New constructions in Ramsey theory

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Thesis defense

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My philosophy on Ramsey theory

Theorem ("Folklore")

Given N points, if half are colored red, then there are $N/2$ red points.

Theorem (Kővári–Sós–Turán 1954)

Given an $N \times N$ grid, if half the points are colored red, then there is a $\log N \times \log N$ red subgrid. This is tight.

Theorem (Szemerédi 1975, Gowers 2001)

Given N points, if half the points are colored red, then there are $\log\log\log\log\log N$ evenly spaced red points. Is this tight?

Theorem (Furstenberg–Katznelson 1978, Nagle–Rödl–Schacht–Skokan 2005, Gowers 2007)

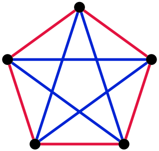
Given an $N \times N$ grid, if half the points are colored red, then there is a $\sqrt{A^{-1}(N)} \times \sqrt{A^{-1}(N)}$ evenly spaced red subgrid. Is this tight?

Any large object contains a large structured subobject. How large?

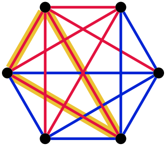
Constructions are crucial for understanding such questions.

Graph Ramsey theory

There is a 2-coloring of the edges of K_5 with no monochromatic triangle



…but every 2-coloring of the edges of K_6 does have a monochromatic triangle.



Ramsey numbers

$r(t)$ = minimum N so that every 2-coloring of the edges of K_N has a monochromatic K_t.

Theorem (Ramsey 1930, Erdős–Szekeres 1935)

$r(t)$ exists (i.e. is finite). In fact, $r(t) < 4^t$.

For a lower bound we need a construction: a coloring of K_N with no monochromatic K_t.

Theorem (Erdős 1947)

$$r(t) > 2^{t/2}.$$

Proof: Let $N = 2^{t/2}$. Consider a random two-coloring of $E(K_N)$.

$$\mathbb{E}[\#\text{monochromatic } K_t] = \binom{N}{t} 2^{1-\binom{t}{2}} < N^t 2^{-\frac{1}{2}t^2} = 1.$$

So there exists a coloring of $E(K_N)$ with < 1 monochromatic K_t. □

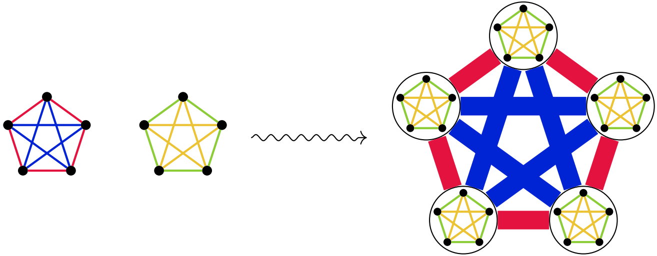
Multicolor Ramsey numbers

$r(t;q) = \min. N$ so that any q-coloring of $E(K_N)$ has monochromatic K_t

Erdős–Szekeres (1935), Erdős (1947):

$$\sqrt{q}^{\,t} < r(t;q) < q^{qt}$$

Product coloring trick: $r(t;q) > 2^{\lfloor \frac{q}{2} \rfloor \frac{t}{2}} \approx \left(2^{\frac{q}{4}}\right)^t$.



Conlon–Ferber (2021): $r(t;q) > \left(2^{\frac{7q}{24}+C}\right)^t$.

W. (2021): $r(t;q) > \left(2^{\frac{3q}{8}-\frac{1}{4}}\right)^t$.

The Conlon–Ferber construction

For $x, y \in \mathbb{F}_2^t$, let $x \cdot y = \sum_{i=1}^t x_i y_i$. We define a graph G_t as follows.

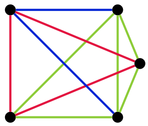
Let $V_t = \{x \in \mathbb{F}_2^t : x \text{ has an even number of 1s}\} = \{x \in \mathbb{F}_2^t : x \cdot x = 0\}$.

For $x, y \in V_t$, make xy adjacent if $x \cdot y = 1$.

Fact 1: G_t contains no K_t (for t even).

Fact 2: G_t has at most $2^{\frac{5}{8}t^2}$ independent sets of size t.

We color the edges of G_t green.



Let $p \approx 2^{-\frac{1}{8}t}$, and keep each vertex of G_t with probability p.

Color all remaining pairs red or blue at random.

$$\mathbb{E}[\#\text{red or blue } K_t] \le p^t \cdot 2^{\frac{5}{8}t^2} \cdot 2^{1-\binom{t}{2}} \approx \left(2^{-\frac{1}{8}t} \cdot 2^{\frac{5}{8}t} \cdot 2^{-\frac{1}{2}t}\right)^t = 1.$$

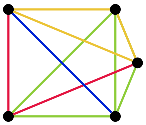
No green K_t by Fact 1, so $r(t;3) > N \approx p|V_t| \approx 2^{\frac{7}{8}t}$.

This works over larger fields, but the bounds aren't very good.

Conlon–Ferber use the product coloring for $q > 4$.

A new approach for more colors

Overlay two random copies of G_t in green and yellow.
The number of sets of size t independent in both copies is $\leq 2^{\frac{1}{4}t^2}$ (because a t-set is independent in either copy with probability $\leq 2^{-\frac{3}{8}t^2}$).
Keep each vertex with probability p (chosen later).
Color all remaining pairs red or blue at random.



$$\mathbb{E}[\#\text{red or blue } K_t] \leq p^t \cdot 2^{\frac{1}{4}t^2} \cdot 2^{1-\binom{t}{2}} \approx \left(p \cdot 2^{\frac{1}{4}t} \cdot 2^{-\frac{1}{2}t}\right)^t.$$

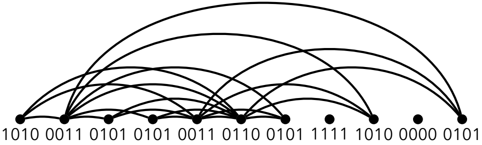
Pick $p \approx 2^{\frac{1}{4}t}$ to obtain $r(t;4) > N \approx p|V_t| \approx 2^{\frac{5}{4}t}$.

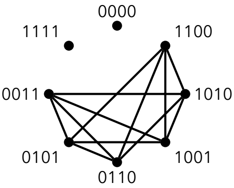
How are we picking $p > 1$???

Random homomorphisms to the rescue

Let p be any positive real number, and let $N = p|V_t|$.

Pick a uniformly random function $f : [N] \to V_t$.





Connect vertices in $[N]$ if their labels are adjacent in G_t to get $\widetilde{G_t}$.

If $p \ll 1$, $\widetilde{G_t}$ looks like keeping vertices from G_t with probability p.

If $p \gg 1$, it looks like a random blowup.

Fact 1: $\widetilde{G_t}$ contains no K_t.

Fact 2: The number of independent sets of size t in $\widetilde{G_t}$ is $\lesssim p^t \cdot 2^{\frac{5}{8}t^2}$.

So the above argument works for any p, if interpreted correctly.

Putting it all together

Theorem (W. 2021)

$$r(t;q) > \left(2^{\frac{3q}{8} - \frac{1}{4}}\right)^t.$$

Proof: Let $p = \left(2^{\frac{3q}{8} - \frac{5}{4}}\right)^t$, let $N = p|V_t|$, and pick $q-2$ random functions $[N] \to V_t$. Overlay the resulting graphs $\widetilde{G_t}$ for the first $q-2$ colors, then color the remaining pairs red or blue at random. □

Theorem (Sawin 2022)

$$r(t;q) > \left(2^{0.383796q - 0.267592}\right)^t.$$

Proof: No reason to use G_t! An appropriately chosen random graph works better as input to the random homomorphism machinery. □

Ramsey numbers of graphs and digraphs

The *Ramsey number* $r(t)$ is the minimum N such that every 2-edge-coloring of K_N contains a monochromatic K_t.

$$2^{t/2} < r(t) < 2^{2t}.$$

The *Ramsey number* $r(H)$ of a graph H is the minimum N such that every 2-coloring of $E(K_N)$ contains a monochromatic copy of H.

Chvátal–Rödl–Szemerédi–Trotter (1983): If H has t vertices and maximum degree Δ, then $r(H) = O_\Delta(t)$.

The *oriented Ramsey number* $\vec{r}(t)$ is the minimum N such that every edge orientation of K_N contains a transitive K_t.

$$2^{t/2} < \vec{r}(t) < 2^t.$$

The *oriented Ramsey number* $\vec{r}(H)$ of an acyclic digraph H is the minimum N such that every N-vertex tournament contains a copy of H.

Bucić–Letzter–Sudakov (2019): If H has t vertices and maximum degree Δ, is it true that $\vec{r}(H) = O_\Delta(t)$?

Theorem (Fox–He–W. 2022)

No! *For any $C > 0$, there exist bounded-degree H with $\vec{r}(H) > t^C$.*

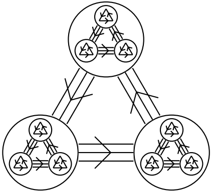
Proof sketch I

Theorem (Fox–He–W. 2022)

There exists a t-vertex acyclic digraph H with bounded maximum degree and $\vec{r}(H) > t^C$.

We need (1) a construction of H, (2) a tournament T on $t^{\log_2(3)-\varepsilon}$ vertices, and (3) a proof that there is no embedding $H \hookrightarrow T$.

For (2): We let T be an iterated blowup of a cyclic triangle.



For (3): Construct H so that in any embedding $H \hookrightarrow T$, some subinterval of $[t]$ of length $\geq 0.49t$ is mapped into a single part.

Ensure that the induced subgraph on this subinterval has the same property, so we can iterate. At each step, $|T|$ drops by a factor of 3, but $|H|$ drops by a factor of 2.01.

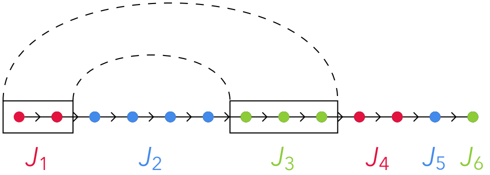
Proof sketch II: interval meshes

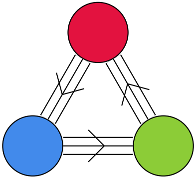
Want: In any embedding $H \hookrightarrow T$, some subinterval of $[t]$ of length $\geq 0.49t$ is mapped into a single part, and this is hereditary.

Definition

H is an *interval mesh* if

- H has a Hamiltonian path $1 \to 2 \to \cdots \to t$.
- For all $1 \leq a < b \leq c < d \leq t$ with $c - b \leq 100 \min(b - a, d - c)$, there is an edge between $[a, b]$ and $[c, d]$.





Thus, $|J_i| > 100 \min(|J_{i-1}|, |J_{i+1}|)$. So $|J_i| \geq 0.49t$ for some i.

Greedy algorithm yields an interval mesh with max degree ≤ 1000.

Size Ramsey numbers

$K_{s,t}$ is the complete bipartite graph with parts of sizes $s \le t$.

Theorem (Erdős–Faudree–Rousseau–Schelp 1978)

There exists a graph G with $O(s^2 t 2^s)$ edges so that every 2-coloring of $E(G)$ contains a monochromatic $K_{s,t}$. $(\star)$

Theorem (Erdős–Rousseau 1993)

Any G with $O(st2^s)$ edges does not have property $(\star)$.

This is proved by considering a uniformly random coloring.

Theorem (Conlon–Fox–W. 2022+)

Any G with $O(s^{2-\frac{s}{t}} t 2^s)$ edges does not have property $(\star)$.
In particular, if $t > Cs \log s$, then the EFRS theorem is best possible.

New construction: Instead of a uniformly random coloring, use a "dyadically iterated hypergeometric" random coloring.

Thank you!