set is S, and we connect two points of S by an edge if they are consecutive points along one of these circles; equivalently, the edges of G_0 are precisely the circular arcs we see when draw this configuration.

Suppose one of the unit circles has m points of S on it. Then this circle contributes m incidences, and additionally it contributes m edges to G_0, since the m points split the circle into m arcs. Therefore, the total number of incidences is precisely $\mathsf{e}(G_0)$, and thus $\mathsf{e}(G_0) = 2u(S)$.

Additionally, note that we already have a drawing of G_0 in the plane, namely the one where every edge is just drawn as the circular arc that defines it. Note that the set of crossing points of edges in this graph is exactly the set of points where the n unit circles cross each other. Since any two circles can cross each other at most twice, we find that this drawing has at most $2\binom{n}{2}$ crossings. Thus,

$$\operatorname{cr}(G_0) \le 2\binom{n}{2} \le n^2.$$

Note that G_0 is not quite an "ordinary" graph, since two vertices may be joined by more than one edge in G_0. This can happen for two reasons. First, if one of the unit circles has only two points on it, then those two points will be joined by two arcs from this circle, and thus by two edges in G_0. Additionally, two different circles may pass through the same two points, which may again give us up to two additional edges between these two points. However, this is it: given two points in the plane, there are at most two unit circles that pass through them. Thus, we find that any pair of vertices in G_0 is joined by at most 4 edges.

We define a new graph G by keeping exactly one of these edges for every pair of vertices in G_0. Thus, G is a subgraph of G_0, and G is a *simple* graph, in the sense that every pair of vertices is joined by at most one edge. G, like G_0, has n vertices. Moreover, since every pair in G_0 was joined by at most 4 edges, we find that $\mathsf{e}(G) \geq \mathsf{e}(G_0)/4$. Finally, since G is a subgraph of G_0, we find that $\operatorname{cr}(G) \leq \operatorname{cr}(G_0) \leq n^2$. We can now apply the crossing number lemma, which tells us that

$$n^2 \geq \operatorname{cr}(G) \geq \frac{\mathsf{e}(G)^3}{64\mathsf{v}(G)^2} \geq \frac{\mathsf{e}(G_0)^3}{4096n^2} = \frac{u(S)^3}{512n^2}.$$

Rearranging, we find that $u(S) \leq 8n^{4/3}$, as claimed.

However, we can only apply the crossing number lemma if $\mathsf{e}(G) > 4\mathsf{v}(G)$. But if not, then

$$u(S) = \frac{\mathsf{e}(G_0)}{2} \leq 2\mathsf{e}(G) \leq 8\mathsf{v}(G) = 8n \leq 8n^{4/3},$$

and we get the same result. □

4.2 The sum-product phenomenon

Given a finite set $A \subset \mathbb{R}$, we define

$$A + A = \{a + b : a, b \in A\} \qquad \text{and} \qquad A \cdot A = \{ab : a, b \in A\}$$

to be the sets of pairwise sums and differences in A, respectively. By considering the sum or product with a single fixed element of A, we can see that $|A + A|, |A \cdot A| \geq |A|$. On the other hand, since we are taking pairs of elements, we also get the simple upper bounds $|A + A|, |A \cdot A| \leq |A|^2$. In general, both of these bounds might be more or less tight: the sizes of $A + A$ and $A \cdot A$ can be anywhere between linear in $|A|$ and quadratic in $|A|$.

However, if the lower bound is nearly tight, then we expect strong additive or multiplicative structure. For instance, one can prove that in fact $|A + A| \geq 2|A| - 1$, and that equality holds if and only if A is an arithmetic progression. Similarly, $|A \cdot A| \geq 2|A| - 1$, with equality if and only if A is a geometric progression.

However, there is a general philosophical notion, backed up by many disparate theorems, which essentially says that the additive and multiplicative structure of $\mathbb{R}$ (or in fact any field) must be totally incompatible. In particular, it is impossible for a set to simultaneously have a lot of additive and a lot of multiplicative structure. One instance of this philosophy is the following conjecture, which is a major open problem.

Conjecture 9 (Erdős–Szemerédi)**.** *For every $\varepsilon > 0$ there exists some $n_0 \in \mathbb{N}$ such that the following holds. If $A \subset \mathbb{R}$ is a finite set with $|A| \geq n_0$, then*

$$\max\{|A + A|, |A \cdot A|\} \geq |A|^{2-\varepsilon}.$$

In other words, this conjecture says that for large sets $A \subset \mathbb{R}$, either $|A+A|$ or $|A \cdot A|$ must be arbitrarily close to its maximal possible size of $|A|^2$. In other words, it is impossible for A to simultaneously have any "appreciable" additive and multiplicative structure.

Despite many years of intensive work, the Erdős–Szemerédi sum-product conjecture remains wide open. The current record (from just two months ago!) is due to Rudnev and Stevens, who proved that

$$\max\{|A+A|, |A \cdot A|\} \geq |A|^{1558/1167}$$

for sufficiently large sets $A \subset \mathbb{R}$.

Elekes previously proved a weaker bound, whose proof uses techniques like the ones we've discussed above.

Theorem 10 (Elekes). *For any set $A \subset \mathbb{R}$,*

$$\max\{|A+A|, |A \cdot A|\} \geq \frac{1}{10}|A|^{5/4}.$$

Elekes's brilliant insight was that this problem, which deals with sets of real numbers, really corresponds to a *geometry* problem, which means that geometric techniques like the crossing number lemma can be applied to it. Without going into too many details, what Elekes does is look at the set

$$P = \{(x, y) \in \mathbb{R}^2 : x \in A + B, y \in A \cdot B\} \subset \mathbb{R}^2.$$

Additionally, he considers lines of the form $y = a(x-b)$ for all pairs $a, b \in A$. Then the point is that every such line goes through many of the points in P, namely all points of the form $(b+c, ac) \in P$, for any $c \in A$. This shows that we have built a configuration of points and lines in the plane with many incidences. Earlier, we proved that a collection of points and unit circles in the plane can't have too many incidences, and the same holds for points and lines; this fact is known as the Szemerédi–Trotter theorem, and it can be proven in the same way as our proof above, using the crossing number lemma. Using it, one can show that the only way for there to be so many incidences between P and these lines is if $|P| \geq \frac{1}{100}|A|^{5/2}$. But since $|P| = |A+A||A \cdot A|$, we conclude the claimed bound.