Ramsey numbers of sparse digraphs

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Joint with Jacob Fox and Xiaoyu He

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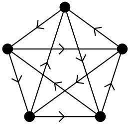

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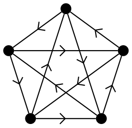

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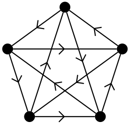

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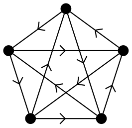

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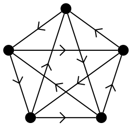

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Rédei's theorem $\iff \vec{r}(P_n) = n$, where $P_n =$ directed n-vertex path.

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So the Ramsey number is exponential if H is dense.

For the rest of the talk, we'll focus on sparse (di)graphs.

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Upshots: H has linear Ramsey number "if and only if" H is sparse. Qualitatively, n and d control $r(H)$.

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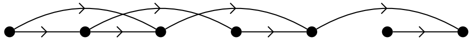

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  So a $o(1)$ fraction of H's edges have length $o(n)$.
- Our construction of a bounded-degree H with $\vec{r}(H) > n^C$ has many edges at every dyadic scale ("interval mesh").

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We need (1) a construction of H, (2) a tournament T on $n^{\log_2(3)-\varepsilon}$ vertices, and (3) a proof that there is no embedding $H \hookrightarrow T$.

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There exists an n-vertex acyclic digraph H with maximum degree ≤ 1000 and $\vec{r}(H) > n^{\log_2(3)-\varepsilon}$.

We need (1) a construction of H, (2) a tournament T on $n^{\log_2(3)-\varepsilon}$ vertices, and (3) a proof that there is no embedding $H \hookrightarrow T$.

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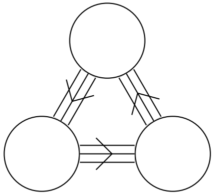

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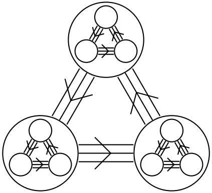

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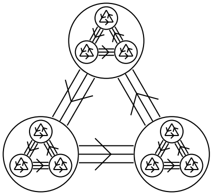

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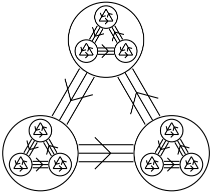

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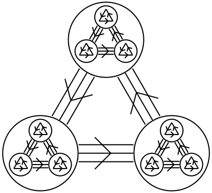

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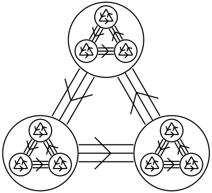

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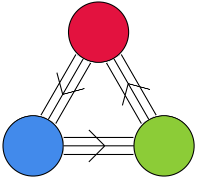

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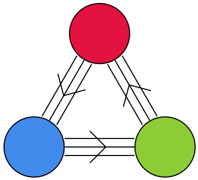

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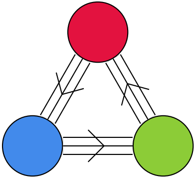

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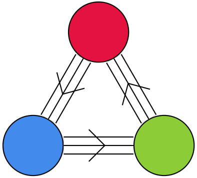

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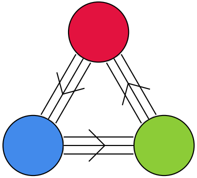

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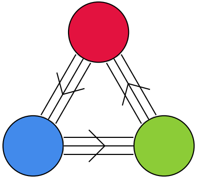

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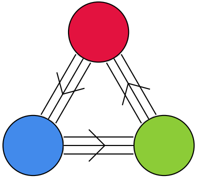

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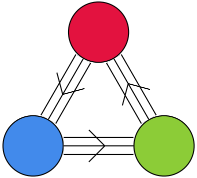

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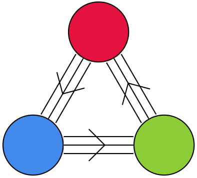

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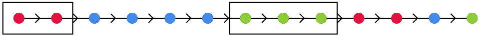

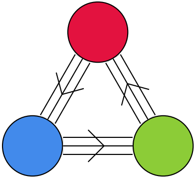

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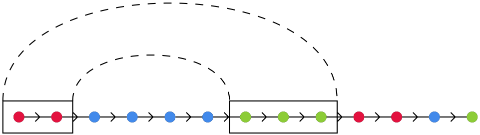

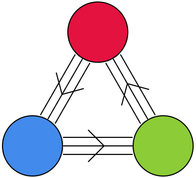

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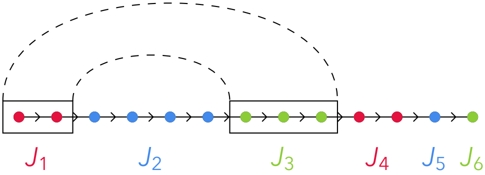

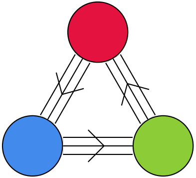

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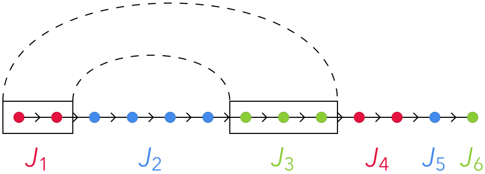

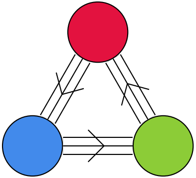

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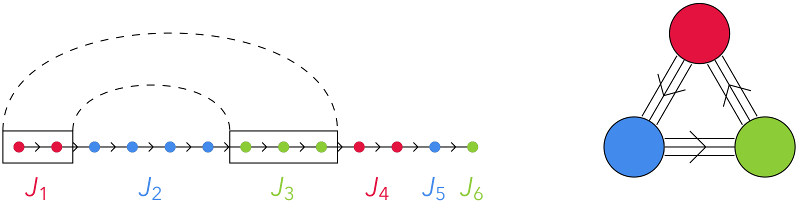

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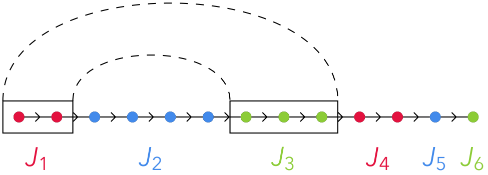

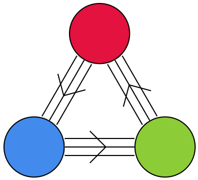

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Greedy algorithm yields an interval mesh with max degree ≤ 1000.

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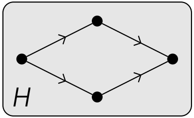

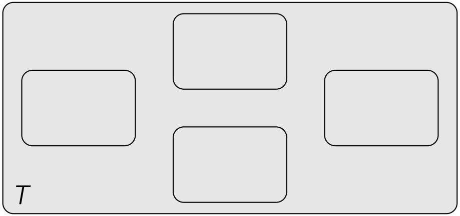

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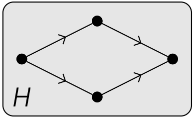

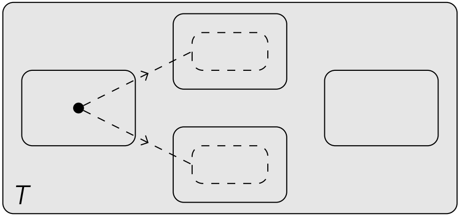

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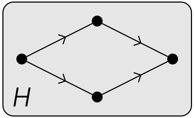

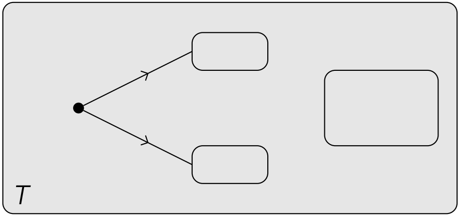

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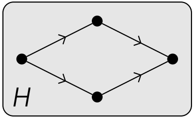

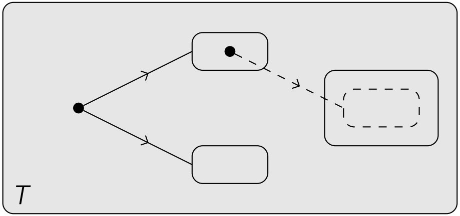

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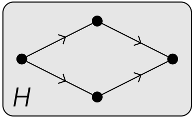

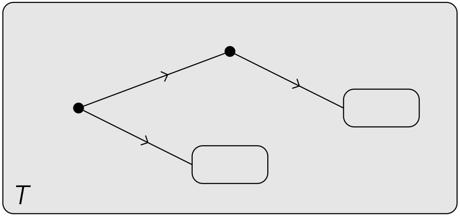

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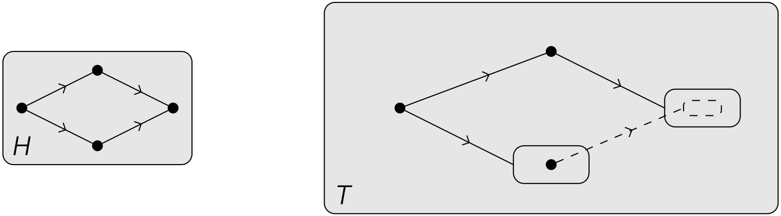

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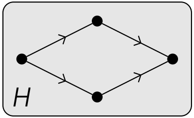

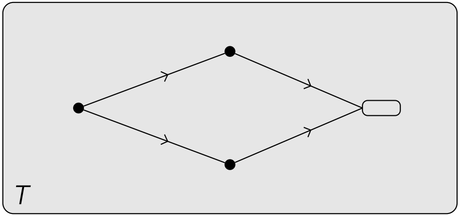

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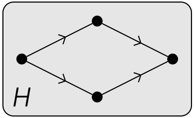

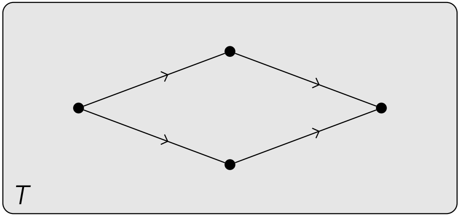

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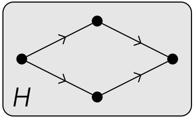

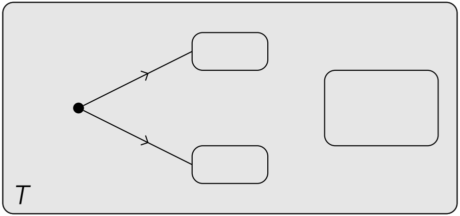

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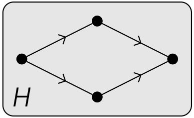

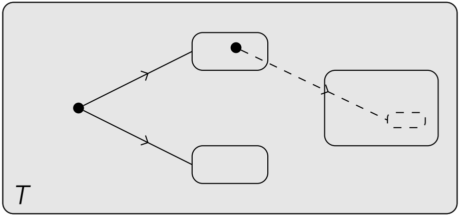

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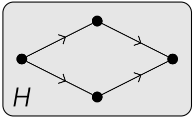

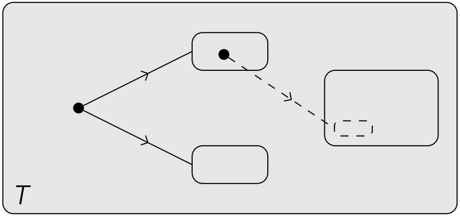

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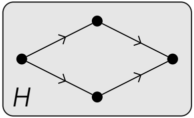

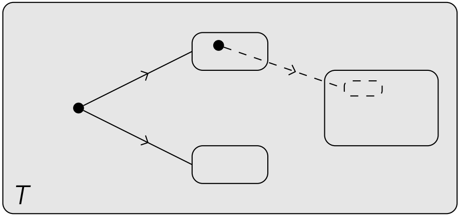

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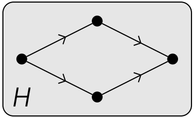

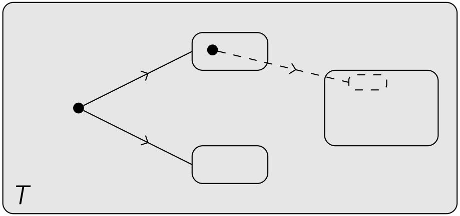

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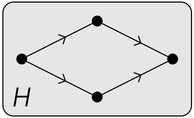

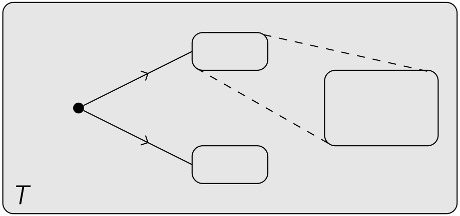

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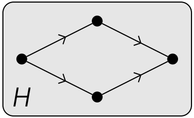

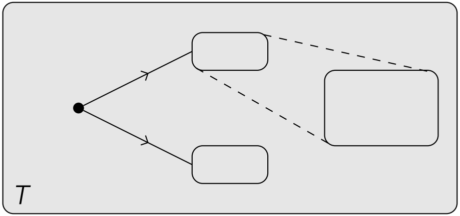

Lemma

If T is H-free, then T contains two large vertex sets with most edges between them oriented the same way.

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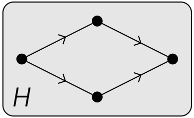

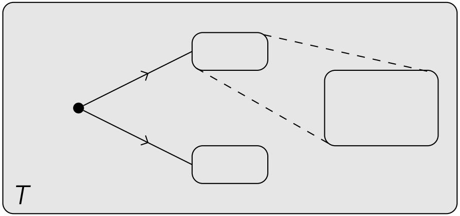

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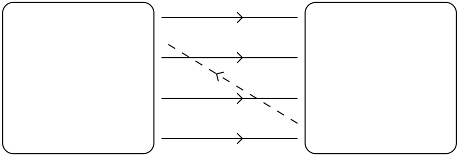

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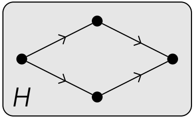

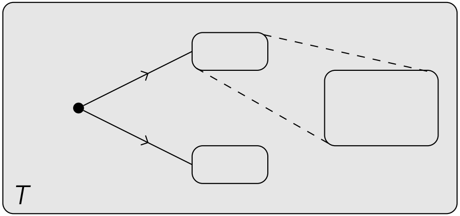

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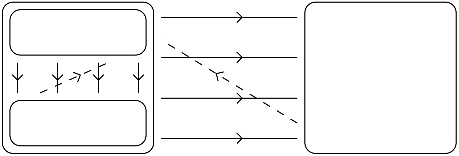

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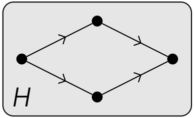

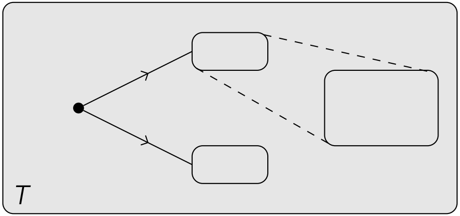

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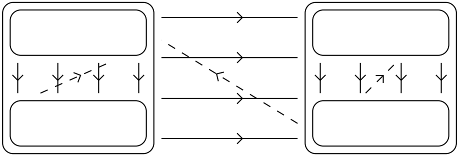

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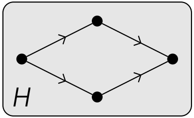

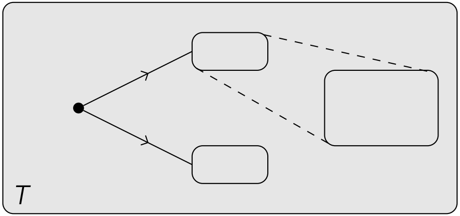

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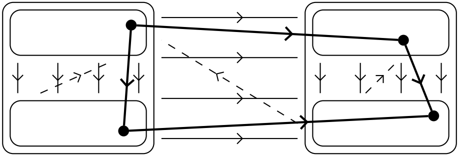

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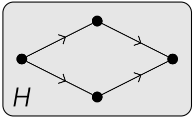

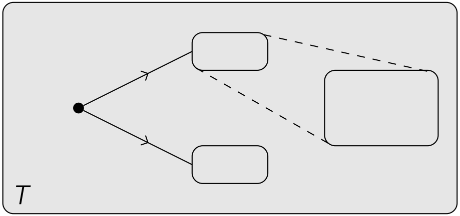

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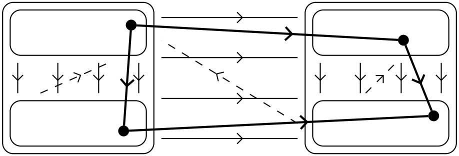

The multiscale complexity of *H* controls the number of iterations.

More colors and ordered Ramsey numbers

Summary: If H has n vertices and maximum degree Δ, then $\vec{r}(H) \leq n^{O_\Delta(\log n)}$, but $\vec{r}(H) \geq n^C$ is possible.

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- Can one combine greedy embedding with existing techniques (e.g. median ordering)?

Thank you!