New constructions in Ramsey theory

Yuval Wigderson

Thesis defense

May 6, 2022

My philosophy on Ramsey theory

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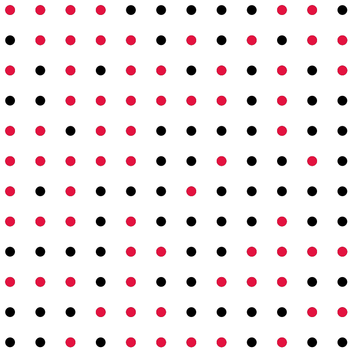

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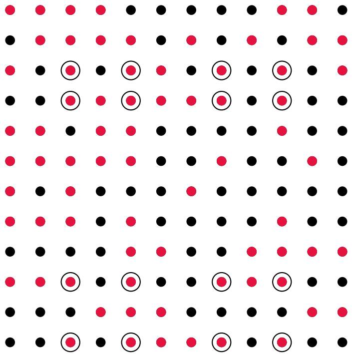

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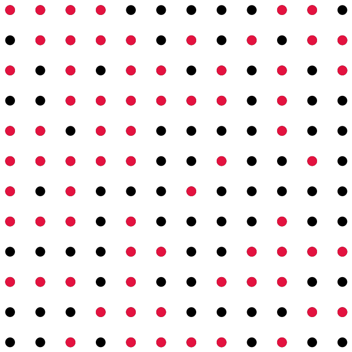

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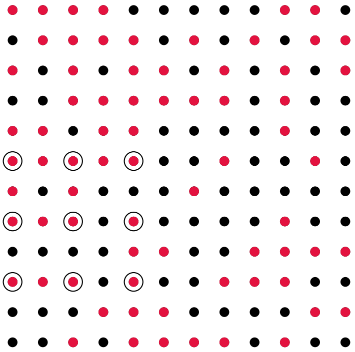

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Constructions are crucial for understanding such questions.

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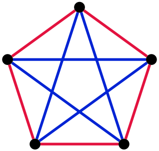

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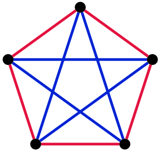

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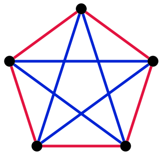

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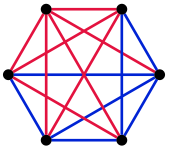

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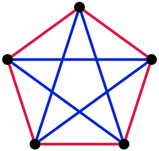

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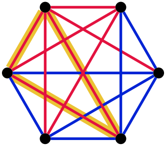

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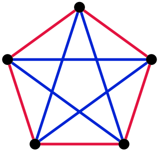

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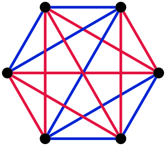

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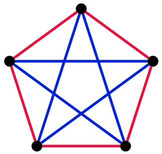

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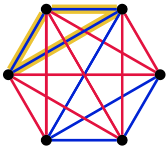

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$r(t) =$ minimum N so that every 2-coloring of the edges of K_N has a monochromatic K_t.

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So there exists a coloring of $E(K_N)$ with < 1 monochromatic K_t. □

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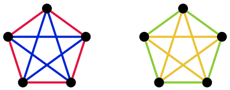

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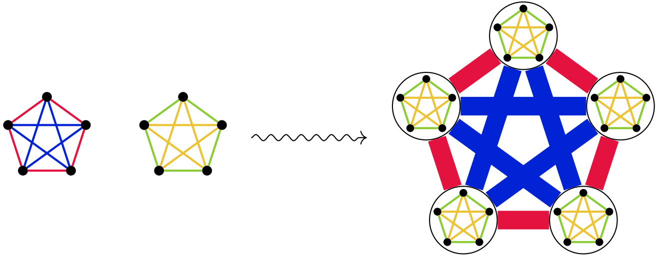

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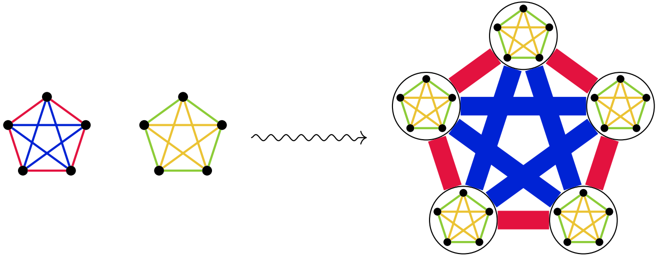

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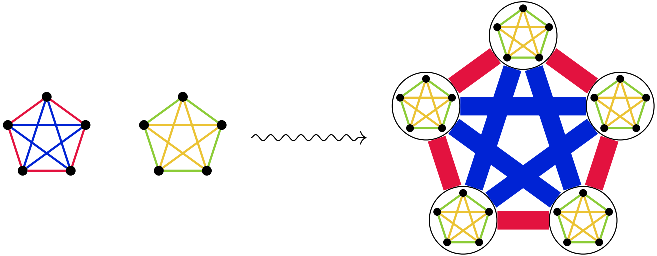

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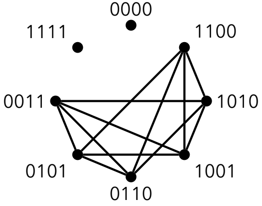

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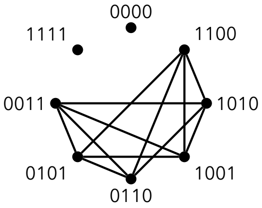

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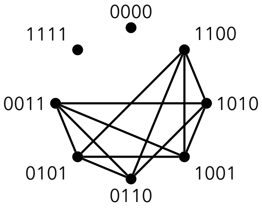

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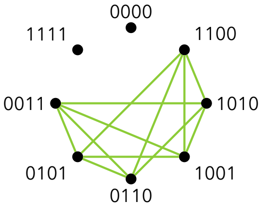

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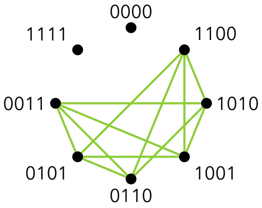

Let $p \approx 2^{-\frac{1}{8}t}$, and keep each vertex of G_t with probability p.

The Conlon–Ferber construction

For $x, y \in \mathbb{F}_2^t$, let $x \cdot y = \sum_{i=1}^t x_i y_i$. We define a graph G_t as follows.

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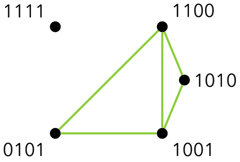

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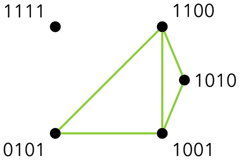

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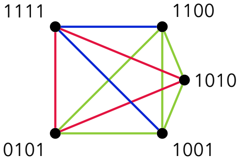

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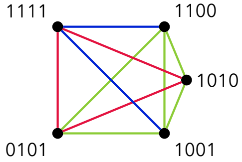

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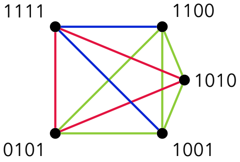

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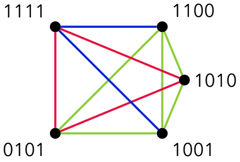

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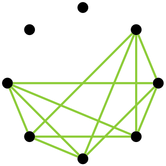
This works over larger fields, but the bounds aren't very good.
Conlon–Ferber use the product coloring for $q > 4$.

A new approach for more colors

Overlay two random copies of G_t in green and yellow.



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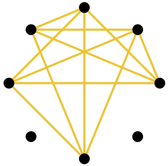

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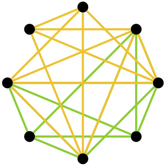

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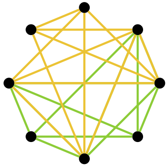

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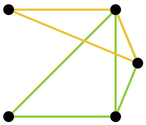

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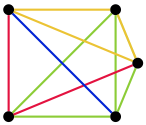

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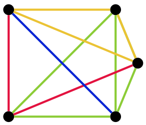

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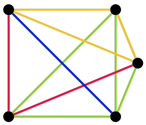

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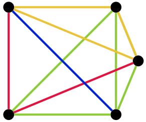

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How are we picking $p > 1$???

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Let p be any positive real number, and let $N = p|V_t|$.

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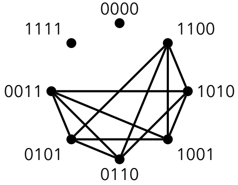

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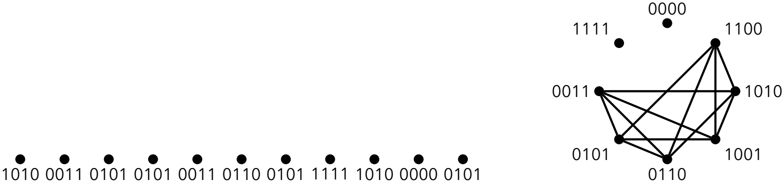

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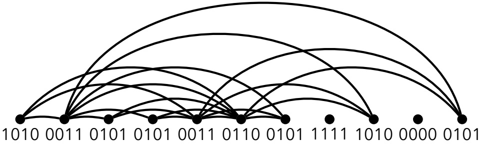

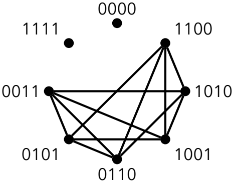

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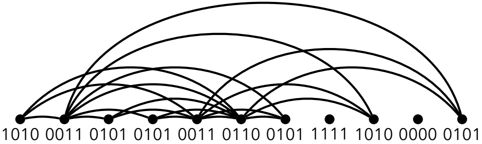

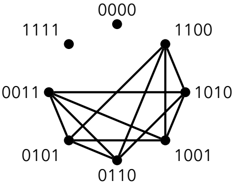

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If $p \ll 1$, $\widetilde{G_t}$ looks like keeping vertices from G_t with probability p.

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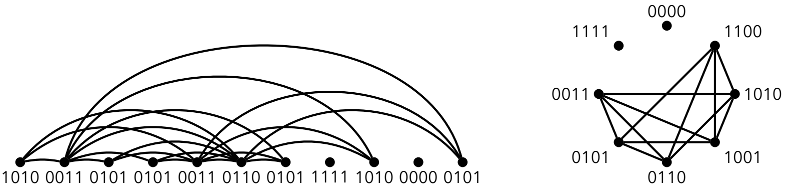

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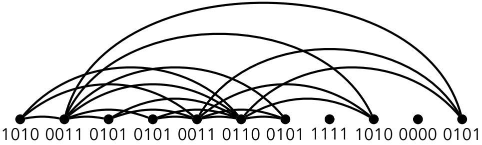

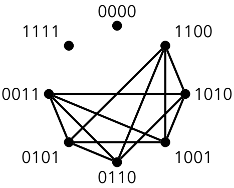

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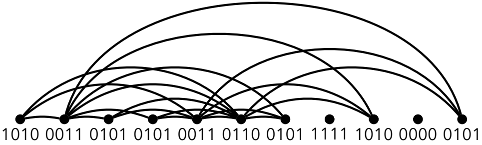

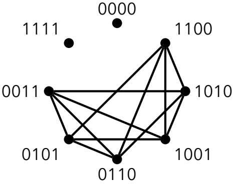

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So the above argument works for any p, if interpreted correctly.

Putting it all together

Theorem (W. 2021)

$$r(t;q) > \left(2^{\frac{3q}{8}-\frac{1}{4}}\right)^t.$$

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Proof: No reason to use G_t! An appropriately chosen random graph works better as input to the random homomorphism machinery. □

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Chvátal–Rödl–Szemerédi–Trotter (1983): If H has t vertices and maximum degree Δ, then $r(H) = O_\Delta(t)$.

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Theorem (Fox–He–W. 2022)

No! *For any $C > 0$, there exist bounded-degree H with $\vec{r}(H) > t^C$.*

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There exists a t-vertex acyclic digraph H with bounded maximum degree and $\vec{r}(H) > t^C$.

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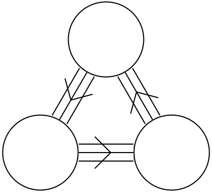

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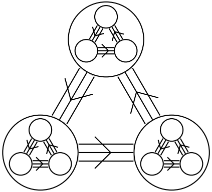

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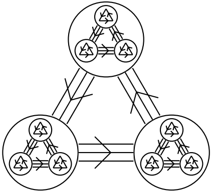

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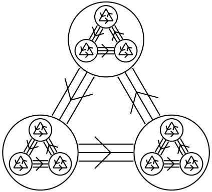

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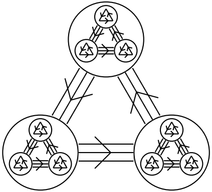

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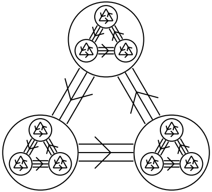

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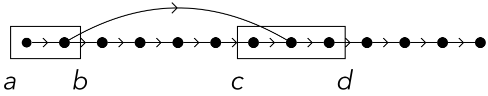

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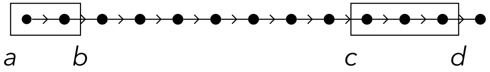

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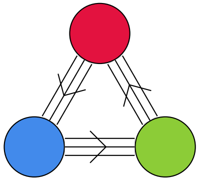

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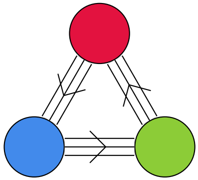

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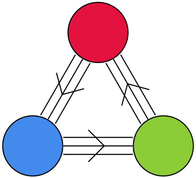

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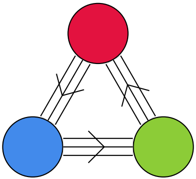

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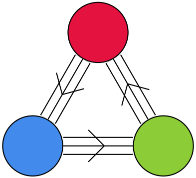

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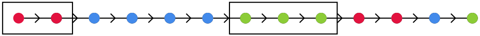

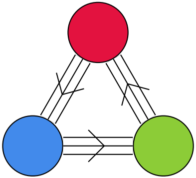

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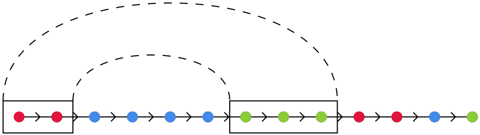

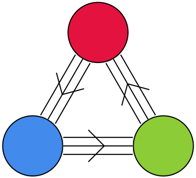

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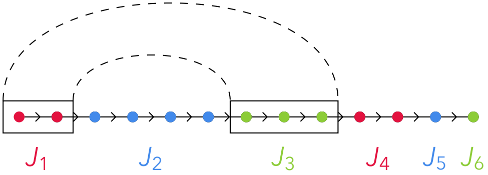

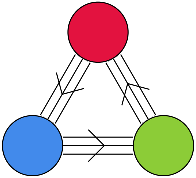

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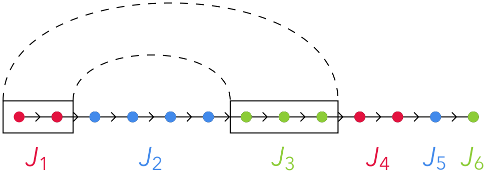

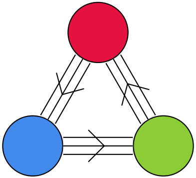

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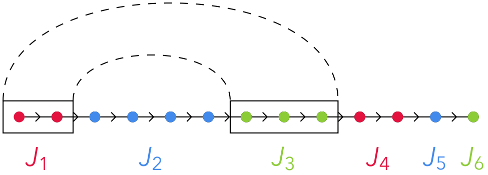

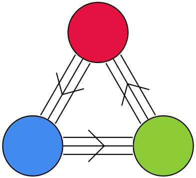

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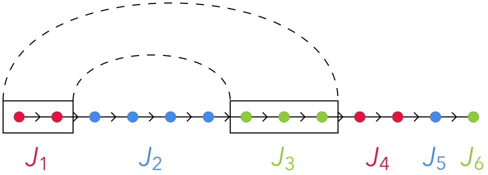

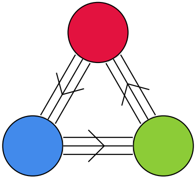

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Greedy algorithm yields an interval mesh with max degree ≤ 1000.

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New construction: Instead of a uniformly random coloring, use a "dyadically iterated hypergeometric" random coloring.

Thank you!